Chapter 6

Analysis of balanced, completely randomized designs.

constant vs random effects

- *fixed (constant) effects* - factor levels affect the mean of the response
- *random effects* - factor levels affect the variability of the response

balanced complete factorial experiments

- Multifactor experiments consist of two or more controllable design factors whose *combinations* of levels are believed to affect the response.
- *Balanced, complete, factorial experiments* have an equal number of repeat tests for all combinations of levels of the design factors.

fixed factor effects

- Factors have *fixed effects* if the levels chosen for inclusion in the experiment are the only ones for which inferences are desired.
- Remarks:
  - (constant) influences are modeled by unknown parameters in statistical models that relate the mean of the response variable to the factor levels
  - fixed factor levels are not randomly selected!
  - we don’t have any assumptions about how intermediate levels affect response

analysis-of-variance (ANOVA)

- ANOVA procedures partition the variation into two basic components:
  1. *Assignable causes* - due to changes in experimental factors or measured covariates.
  2. *Random variation* - due to uncontrolled effects, including chance causes and measurement errors.

fixed-effects model assumptions

1. The levels of all factors in the experiments represent the only levels for which inferences are desired.
2. The analysis-of-variance model contains parameters (unknown constants) for all main effects and interactions in the experiment.
3. The experimental errors are statistically independent.
4. The experimental errors are satisfactorily modeled by the normal probability distribution with zero mean and (unknown) constant standard deviation.

components of the model

- effects of the assignable causes
  - main effects (individual factor levels)
  - interactions (combination of two or more factor levels)
- random error effects

Remarks:

- A two-factor interaction is present in a model only if the effects of two factors on the response cannot be adequately modeled by the main effects. Similarly, a three-factor interaction (...).
- We must impose constraints on the values of the parameters to make representation unique.

model for a three factor experiment

$$y_{ijkl} = \mu_{ijk} + e_{ijkl}$$
$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk}$$

Right side is not unique, we must impose constraints on the values of parameters, for example:

$$\Sigma_i \alpha_i = 0,\ \ldots,\ \Sigma_i (\alpha\beta)_{ij} = 0,\ \Sigma_j (\alpha\beta)_{ij} = 0, \ldots$$

parameters as averages

$$\mu = \bar{\mu}_{\bullet\bullet\bullet}$$

$$\alpha_i = \bar{\mu}_{i\bullet\bullet} - \bar{\mu}_{\bullet\bullet\bullet}$$

$$\ldots$$

Remark: main-effect and interaction parameters might be introduced in many ways.

hierarchical model

A statistical model is hierarchical if an interaction term involving k factors is included only when the main effects and lower-order interactions involving the k factors are also included in the model.

analysis-of-variance tables

- Assume we have a complete factorial experiment with three factors: A (a levels, subscript i), B (b levels, subscript j), C (c levels, subscript k) and r repeat test for each combination of the factors.
- We need to come up with a suitable measure of variation for which a partitioning into components can be accomplished
- *total sum of squares* TTS:

$$TSS = \Sigma_{i=1}^{n}(y_i - \bar{y})^2$$

some calculations

$$
\begin{aligned}
TSS &= \Sigma_i \Sigma_j \Sigma_k \Sigma_l (y_{ijkl} - \bar{y}_{\bullet\bullet\bullet\bullet})^2 \\
&= r \Sigma_i \Sigma_j \Sigma_k (y_{ijk\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet})^2 + \Sigma_i \Sigma_j \Sigma_k \Sigma_l (y_{ijkl} - \bar{y}_{ijk\bullet})^2 \\
&= MSS + SS_E \\
&= SS_A + SS_B + SS_A B + SS_A C + SS_B C + SS_A BC + SS_E
\end{aligned}
$$

For example:

$$SS_A = bcr \Sigma_{i=1}^{a} (\bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet})^2$$

Source of Variation	df	Sum of Squares	Mean Square	F-Value
Temperature T	1	2116.00	2116.00	264.50
Concentration Co	1	100.00	100.00	12.50
Catalyst Ca	1	9.00	9.00	1.13
T × Co	1	9.00	9.00	1.13
T × Ca	1	400.00	400.00	50.00
Co × Ca	1	0.00	0.00	0.00
T × Co × Ca	1	1.00	1.00	0.13
Error	8	64.00	8.00	
Total	15	2699.00		

parameter estimation - error standard deviation

- key assumption: the model errors can be considered to follow a common probability distribution (usually the normal probability distribution)
- for *saturated* model and complete factorial experiment with repeated tests, the error mean square is

$$s_e^2 = MS_E = \frac{1}{abc} \Sigma_i \Sigma_j \Sigma_k s_{ijk}^2.$$

Remark: if there are no repeat tests and the model is saturated, there is no estimate of experimental error available (unless some of the parameters acan be assumed to be zero).

parameter estimation - effects parameters

- for fixed-effects ANOVA model, the model means are estimated by the corresponding response averages
- for example $\bar{y}_{ijk\bullet}$ estimate the model means μ_{ijk}
- we can also compare factor-level means
- we can place confidence intervals as discussed in Chapter 3

Remark: in general, estimation of the individual parameters depends on how they were defined (how they were related to the model means)

quantitative factor levels

- if factor levels are quantitative, we can assess whether factor effects are linear, quadratic, cubic or possibly of higher order
- we have to assume a smooth functional relationship between the factor levels and the mean of the response variable

statistical tests

- an alternative methodology to the interval-estimation procedures
- tests of the hypotheses on individual model means are extensions of the single-sample t-tests
- separate t-tests for each main effect or interaction effect can cause problems with Type I errors, we can use F-tests

		ANOVA		
Source of Variation	Degrees of Freedom (df)	Sum of Squares	Mean Square	*F*-Value
A	$a-1$	SS_A	$MS_A = SS_A/df(A)$	$F_A = MS_A/MS_E$
B	$b-1$	SS_B	$MS_B = SS_B/df(B)$	$F_B = MS_B/MS_E$
C	$c-1$	SS_C	$MS_C = SS_C/df(C)$	$F_C = MS_C/MS_E$
AB	$(a-1)(b-1)$	SS_{AB}	$MS_{AB} = SS_{AB}/df(AB)$	$F_{AB} = MS_{AB}/MS_E$
AC	$(a-1)(c-1)$	SS_{AC}	$MS_{AC} = SS_{AC}/df(AC)$	$F_{AC} = MS_{AC}/MS_E$
BC	$(b-1)(c-1)$	SS_{BC}	$MS_{BC} = SS_{BC}/df(BC)$	$F_{BC} = MS_{BC}/MS_E$
ABC	$(a-1)(b-1)(c-1)$	SS_{ABC}	$MS_{ABC} = SS_{ABC}/df(ABC)$	$F_{ABC} = MS_{ABC}/MS_E$
Error	$abc(r-1)$	SS_E	$MS_E = SS_E/df(\text{Error})$	
Total	$abcr-1$	TSS		

multiple comparisons

- sometimes we want to know what effects are accounting for the results obtained in an experiment
- we are not interested in the existence of overall experimental effects, so ANOVA table is not very useful
- we can explore in more detail on groups of means for which F-test has shown significance

multiple comparisons continued

- general comparison of means
- comparisons based on t-statistics
  - Fisher's LSD (if two averages are significantly different, given different number of observations)
  - Bonferroni comparisons (if linear combination of means is equal 0)
  - Tukey's TSD (if two averages are significantly different, given the same number of observations)

graphical comparisons

- cube plots (5.3)
- trellis plots (6.4)
- interaction plots (6.5)
- leat significant interval plot (6.6)