STOCHASTIC LOCAL SEARCH

FOUNDATIONS AND APPLICATIONS

Search Space Structure and SLS Performance

Holger H. Hoos & Thomas Stützle

Outline

1. Fundamental Search Space Properties
2. Search Landscapes and Local Minima
3. Fitness-Distance Correlation
4. Ruggedness
5. Plateaux
6. Barriers and Basins

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Note: The diameter of a given search space depends on the *neighbourhood size*.

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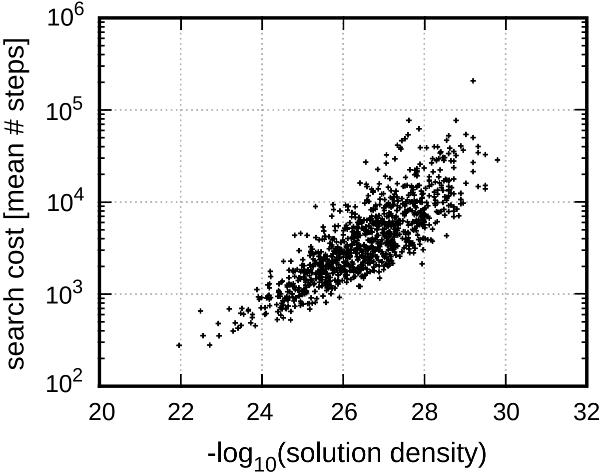
  - counting algorithms (often variants of complete algorithms).
- In many cases, (optimal) solutions tend to be clustered; this is reflected in uneven distributions of pairwise distances between solutions.

Example: Correlation between solution density and search cost for GWSAT over set of hard Random-3-SAT instances:



Search Landscapes

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the *search landscape of π, $L(\pi)$*, is defined as $L(\pi) := (S(\pi), N(\pi), g(\pi))$.

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Classification of search positions (according to evaluation function values of direct neighbours):

position type	$>$	$=$	$<$
SLMIN (strict local min)	$+$	–	–
LMIN (local min)	$+$	$+$	–

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LMAX (local max)	–	+	+
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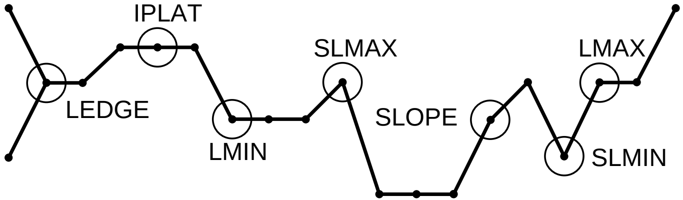
Example for various types of search positions:



Example: Complete distribution of position types for hard Random-3-SAT instances

instance	*avg sc*	SLMIN	LMIN	IPLAT
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`uf20-91/medium`	83.25	$<$ 0.01%	0.13%	0%
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`uf20-91/easy`	0.59%	99.27%	0.04%	$< 0.01\%$
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(based on exhaustive enumeration of search space;
sc refers to search cost for GWSAT)

Example: Sampled distribution of position types for hard Random-3-SAT instances

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`uf50-218/medium`	615.25	0%	47.29%	0%
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Solution: Approximation based on sampling or estimation from other measures (such as autocorrelation measures, see below).

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- Sample set of purportedly globally optimal tours using multiple independent runs of high-performance TSP algorithm.
- Measure minimal pairwise distances between local minima and respective closest optimal tour (using bond distance).

Empirical results:

Instance	avg sq [%]	avg d_{lmin}	avg d_{opt}
	Results for 3-opt		
`rat783`	3.45	197.8	185.9
`pr1002`	3.58	242.0	208.6
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(based on local minima collected from 1 000/200 runs of 3-opt/ILS)

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  ⇒ *Higher-quality local minima tend to be concentrated in smaller regions of the search space.*

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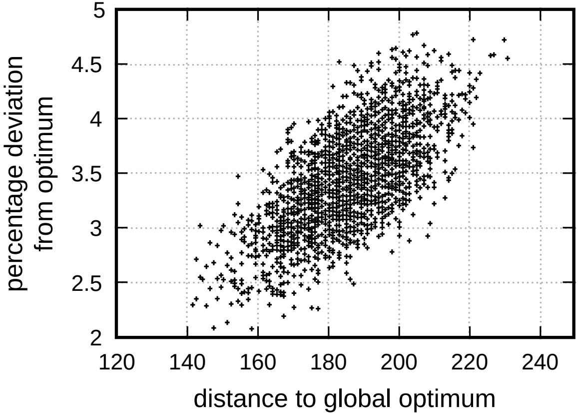
- r_{fdc} is calculated based on a sample of m candidate solutions (typically: set of local optima found over multiple runs of an iterative improvement algorithm).
- *Fitness-distance plots*, *i.e.*, scatter plots of the (g_i, d_i) pairs underlying an estimate of r_{fdc}, are often useful to graphically illustrate fitness distance correlations.

Example: FDC plot for TSPLIB instance `rat783`, based on 2500 local optima obtained from a 3-opt algorithm



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Low FDC (r_{fdc} close to zero):

- global structure of landscape does not provide guidance for local search;
- typical for very hard combinatorial problems, such as certain types of QAP (Quadratic Assignment Problem) instances.

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- *a posteriori* method, requires set of (optimal) solutions, **but:** results often generalise to larger instance classes;
- optimal solutions are often not known, using best known solutions can lead to erroneous results;
- can give misleading results when used as the sole basis for assessing problem or instance difficulty.

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Example: Smooth *vs* rugged search landscape

Ruggedness

Idea: Rugged search landscapes, *i.e.*, landscapes with high variability in evaluation function value between neighbouring search positions, are hard to search.

Example: Smooth *vs* rugged search landscape

Note: Landscape ruggedness is closely related to local minima density: rugged landscapes tend to have many local minima.

The ruggedness of a landscape L can be measured by means of the *empirical autocorrelation function* $r(i)$:

$$r(i) := \frac{1/(m-i) \cdot \sum_{k=1}^{m-i}(g_k - \bar{g}) \cdot (g_{k+i} - \bar{g})}{1/m \cdot \sum_{k=1}^{m}(g_k - \bar{g})^2}$$

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Note: $r(i)$ and ξ depend on the given neighbourhood relation.

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- There are other measures of ruggedness, such as *(empirical) correlation length*.

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  - classifying the diffculty of combinatorial problems.

Plateaux

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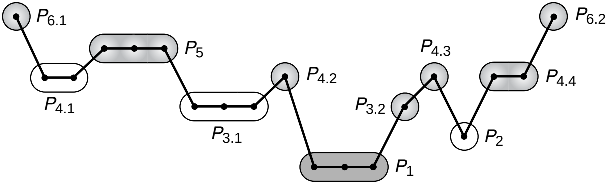

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- *Plateau region:* region in which all positions have the same level, *i.e.*, evaluation function value, l.
- *Plateau:* maximally extended plateau region, *i.e.*, plateau region in which no border position has any direct neighbours at the plateau level l.

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- *Open / closed plateau:* plateau with / without exits.

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- *number of exits*, *exit density*
- *distribution of exits within a plateau*, *exit distance distribution* (in particular: avg./max. distance to closest exit)

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- The diameter of plateaux, particularly at higher levels, is comparable to the diameter of search space. (In particular: plateaux tend to span large parts of the search space, but are quite well connected internally.)
- For open plateaux, exits tend to be clustered, but the average exit distance is typically relatively small.

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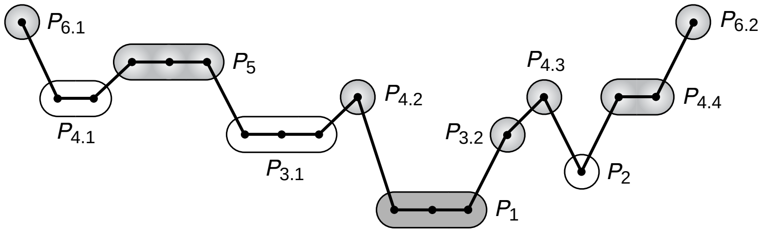
Plateau connection graphs (PCGs):

- *Vertices:* plateaux of given landscape
- *Edges (directed):* connect plateaux that are directly connected by one or more exit.
- Additionally, *edge weights* can be used to indicate the relative numbers of exits from one plateau to its PCG neighbours.

Example: Simple neutral search landscape L ...



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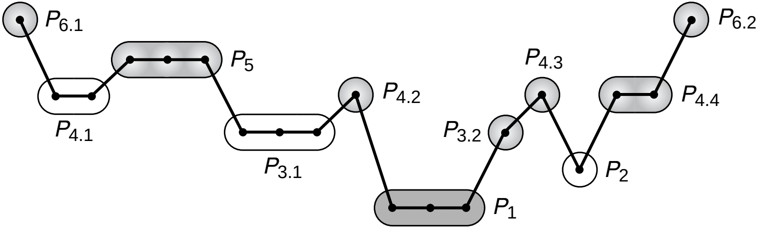
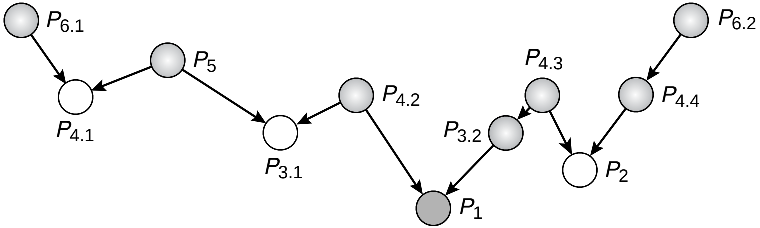
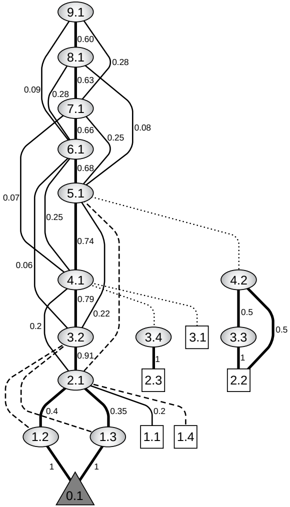


Note: The plateaux form a partition of L, *i.e.* every position in L is part of exactly one (possibly degenerate) plateau.

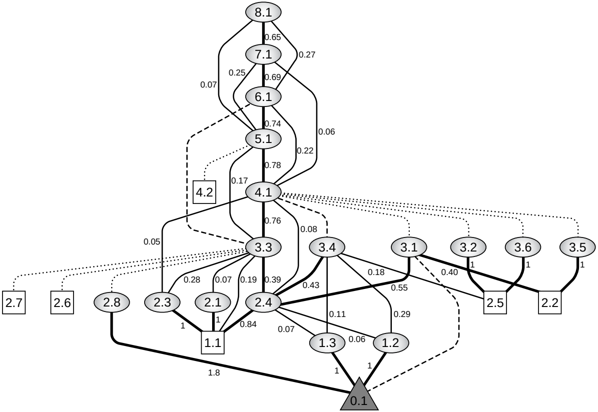
Example: . . . and the respective plateau connection graph



Example: PCG of *easy* Random 3-SAT instance



Example: PCG of *hard* Random 3-SAT instance



Barriers and Basins

Observation:

The *difficulty of escaping* from closed plateaux or strict local minima is related to the *height of the barrier*, *i.e.*, the difference in evaluation function, that needs to be overcome in order to reach better search positions:

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The *difficulty of escaping* from closed plateaux or strict local minima is related to the *height of the barrier*, *i.e.*, the difference in evaluation function, that needs to be overcome in order to reach better search positions:

Higher barriers are typically more difficult to overcome (this holds, *e.g.*, for Probabilistic Iterative Improvement or Simulated Annealing).

Definition:

- Positions s, s' are *mutually accessible at level l* iff there is a path connecting s' and s in the neighbourhood graph that visits only positions t with $g(t) \leq l$.

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  the difference between the level of s and $bl(s, s')$ is called the *barrier height between s and s'*.

- The *depth of a position s* is the minimal barrier height between s and any position s' at a level lower than s, *i.e.*, for which $g(s') < g(s)$.

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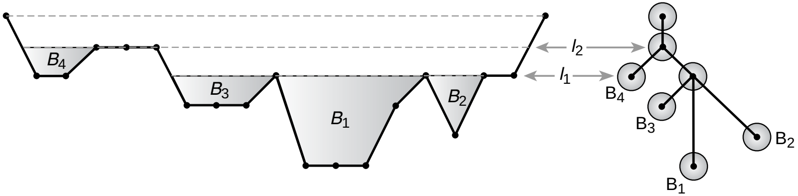
- Basins of a given landscape form a *hierarchy*, *i.e.*, two basins are either disjoint, or one is contained in the other.

- Basin hierarchies can be formally represented as *basin trees*.

Example: Basins in a simple search landscape and corresponding basin tree



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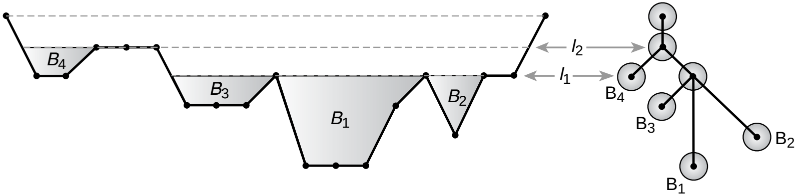

Note: The basin tree only represents basins just below the critical levels at which neighbouring basins are joined (by a *saddle*).

Note:

- Like plateau connection graphs, basin trees can provide much deeper insights into SLS behaviour and problem hardness than global measures of search space structure, such as FDC or ACC.

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- Like plateau connection graphs, basin trees can provide much deeper insights into SLS behaviour and problem hardness than global measures of search space structure, such as FDC or ACC.
- **But:** This type of analysis is computationally expensive, since it requires enumeration (or sampling) of large parts of the search space.