Preconditioning for Two-Phase Flows

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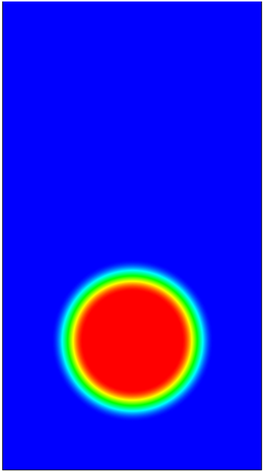
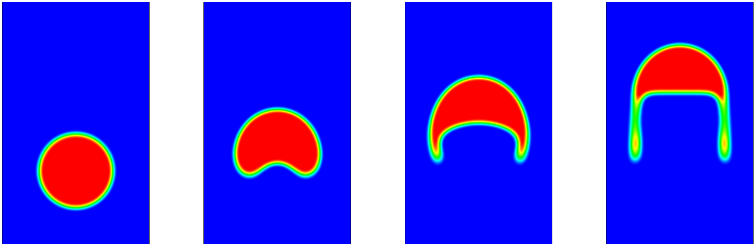
Jointly with Martin Stoll (moving to TU Chemnitz) and Christian Kahle (TU Munich).



Motivation

[Abels/Garcke/Grün '12] proposed a diffuse interface model for incompressible two-phase flows with different densities.



t=0 t=1 t=2 t=3

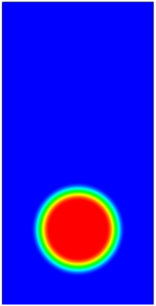
[Garcke/Hinze/Kahle '16] developed a thermodynamically consistent discretization scheme for that model.

Our Contributions

- Fully iterative solution of the large and sparse linear systems.
- Development of robust preconditioners.

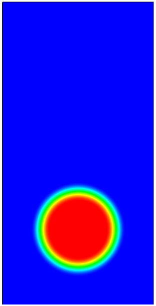
Setting

- two-phase flow



Setting

- two-phase flow
- two immiscible, incompressible viscous fluids



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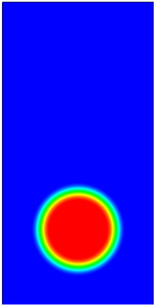

Setting

- two-phase flow
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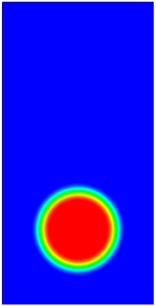
- different physical properties
  - densities $\tilde{\rho}_1$, $\tilde{\rho}_2$
  - viscosities $\tilde{\eta}_1$, $\tilde{\eta}_2$



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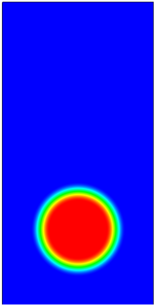

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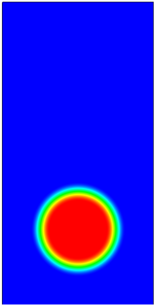
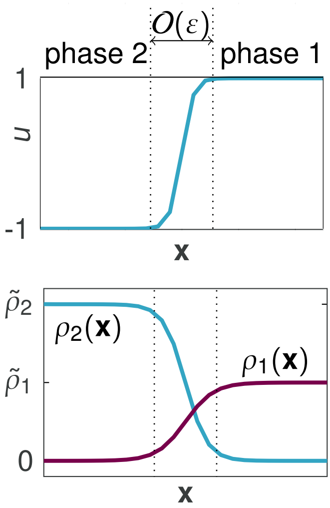
- diffuse interface model
  - to cope with topological changes



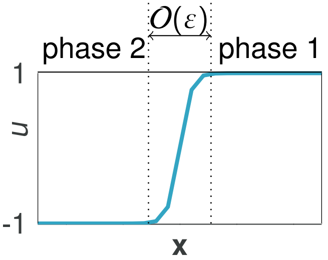
- continuous order-parameter u indicating the two phases

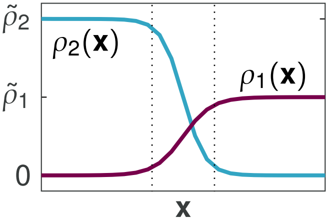


Diffuse Interface Approach

- continuous order-parameter u indicating the two phases

$$u(\mathbf{x}) = \frac{\rho_1(\mathbf{x})}{\tilde{\rho}_1} - \frac{\rho_2(\mathbf{x})}{\tilde{\rho}_2}$$





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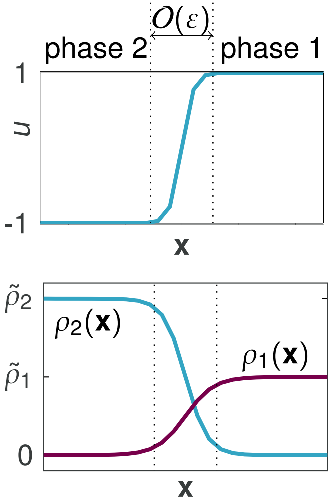
$$\in \begin{cases} \{-1\} & \text{if } \mathbf{x} \text{ in phase 2} \\ (-1,1) & \text{if } \mathbf{x} \text{ in interface} \\ \{1\} & \text{if } \mathbf{x} \text{ in phase 1} \end{cases}$$



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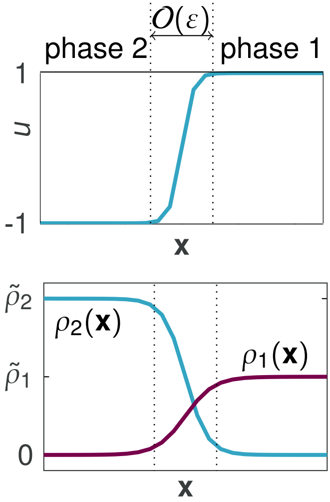
- interface of small thickness $\mathcal{O}(\varepsilon)$



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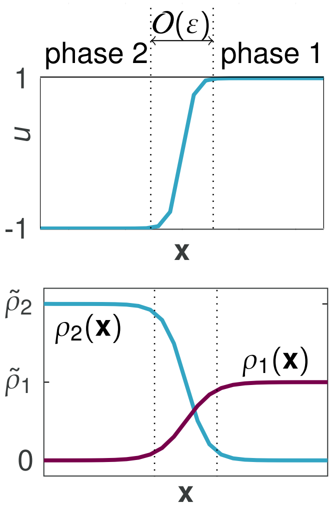
- interface of small thickness $\mathcal{O}(\varepsilon)$
- mixing inside the interface



Cahn–Hilliard Part

Cahn–Hilliard equations

$$\begin{cases} \partial_t u = \operatorname{div}(m\nabla w) \\ w = -\sigma\varepsilon\Delta u + \sigma\varepsilon^{-1}\psi_s'(u) \end{cases}$$

- w chemical potential
- $m(u)$ mobility
- σ surface tension
- ε interfacial width
- $\psi_s(u) = \frac{1}{2}(1-u^2) + \frac{s}{2}(\max^2(0, u-1) + \min^2(0, u+1))$
- s regularization parameter

- Boundary conditions
  - $\nabla w \cdot \mathbf{n} = 0$
  - $\nabla u \cdot \mathbf{n} = 0$

Convective Cahn–Hilliard equations

$$\begin{cases} \mathbf{v} \cdot \nabla u + \partial_t u &= \operatorname{div}(m \nabla w) \\ w &= -\sigma\varepsilon\Delta u + \sigma\varepsilon^{-1}\psi_s'(u) \end{cases}$$

- w chemical potential
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- s regularization parameter
- $\mathbf{v}$ velocity

- Boundary conditions
  - $\nabla w \cdot \mathbf{n} = 0$
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Navier–Stokes equation

$$\begin{cases} \rho\partial_t \mathbf{v} + ((\rho\mathbf{v} \qquad) \cdot \nabla)\mathbf{v} - \operatorname{div}(2\eta D\mathbf{v}) + \nabla p &= \rho\mathbf{g} \\ \operatorname{div}\mathbf{v} &= 0 \end{cases}$$

- $\rho(u)$ density
- $\eta(u)$ viscosity
- $\mathbf{g}$ gravitation
- $2D\mathbf{v} = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$ strain tensor

- Boundary conditions
  - $\mathbf{v} = \mathbf{f}$ with $\mathbf{f} \cdot \mathbf{n} = 0$

Navier–Stokes Part

Navier–Stokes equation + surface tension force

$$\begin{cases} \rho\partial_t \mathbf{v} + ((\rho\mathbf{v} + J)\cdot\nabla)\mathbf{v} - \operatorname{div}(2\eta D\mathbf{v}) + \nabla p &= \rho\mathbf{g} + w\nabla u \\ \operatorname{div}\mathbf{v} &= 0 \end{cases}$$

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- $2D\mathbf{v} = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$ strain tensor
- $J = -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m(u)\nabla w$

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  - $\mathbf{v} = \mathbf{f}$ with $\mathbf{f}\cdot\mathbf{n} = 0$

Coupled System

Cahn–Hilliard Navier–Stokes system by [ABELS/GARCKE/GRÜN '12]:

$$\begin{cases} \rho\partial_t \mathbf{v} + ((\rho\mathbf{v} + J)\cdot\nabla)\mathbf{v} - \operatorname{div}(2\eta D\mathbf{v}) + \nabla p &= \rho\mathbf{g} + w\nabla u \\ \operatorname{div}\mathbf{v} &= 0 \\ \mathbf{v}\cdot\nabla u + \partial_t u &= \operatorname{div}(m\nabla w) \\ w &= -\sigma\varepsilon\Delta u + \sigma\varepsilon^{-1}\psi_s'(u) \end{cases}$$

- $w\nabla u$ capillary force
- $\mathbf{v}\cdot\nabla u$ convection term
- $J = -\frac{\tilde\rho_2-\tilde\rho_1}{2} m(u)\nabla w$ crucial for thermodynamical consistency

Discretization

[Garcke/Hinze/Kahle '16] developed a discretization scheme:

- thermodynamical consistent

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- thermodynamical consistent
- adaptive spatial discretization
- Taylor-Hood LBB-stable $P2 - P1$ finite elements for the velocity-pressure field
- $P1$ finite elements for the phase field and chemical potential

Fully coupled linear system

$$\mathcal{A}\mathbf{z} = \left(\begin{array}{ccc|cc} F_{11} & F_{12} & B_1^T & I_1 & 0 \\ F_{21} & F_{22} & B_2^T & I_2 & 0 \\ B_1 & B_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{11} & C_{12} \\ T_1 & T_2 & 0 & C_{21} & C_{22} \end{array}\right) \begin{pmatrix} \delta v_1 \\ \delta v_2 \\ \delta p \\ \delta w \\ \delta u \end{pmatrix} = \begin{pmatrix} r_{v_1} \\ r_{v_2} \\ r_p \\ r_w \\ r_u \end{pmatrix}$$

- **F** discrete convection-diffusion operator
- **B** negative discrete divergence operator
- **I** interfacial force coupling
- **T** interfacial transport coupling
- C_{11}, C_{22} mass matrices
- C_{12} diffusion operator + regularization
- C_{21} mobility dependent diffusion operator

Outer Preconditioner

For our original problem with

$$\mathcal{A} = \left(\begin{array}{ccc|cc} F_{11} & F_{12} & B_1^T & I_1 & 0 \\ F_{21} & F_{22} & B_2^T & I_2 & 0 \\ B_1 & B_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{11} & C_{12} \\ T_1 & T_2 & 0 & C_{21} & C_{22} \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A}_{NS} & \mathcal{C}_I \\ \hline \mathcal{C}_T & \mathcal{A}_{CH} \end{array}\right),$$

we consider the right preconditioned matrix $\mathcal{A}\mathcal{P}_{out}^{-1}$ with

$$\mathcal{P}_{out} = \left(\begin{array}{c|c} \hat{\mathcal{A}}_{NS} & \mathcal{C}_I \\ \hline 0 & \hat{\mathcal{S}} \end{array}\right)$$

where $\hat{\mathcal{A}}_{NS} \approx \mathcal{A}_{NS}$ and $\hat{\mathcal{S}} \approx \mathcal{S} = \mathcal{A}_{CH} - \mathcal{C}_T \mathcal{A}_{NS}^{-1} \mathcal{C}_I$.

Why is $\mathcal{P}_{out}$ a good starting point?

For any nonsingular matrix of the form

$$\mathcal{K} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

as our coupled problem, [Murphy/Golub/Wathen '00], [Ipsen '01] showed exactly one eigenvalue at 1 for the preconditioned system $\mathcal{K}\mathcal{P}_{\mathcal{K}}^{-1}$ with

$$\mathcal{P}_{\mathcal{K}} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B^T \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & S \end{pmatrix}.$$

Back to our CH NS Preconditioner

For our original problem with

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Practical Preconditioner for $\mathcal{A}_{NS}$

For the Navier–Stokes problem

$$\mathcal{A}_{NS} = \begin{pmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{pmatrix},$$

[Elman/Silvester/Wathen '05] showed

$$\hat{\mathcal{A}}_{NS} = \begin{pmatrix} \hat{\mathbf{F}} & \mathbf{B}^T \\ \mathbf{0} & \hat{S}_{NS} \end{pmatrix} \qquad \hat{S}_{NS} = A_p F_p^{-1} Q_p$$

as a good practical approximation, where

- A_p pressure space Laplacian
- F_p pressure space convection-diffusion operator
- Q_p pressure space mass matrix

Back to our CH NS Preconditioner

For our original problem with

$$\mathcal{A} = \left(\begin{array}{ccc|cc} F_{11} & F_{12} & B_1^T & I_1 & 0 \\ F_{21} & F_{22} & B_2^T & I_2 & 0 \\ B_1 & B_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{11} & C_{12} \\ T_1 & T_2 & 0 & C_{21} & C_{22} \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A}_{NS} & \mathcal{C}_I \\ \hline \mathcal{C}_T & \mathcal{A}_{CH} \end{array}\right),$$

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Preconditioner for the Schur complement $\mathcal{S}$

For

$$\mathcal{A} = \left(\begin{array}{c|c} \mathcal{A}_{NS} & \mathcal{C}_I \\ \hline \mathcal{C}_T & \mathcal{A}_{CH} \end{array}\right) \qquad \mathcal{P}_{out} = \left(\begin{array}{c|c} \hat{\mathcal{A}}_{NS} & \mathcal{C}_I \\ \hline 0 & \hat{\mathcal{S}} \end{array}\right)$$

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we have that

$$\begin{aligned} \mathcal{S} &= \mathcal{A}_{CH} - \mathcal{C}_T \mathcal{A}_{NS}^{-1} \mathcal{C}_I \\ &\approx \mathcal{A}_{CH} - \mathcal{C}_T \hat{\mathcal{A}}_{NS}^{-1} \mathcal{C}_I \\ &= \mathcal{A}_{CH} \\ &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}. \end{aligned}$$

Practical Preconditioner for $\mathcal{S}$

To solve with the Schur complement approximation

$$\mathcal{A}_{CH} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \approx \mathcal{S},$$

we apply preconditioned GMRES as inner iteration, with the inner preconditioner

$$\mathcal{P}_{in} = \begin{pmatrix} C_{11} & C_{12} \\ 0 & -\hat{S}_{CH} \end{pmatrix}.$$

Practical Preconditioner for $\mathcal{S}$

In order to approximate the Schur complement

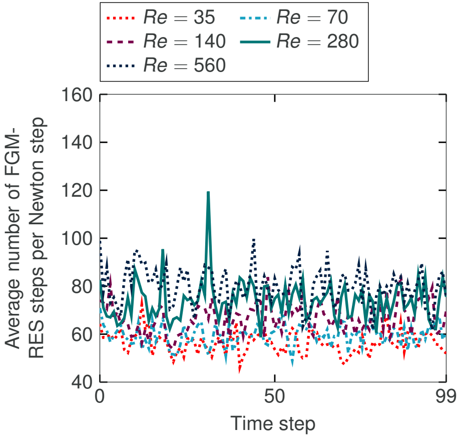
$$S_{CH} = C_{22} - C_{21} C_{11}^{-1} C_{12} \,,$$

we follow the matching strategy, proposed by [Pearson/Wathen '12]. We construct an approximation

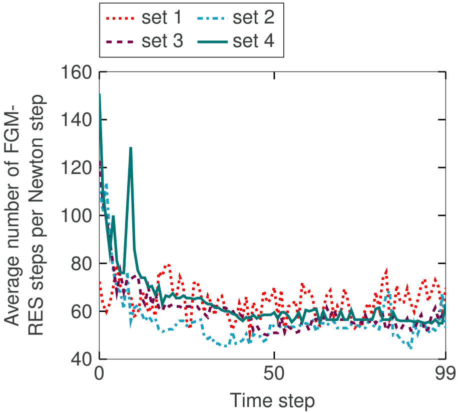
$$\begin{aligned} \hat{S}_{CH} &= (C_{11} + \alpha C_{21}) C_{11}^{-1} (C_{11} - \beta C_{12}) \\ &= C_{22} - \alpha\beta C_{21} C_{11}^{-1} C_{12} + \alpha C_{21} - \beta C_{12} \end{aligned}$$

with $\alpha, \beta > 0$ being chosen such that the exact Schur complement is captured as close as possible; see [B./Kahle/Stoll '17].

Iteration Numbers



(e) Varying the Reynolds number.



(f) Varying a set of parameters.

Conclusions

- Preconditioning is essential.
- Many well-studied ingredients can be used.
- Numerically robustness w.r.t. the mesh size and Reynolds number (here). In our paper also w.r.t. other parameters.

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THANK YOU!