Convergence of the multiplicative Schwarz method for convection-diffusion problems discretized on a Shishkin mesh

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joint work with

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Problem formulation

We consider **singularly-perturbed** elliptic B.V.P's given by:

$$-\epsilon u'' + \omega u' + \beta u = f \text{ in } \Omega = (0,1), \quad u(0) = u_0, \quad u(1) = u_1.$$

- The presence of **boundary layers** makes standard discretization techniques fail.
- Adaptive meshes or stabilization techniques need to be used.
- We consider the **Shishkin mesh**: a piecewise equidistant mesh.
- We will analyze the **multiplicative Schwarz method** in the context of domain decomposition methods.

The Shishkin mesh

The 1D Shishkin mesh divides the domain into two subdomains:

Use of FDMs on this mesh leads to discrete operators of the form:

$$\left(\begin{array}{cccc|c|cccc}
a_H & b_H & & & & & & & \\
c_H & \ddots & \ddots & & & & & & \\
 & \ddots & \ddots & b_H & & & & & \\
 & & c_H & a_H & b_H & & & & \\
\hline
 & & & c & a & b & & & \\
\hline
 & & & & c_h & a_h & b_h & & \\
 & & & & & c_h & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & b_h \\
 & & & & & & & c_h & a_h
\end{array}\right).$$

Properties of the discrete operator

We thus need to obtain solutions to algebraic systems of the form:

$$\boxed{\mathbf{A}\mathbf{u}^N = \mathbf{f}^N}$$

where the discretized convection-diffusion matrix A is non-symmetric, nonnormal and typically ill conditioned.

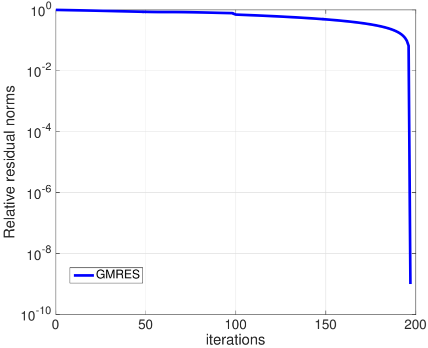
Numerical example: N=198, $\epsilon = 10^{-8}$, $\omega = 1$, $\beta = 0$, and boundary conditions $u(0) = u(1) = 0$, we have:

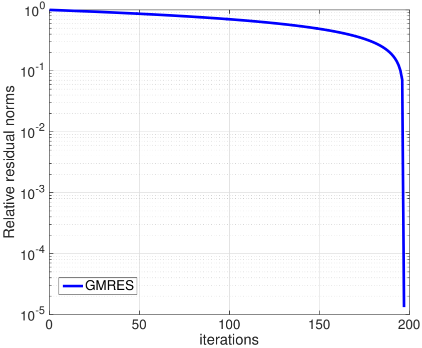
	$\kappa_2(A)$	$\kappa_2(Y)$
FDM upwind	4.0500×10^{10}	1.9016×10^{17}

where $A = Y \Lambda Y^{-1}$.

Poor performance of standard solution methods

The unpreconditioned GMRES method performs poorly when used to solve these types of problems:



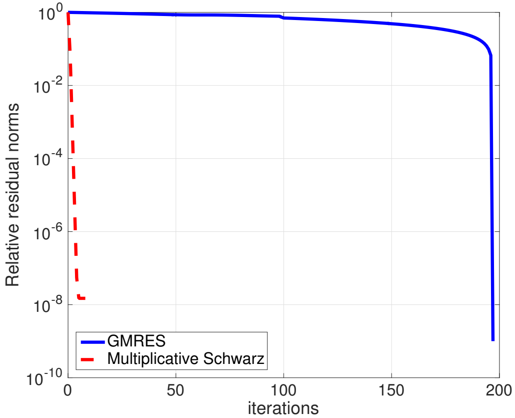


N=198, $\epsilon = 10^{-4}$ and 10^{-8}, $\omega = 1$, $\beta = 0$, $u(0) = u(1) = 0$.

Other solution methods, like the unpreconditioned BiCGSTAB, perform in a similar way.

Special solution techniques are needed

On the other hand the multiplicative Schwarz method applied to a Shishkin mesh discretization works well:



N=198, $\epsilon = 10^{-4}$, $\omega = 1$, $\beta = 0$, $u(0) = u(1) = 0$.

Can we prove this analytically?

The multiplicative Schwarz method

- The multiplicative Schwarz iterative scheme is

$$x^{k+1} = Tx^k + v, \quad T = (I - P_2)(I - P_1), \quad k = 0, 1, 2, \ldots,$$

  where the vector v is defined such that the scheme is **consistent**, i.e., $x = Tx + v$.
- The focus on each local domain is achieved using restriction operators:

$$R_1 = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & I_n \end{bmatrix}.$$

- The matrices corresponding to the solves on the two separate subdomains are:

$$P_i = R_i^T (R_i A R_i^T)^{-1} R_i A, \quad i = 1, 2.$$

Error at each step

The error of the multiplicative Schwarz iteration is given by

$$e^{k+1} = u^N - x^{k+1} = (Tu^N + v) - (Tx^k + v) = Te^k,$$

and hence $e^{k+1} = T^{k+1}e^0$ by induction.

For any consistent norm $\|\cdot\|$, we have

$$\|e^{k+1}\| \le \|T^{k+1}\|\,\|e^0\| \le \|T\|^{k+1}\,\|e^0\|.$$

Convergence based on structure

Exploiting the structure of the iteration matrix T we show:

Lemma [E., Liesen , Szyld, Tichý, 2016]

The iteration matrix in the multiplicative Schwarz iteration is given by

$$T = \left[\begin{array}{c|c|c} & t_1 & \\ & \vdots & \\ 0\ldots0 & t_{n+1} & 0\ldots0 \\ & \vdots & \\ & t_{N-1} & \end{array}\right] = t\, e_{n+1}^T .$$

Therefore, $T^2 = t\,(e_{n+1}^T\, t)\, e_{n+1}^T = t_{n+1}\, T$, and

$$\|\, T^{k+1} \| \;=\; |t_{n+1}|^k \, \|\, T\, \| \, .$$

How can we bound $|t_{n+1}|$, and $\|\, T\|$ in a convenient norm ($\|\cdot\|_\infty$)?

Convergence analysis

Details

$$A = \left[\begin{array}{c|c|c} A_H & & \\ & b_H & \\ \hline & & \\[-2ex] \qquad\qquad c & a & b \qquad\qquad \\ \hline & c_h & \\ & & A_h \end{array}\right].$$

Let $m \equiv n-1$, an d $\rho \equiv |t_{n+1}|$ be the **convergence factor**. Then,

$$\rho = \left| \frac{b b_H \left(A_H^{-1}\right)_{m,m}}{a - c b_H \left(A_H^{-1}\right)_{m,m}} \right| \left| \frac{c c_h \left(A_h^{-1}\right)_{1,1}}{a - b c_h \left(A_h^{-1}\right)_{1,1}} \right| .$$

Convergence analysis

Bounding $\left(A_H^{-1}\right)_{m,m}$ and $\left(A_h^{-1}\right)_{1,1}$

A matrix $B = [b_{i,j}]$ is called a nonsingular ***M*-matrix** when

- B is nonsingular,
- $b_{i,i} > 0$ for all i, $b_{i,j} \leq 0$ for all $i \neq j$,
- and $B^{-1} \geq 0$ (elementwise).

If A_H and A_h are nonsingular ***M*-matrices**, then using [Nabben 1999],

$$\left(A_H^{-1}\right)_{m,m} \leq \min\left\{\frac{1}{|b_H|}, \frac{1}{|c_H|}\right\},$$

$$\left(A_h^{-1}\right)_{1,1} \leq \min\left\{\frac{1}{|b_h|}, \frac{1}{|c_h|}\right\}.$$

A sufficient condition: The sign conditions & irreducibly diagonal dominant $\Rightarrow$ nonsingular M-matrix. [Meurant, 1996], [Hackbusch, 2010]

Convergence analysis

The upwind scheme

The matrices A_H **and** A_h **are** M**-matrices**, and using the rank-one structure of the iteration matrix, we know that the error satisfies:

$$\frac{\|e^{(k+1)}\|_\infty}{\|e^{(0)}\|_\infty} \leq \rho^k \|T\|_\infty.$$

Theorem [E., Liesen , Szyld, Tichý, 2016]

For the upwind scheme we have

$$\rho \leq \frac{\epsilon}{\epsilon + \omega H}, \quad \text{and} \quad \|T\|_\infty \leq \frac{\epsilon}{\epsilon + \omega H}.$$

Convergence analysis

The central difference scheme

- A_h is still an M-matrix.
- If $\omega H > 2\epsilon$, i.e. $b_H > 0$, then A_H is not an M-matrix
  ...the most common situation from a practical point of view.
- Recall

$$\rho = \left| \frac{b b_H \left(A_H^{-1}\right)_{m,m}}{a - c b_H \left(A_H^{-1}\right)_{m,m}} \right| \left| \frac{c c_h \left(A_h^{-1}\right)_{1,1}}{a - b c_h \left(A_h^{-1}\right)_{1,1}} \right| .$$

- How to bound $\left(A_H^{-1}\right)_{m,m}$? ... results by [Usmani 1994]
- We proved: If $m = N/2 - 1$ is even, then

$$b_H (A_H^{-1})_{m,m} \leq \frac{1 - \left|\frac{b_H}{c_H}\right|^m}{\left|\frac{c_H}{b_H}\right| + \left|\frac{b_H}{c_H}\right|^m} < \frac{2m\epsilon}{\epsilon + \frac{\omega H}{2}}.$$

Convergence analysis

The central difference scheme

A_h **is** M**-matrix**, if $\omega H > 2\epsilon$, A_H is not an M-matrix.

Theorem [E., Liesen , Szyld, Tichý, 2016]

Let $m = N/2 - 1$ be even, and let $\omega H > 2\epsilon$. For the central difference scheme we have

$$\rho < \frac{2m\epsilon}{\epsilon + \frac{\omega H}{2}} < N \frac{\epsilon}{\epsilon + \frac{\omega}{N}}, \quad \text{and} \quad \|T\|_\infty < 2.$$

- Standard convergence results: asymptotic and only for symmetric problems using the CG method.
- Our results: for a class of nonsymmetric, nonnormal problems and descriptive from the first step.

Numerical examples

Consider

$$-\epsilon u'' + u' = 1, \quad u(0) = 0, \quad u(1) = 0,$$

i.e.

$$\omega = 1, \quad \beta = 0, \quad f(x) \equiv 1\,.$$

For various values of the diffusion parameter ϵ we have:

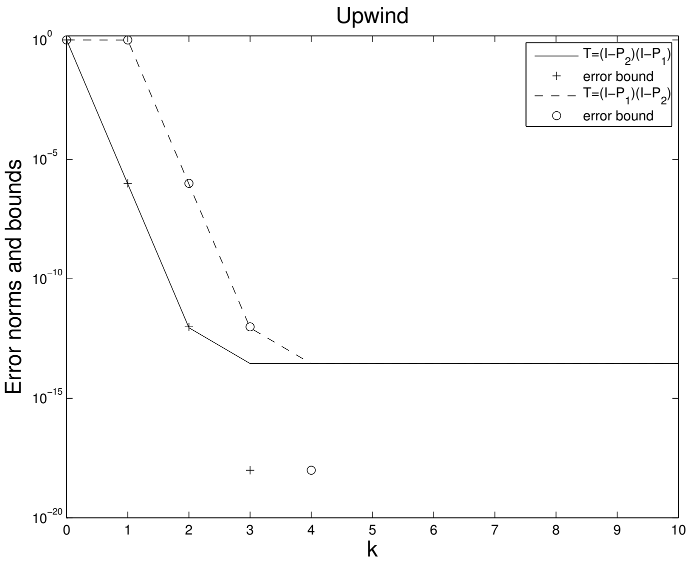
ϵ	ρ_{up}	our bound	ρ_{cd}	our bound
10^{-8}	9.4×10^{-7}	9.9×10^{-7}	1.8×10^{-4}	3.9×10^{-4}
10^{-6}	9.4×10^{-5}	9.9×10^{-5}	1.8×10^{-2}	3.9×10^{-2}
10^{-4}	9.3×10^{-3}	9.8×10^{-3}	8.3×10^{-1}	3.8×10^{-0}

We can see that the bounds fit closely the numerical results.

$$\rho_{up} < \frac{\epsilon}{\epsilon + \omega H}. \qquad \rho_{cd} < \frac{2m\epsilon}{\epsilon + \frac{\omega H}{2}}\,.$$

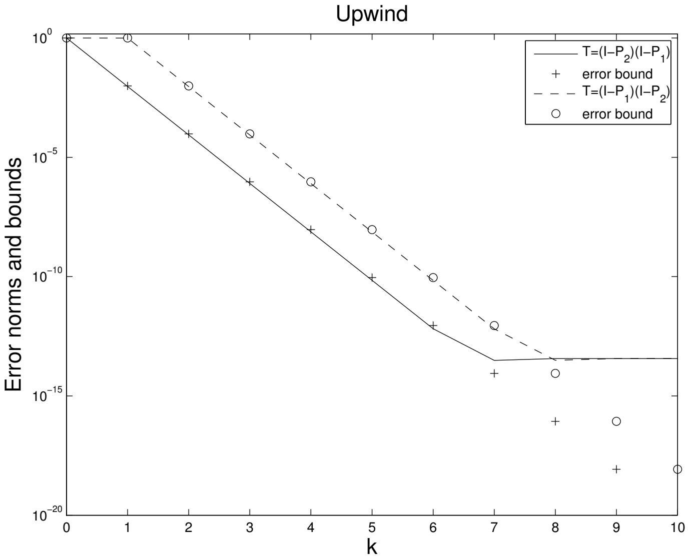
Numerical examples

Upwind, $\epsilon = 10^{-8}$



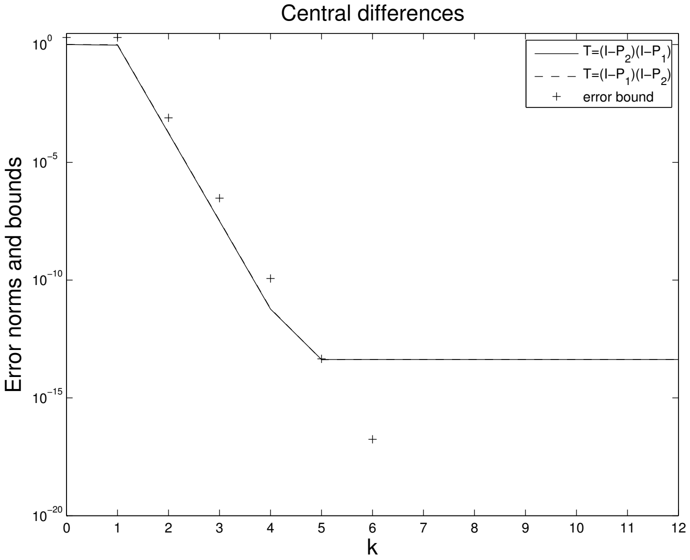
Numerical examples

Upwind, $\epsilon = 10^{-4}$



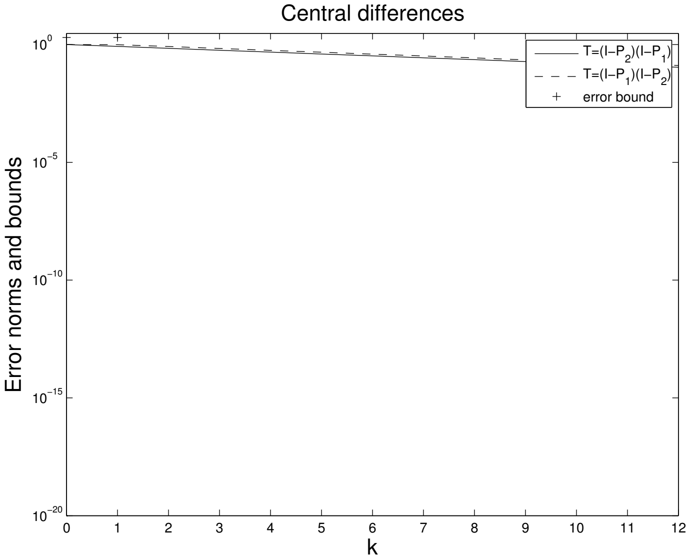
Numerical examples

Central differences, $\epsilon = 10^{-8}$



Numerical examples

Central differences, $\epsilon = 10^{-4}$



Schwarz method as a preconditioner

We have the consistent scheme

$$x^{k+1} \; = \; T\,x^k + v.$$

Hence, x solves $Ax = b$ and also "the **preconditioned** system"

$$(I - T)x = v.$$

We can formally define a **preconditioner** M such

$$Ax = b \quad \Leftrightarrow M^{-1}Ax = M^{-1}b \quad \Leftrightarrow (I - T)x = v.$$

Clearly $M = A(I - T)^{-1}$. Then

$$\begin{aligned} x^{k+1} \; &= \; x^k + (I - T)(x - x^k) \\ &= \; x^k + M^{-1}r^k. \end{aligned}$$

Schwarz method as a preconditioner for GMRES

- The multiplicative **Schwarz** method as well as **GMRES** applied to the preconditioned system obtain their approximations from **the same Krylov subspace**.
- In terms of the residual norm, the preconditioned **GMRES** will **always** converge **faster** than the multiplicative Schwarz.
- Moreover, in this case, the iteration matrix T has **rank-one structure**, and

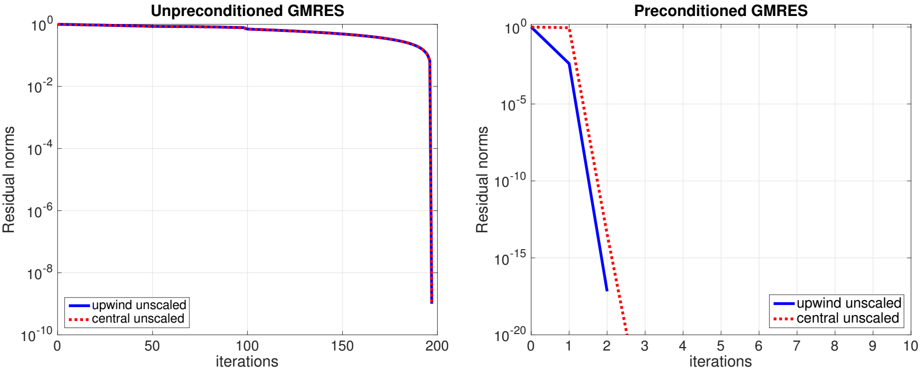
$$\dim\left(\mathcal{K}_k(I - T, r_0)\right) \leq 2.$$

- Therefore, GMRES converges in **at most 2 steps**, motivating the use of the preconditioner for higher dimensional cases.
- **Practical point of view:** when using inexact solves convergence may deteriorate.

Numerical example

Upwind Finite Differences, $N = 198$, $\epsilon = 10^{-4}$, $\omega = 1$, $\beta = 0$

We can compare the behavior of the unpreconditioned and preconditioned GMRES method:



Summary & Conclusions

- We considered finite difference discretizations of the singularly-perturbed convection-diffusion-reaction equation posed on a **Shishkin mesh**.
- Exploiting the **structure** of the discretized operators we have developed descriptive **bounds** that describe convergence from the first iteration.
- For the upwind and central finite differences we proved rapid convergence of the multiplicative Schwarz method in the most relevant cases.
- Due to the **rank-one structure** of T, the preconditioned GMRES converges in **two steps**.
- Details in: C. Echeverría, J. Liesen, D. Szyld, and P. Tichý,
  [Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on a Shishkin mesh, 2016, submitted]