Thin-Plate Spline Interpolation: Convergence Analysis and H-Matrix Solution Techniques

Jens Markus Melenk [1]

Maike Löhndorf

Radial basis functions provide a versatile tool for scattered data interpolation. One of the basic questions is the interpolation problem: Given N data points x_i with corresponding values f_i, find the function If of the form $If(x) = \sum_{i=1}^{N} c_i \phi(|x - x_i|) + \pi(x)$ that interpolates the given data f_i in the points x_i. One possible choice of the function ϕ is that of polyharmonic splines, i.e., ϕ is the fundamental solution of the iterated Laplacian Δ^m. In the case $d = 2 = m$, the function ϕ is the fundamental solution of the biharmonic equation and called thin-plate spline.

Existence, uniqueness, and optimal rates of convergence for quasi-uniformly distributed data points x_i were established in fundamental papers by Duchon and Meinguet. Convergence here means that the interpolation data $f_i = f(x_i)$ originate from a function $f \in H^m(\Omega)$ and the error $f - If$ is considered. We extend this classical theory to functions $f \in H^k(\Omega)$ with $k > m$. Specifically, we show that optimal convergence rates can be obtained for $f \in H^k(\Omega)$ in the range $k \in [m, m+1/2)$. Boundary effects limit the achievable convergence in the regime $k > m+1/2$; however, we show how further improvements in the convergence rate can be obtained by condensing data points x_i near the boundary. The linear system of equations that describes the interpolation problem is a dense system due to the non-locality of ϕ, which calls for data-sparse techniques. We propose to employ H-matrix techniques, which were introduced by W. Hackbusch in 1999. This data-sparse format allows for compressing the linear system to log-linear storage requirement. More importantly, the H-matrix format comes with an (approximate) arithmetic including factorizations in this format in log-linear complexity, thus opening the door to solving very large interpolation problems.

Numerical examples corroborate the theoretical assertions.

[1] Technische Universität Wien, Austria (melenk@tuwien.ac.at)