Preconditioned GMRES-based Iterative Refinement for the Solution of Sparse Ill-Conditioned Linear Systems

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Iterative refinement is a long-standing technique for improving the accuracy of a computed solution to a nonsingular linear system $Ax = b$ obtained via LU factorization. Existing results guarantee convergence if the matrix A has condition number safely less than the reciprocal of the unit roundoff u. We identify a mechanism that allows iterative refinement to produce solutions with normwise relative error of order u to systems with condition numbers of order u^{-1} or larger, provided that the update equation is solved with a relative error sufficiently less than 1.

Building on the analysis, we develop a GMRES-based iterative refinement method (GMRES-IR) that makes use of the computed LU factors as preconditioners in accurately solving the update equation. GMRES-IR exploits the fact that even if A is extremely ill conditioned the LU factors contain enough information that preconditioning can greatly reduce the condition number of A. Our rounding error analysis and numerical experiments show that GMRES-IR can succeed where standard refinement fails, and that it can provide accurate solutions to systems with condition numbers of order u^{-1} and greater.

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