Iterative Solution of Double Saddle Point Problems

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This talk will describe several approaches for the iterative solution of large, sparse, indefinite linear systems of equations of the form

$$\mathcal{A}u \equiv \begin{bmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & -D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD), $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times p}$ is symmetric positive semidefinite (SPS) and possibly zero. We assume that $n \geq m + p$.

Linear systems of the form (1) arise frequently from mixed and mixed-hybrid formulations of second-order elliptic equations [3, Sect. 7.2],[5] and elasticity [3, Sect. 9.3.1] problems. Numerical methods in constrained optimization [6] and liquid crystal modeling [7] also lead to sequences of linear systems of the type (1). We further mention that finite element models of certain incompressible flow problems arising in the analysis of non-Newtonian fluids and in geophysics lead to large linear systems with coefficient matrices of the form

$$\mathcal{B} = \begin{bmatrix} A & C^T & B^T \\ C & -D & 0 \\ B & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} -D & C & 0 \\ C^T & A & B^T \\ 0 & B & 0 \end{bmatrix};$$

see, e.g., [1] and [4], respectively. It is easy to see that both $\mathcal{B}$ and $\mathcal{C}$ can be brought into the same form as matrix $\mathcal{A}$ in (1) by means of symmetric permutations (row and column interchanges).

Different partitionings of the matrix $\mathcal{A}$ in (1) lead to different Uzawa-type methods and block preconditioners for MINRES and (F)GMRES. Theoretical results on convergence of iterative solvers and on the eigenvalue distribution of preconditioned matrices will be discussed. The theory will be illustrated by means of numerical experiments on two sets of problems, one arising from the mixed-hybrid finite element modeling of potential flow problems [5], the other from liquid crystal director modeling.

This is joint work with Fatemeh Panjeh Ali Beik (Vali-e-Asr University of Rafsanjan).

References

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