

Remedial Measures, Brown-Forsythe test, F test

Frank Wood

October 10, 2011

Remedial Measures

- ▶ How do we know that the regression function is a good explainer of the observed data?
 - Plotting
 - Tests
- ▶ What if it is not? What can we do about it?
 - Transformation of variables(next lecture)

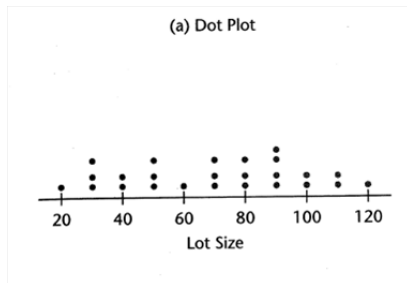
Graphical Diagnostics for the Predictor Variable

- ▶ Dot Plot
 - Useful for visualizing distribution of inputs
- ▶ Sequence Plot
 - Useful for visualizing dependencies between error terms
- ▶ Box Plot - Useful for visualizing distribution of inputs

Toluca manufacturing example again: production time vs. lot size

Dot Plot

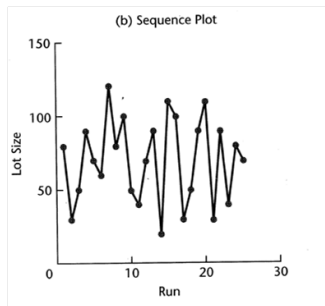
Figure:



- ▶ How many observations per input value?
- ▶ Range of inputs?

Sequence Plot

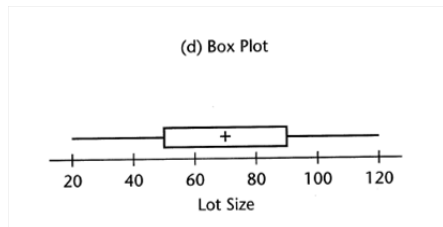
Figure:



- ▶ If observations are made over time, is there a correlation between input and position in observation sequence?

Box Plot

Figure:



- ▶ Shows
 - Median
 - 1st and 3rd quartiles
 - Maximum and minimum

Residuals

- ▶ Remember, the definition of residuals:

$$e_i = Y_i - \hat{Y}_i$$

- ▶ And the difference between that and the unknown true error

$$\epsilon = Y_i - E(Y_i)$$

- ▶ In a normal regression model the ϵ_i 's are assumed to be iid $N(0, \sigma^2)$ random variables. The observed residuals e_i should reflect these properties.

Remember: residual properties

- ▶ Mean

$$\bar{e}_i = \frac{\sum e_i}{n} = 0$$

- ▶ Variance

$$s^2 = \frac{(e_i - \bar{e})^2}{n-2} = \frac{SSE}{n-2} = MSE$$

Nonindependence of Residuals

- ▶ The residuals e_i are not independent random variables - The fitted values \hat{Y}_i are based on the same fitted regression line.
 - The residuals are subject to two constraints
 1. - Sum of the e_i 's equals 0
 2. - Sum of the products $X_i e_i$'s equals 0
- ▶ When the sample size is large in comparison to the number of parameters in the regression model, the dependency effect among the residuals e_i can reasonably safely be ignored.

Definition: semistudentized residuals

- ▶ It may be useful sometimes to look at a standardized set of residuals, for instance in outlier detection.
- ▶ Like usual, since the standard deviation of ϵ_i is σ (itself estimated by square root of MSE) a natural form of standardization to consider is

$$e_i^* = \frac{e_i}{\sqrt{MSE}}$$

- ▶ This is called a semistudentized residual.

Departures from Model...

To be studied by residuals

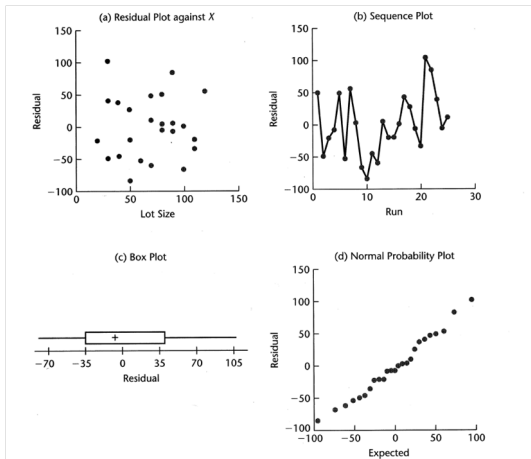
- ▶ Regression function not linear
- ▶ Error terms do not have constant variance
- ▶ Error terms are not independent
- ▶ Model fits all but one or a few outlier observations
- ▶ Error terms are not normally distributed
- ▶ One or more predictor variables have been omitted from the model

Diagnostics for Residuals

- ▶ Plot of residuals against predictor variable
- ▶ Plot of absolute or squared residuals against predictor variable
- ▶ Plot of residuals against fitted values
- ▶ Plot of residuals against time or other sequence
- ▶ Plot of residuals against omitted predictor variables
- ▶ Box plot of residuals
- ▶ Normal probability plot of residuals

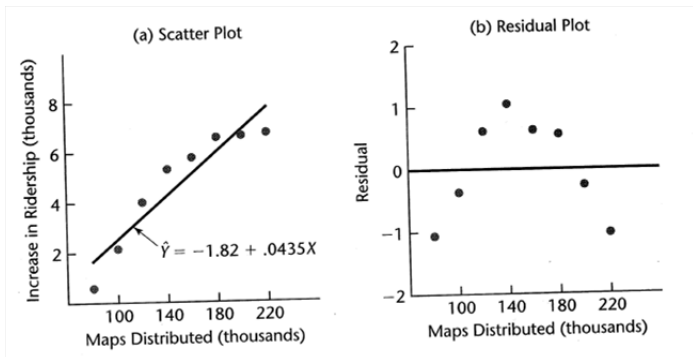
Diagnostic Residual Plots

Figure: Toluca example: labor time vs. lot size



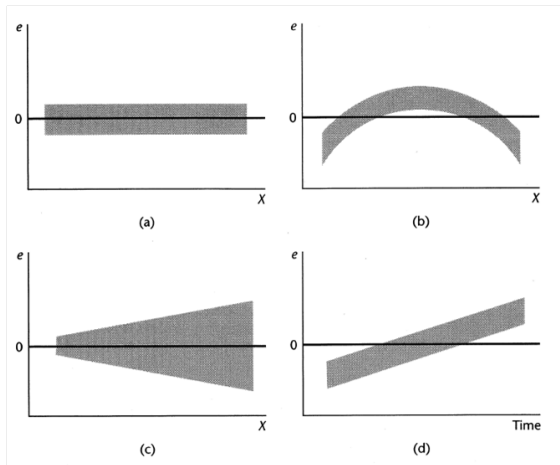
Scatter and Residual Plot

Figure: Transit example : ridership increase vs. num. maps distributed

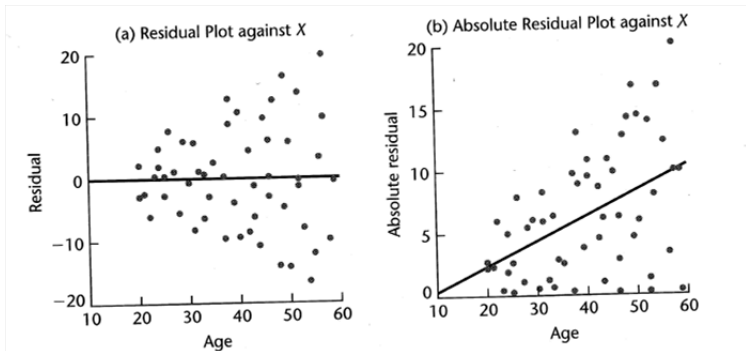


Prototype Residual Plots

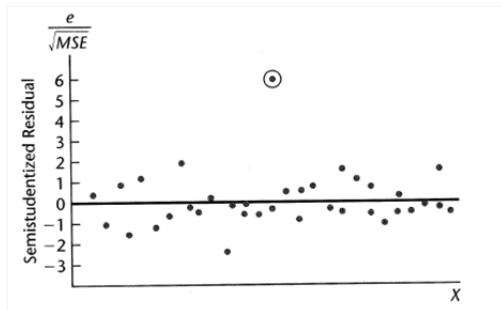
Figure: Indicate residual plots



Nonconstancy of Error Variance

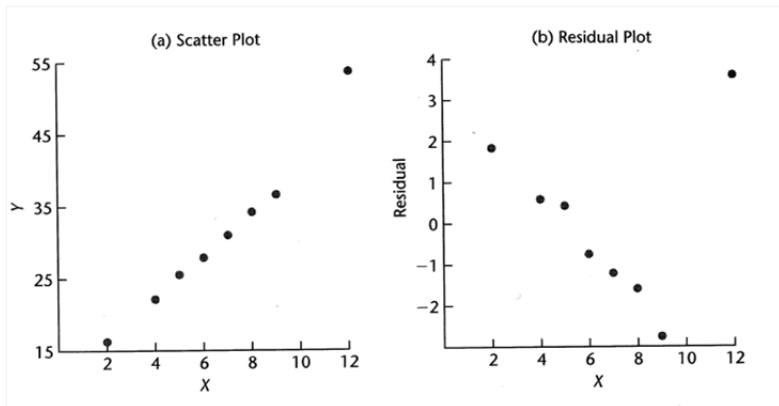


Presence of Outliers

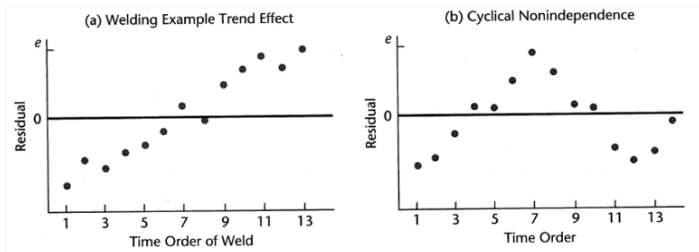


Outliers can strongly effect the fitted values of the regression line.

Outlier effect on residuals



Nonindependence of Error Terms

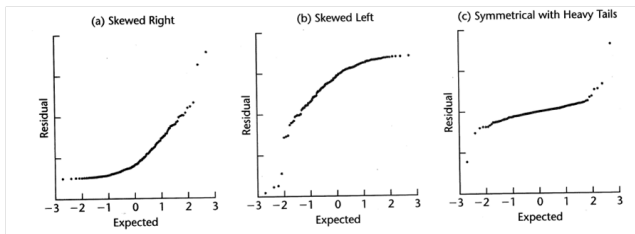


Sequential observations can exhibit observable trends in error distribution.

Non-normality of Error Terms

- ▶ Distribution plots
- ▶ Comparison of Frequencies
- ▶ Normal probability plot
 - Q-Q plot with numerical quantiles on the horizontal axis

Figure: Examples of non-normality in distribution of error terms



Normal probability plot

- ▶ For a $N(0, MSE^{1/2})$ random variable, a good approximation of the expected value of the k -th smallest observation in a random sample of size n is

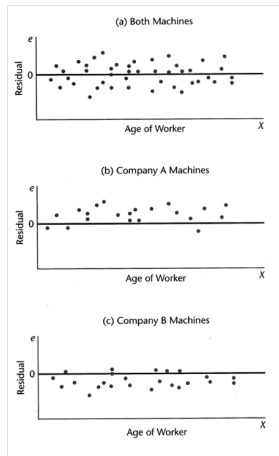
$$\sqrt{MSE} \left[z \left(\frac{k-.375}{n+.25} \right) \right]$$

- ▶ A normal probability plot consists of plotting the expected value of the k -th smallest observation against the observed k -th smallest observation

Omission of Important Predictor Variables

- ▶ Example
 - Qualitative variable
 - Type of machine
- ▶ Partitioning data can reveal dependence on omitted variable(s)
- ▶ Works for quantitative variables as well
- ▶ Can suggest that inclusion of other inputs is important

Figure:



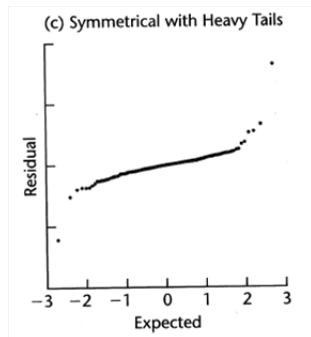
Tests Involving Residuals

- ▶ Tests for randomness
- ▶ Tests for constancy of variance
- ▶ Tests for outliers
- ▶ Tests for normality of error distribution

Correlation Test for Normality of Error Distribution

- ▶ A formal test for the normality of the error terms can be developed in terms of the correlation between the ordered observed errors
- ▶ Tables(B.6 in the book) given critical values for the null hypothesis (normally distributed errors) holding.

Figure:



Correlation Test for Normality of Error Distribution

For one way to run this correlation test let

$$\{e_{\sigma(1)}, \dots, e_{\sigma(n)}\}$$

be the permutation of the observed errors such that $e_{\sigma(j)} \leq e_{\sigma(j)} \forall j$ and $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ is a permutation. Let

$$\{r_1, \dots, r_k, \dots, r_n\}$$

be the expected value of the k^{th} residual under the normality assumption, i.e.

$$r_k \approx \sqrt{MSE} \left[z \left(\frac{k - .375}{n + .25} \right) \right]$$

Then compute the (sample) correlation between these sets of random variables. The sample correlation can be found by replacing the covariance and variance functions with their sample estimates in

$$\rho(Y, Z) = \frac{\sigma\{Y, Z\}}{\sigma\{Y\}\sigma\{Z\}}$$

Tests for Constancy of Error Variance

- ▶ Brown-Forsythe test does not depend normality of error terms. The Brown-Forsythe test is applicable to simple linear regression when
 - ▶ The variance of the error terms either increases or decreases with X (“megaphone” residual plot)
 - ▶ Sample size is large enough to ignore dependencies between the residuals
- ▶ The Brown-Forsythe test is essentially a t-test for testing whether the means of two normally distributed populations are the same where the populations are the absolute deviations between the prediction and the observed output space in two non-overlapping partitions of the input space.

Brown-Forsythe Test

- ▶ Divide X into X_1 (the low values of X) and X_2 (the high values of X)
- ▶ Let e_{i1} be the error terms for X_1 and vice versa
- ▶ let $n = n_1 + n_2$
- ▶ The Brown-Forsythe test uses the absolute deviations of the residuals around their group median

$$d_{1i} = |e_{1i} - \tilde{e}_1|$$

Brown-Forsythe Test

- ▶ The test statistic for comparing the means of the absolute deviations of the residuals around the group medians is

$$t_{BF}^* = \frac{\bar{d}_1 - \bar{d}_2}{s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where

$$s^2 = \frac{\sum (d_{i1} - \bar{d}_1)^2 + \sum (d_{i2} - \bar{d}_2)^2}{n-2}$$

Brown-Forsythe Test

- ▶ If n_1 and n_2 are not extremely small

$$t_{BF}^* \sim t(n-2)$$

approximately

- ▶ From this confidence intervals and tests can be constructed.

F test for lack of fit

- ▶ Formal test for determining whether a specific type of regression function adequately fits the data.
- ▶ Assumptions (usual):
 - observations $Y|X$ are
 1. i.i.d.
 2. normally distributed
 3. same variance σ^2
- ▶ Requires: repeat observations at one or more X levels (called replicates)

Example

- ▶ 12 similar branches of a bank offered gifts for setting up money market accounts
- ▶ Minimum initial deposits were specific to qualify for the gift
- ▶ Value of gift was proportional to the specified minimum deposit
- ▶ Interested in: relationship between specified minimum deposit and number of new accounts opened

F Test Example Data and ANOVA Table

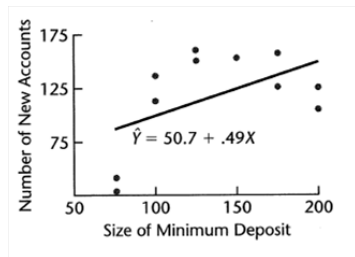
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(a) Data					
Branch	Size of Minimum Deposit (dollars)	Number of New Accounts	Branch	Size of Minimum Deposit (dollars)	Number of New Accounts
i	X_i	Y_i	i	X_i	Y_i
1	125	160	7	75	42
2	100	112	8	175	124
3	200	124	9	125	150
4	75	28	10	200	104
5	150	152	11	100	136
6	175	156			

(b) ANOVA Table			
Source of Variation	SS	df	MS
Regression	5,141.3	1	5,141.3
Error	14,741.6	9	1,638.0
Total	19,882.9	10	

Fit

Figure:



Data Arranged To Highlight Replicates

Figure:

	Size of Minimum Deposit (dollars)					
Replicate	$j = 1$ $X_1 = 75$	$j = 2$ $X_2 = 100$	$j = 3$ $X_3 = 125$	$j = 4$ $X_4 = 150$	$j = 5$ $X_5 = 175$	$j = 6$ $X_6 = 200$
$i = 1$	28	112	160	152	156	124
$i = 2$	42	136	150		124	104
Mean \bar{Y}_j	35	124	155	152	140	114

- ▶ The observed value of the response variable for the i -th replicate for the j -th level of X is Y_{ij}
- ▶ The mean of the Y observations at the level $X = X_j$ is \bar{Y}_j

Full Model vs. Regression Model

- ▶ The full model is

$$Y_{ij} = \mu_j + \epsilon_{ij} \quad \text{Full model}$$

where

- μ_j are parameters $j = 1, \dots, c$
 - ϵ_{ij} are iid $N(0, \sigma^2)$
- ▶ Since the error terms have expectation zero

$$E(Y_{ij}) = \mu_j$$

Full Model

- ▶ In the full model there is a different mean (a free parameter) for each X_i ;
- ▶ In the regression model the mean responses are constrained to lie on a line

$$E(Y) = \beta_0 + \beta_1 X$$

Fitting the Full Model

- ▶ The estimators of μ_j are simply

$$\hat{\mu}_j = \bar{Y}_j$$

- ▶ The error sum of squares of the full model therefore is

$$SSE(F) = \sum \sum (Y_{ij} - \bar{Y}_j)^2 == SSPE$$

Degrees of Freedom

- ▶ Ordinary total sum of squares had $n-1$ degrees of freedom.
- ▶ Each of the j terms is a ordinary total sum of squares
 - Each then has $n_j - 1$ degrees of freedom
- ▶ The number of degrees of freedom of SSPE is the sum of the component degrees of freedom

$$df_F = \sum_j (n_j - 1) = \sum_j n_j - c = n - c$$

General Linear Test

- ▶ Remember: the general linear test proposes a reduced model null hypotheses
 - this will be our normal regression model
- ▶ The full model will be as described (one independent mean for each level of X)

$$H_0 : E(Y) = \beta_0 + \beta_1 X$$

$$H_a : E(Y) \neq \beta_0 + \beta_1 X$$

SSE For Reduced Model

The SSE for the reduced model is as before
- remember

$$\begin{aligned}SSE(R) &= \sum_i \sum_j [Y_{ij} - (b_0 + b_1 X_j)]^2 \\ &= \sum_i \sum_j (Y_{ij} - \hat{Y}_{ij})^2\end{aligned}$$

- and has $n-2$ degrees of freedom $df_R = n - 2$

SSE(R)

Figure:

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Source of Variation	SS	df	MS
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F Test Statistic

From the general linear test approach

$$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F}$$
$$F^* = \frac{SSE - SSPE}{(n-2) - (n-c)} \div \frac{SSPE}{n-c}$$

where a little algebra takes us to the next slide

F Test Rule

- ▶ From the F test we know that large values of F^* lead us to reject the null hypothesis:

If $F^* \leq F(1 - \alpha; c - 2, n - c)$, conclude H_0

If $F^* > F(1 - \alpha; c - 2, n - c)$, conclude H_a

- ▶ For this example we have

$$SSPE = 1,148.0 \qquad n - c = 11 - 6 = 5$$

$$SSE = 14,741.6$$

$$SSLF = 14,741.6 - 1,148.0 = 13,593.6 \qquad c - 2 = 6 - 2 = 4$$

$$\begin{aligned} F^* &= \frac{13,593.6}{4} \div \frac{1,148.0}{5} \\ &= \frac{3,398.4}{229.6} = 14.80 \end{aligned}$$

Example Conclusion

- ▶ If we set the significance level to $\alpha = .01$
- ▶ And look up the value of the F inv-cdf $F(.99, 4, 5) = 11.4$
- ▶ We can conclude that the null hypothesis should be rejected.