Regression Introduction and Estimation Review

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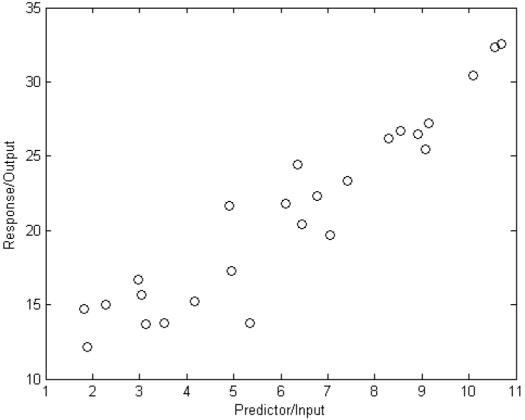
Quick Example - Scatter Plot



Use *linear_regression/demo.m*

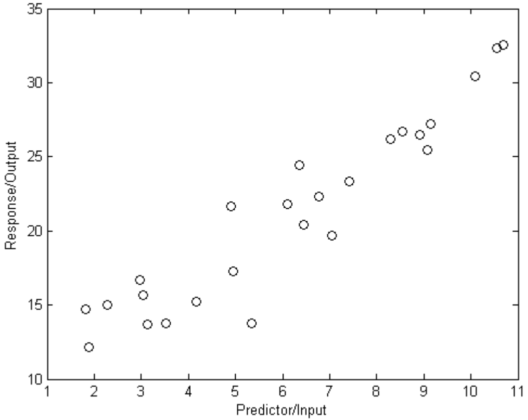
Linear Regression

- Want to find parameters for a function of the form

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Distribution of error random variable not specified

Quick Example - Scatter Plot



Formal Statement of Model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Y_i value of the response variable in the i^{th} trial
- β_0 and β_1 are parameters
- X_i is a known constant, the value of the predictor variable in the i^{th} trial
- ϵ_i is a random error term with mean $E\{\epsilon_i\} = 0$ and finite variance $\sigma^2\{\epsilon_i\} = \sigma^2$
- $i = 1, \ldots, n$

Properties

- The response Y_i is the sum of two components
  - Constant term $\beta_0 + \beta_1 X_i$
  - Random term ϵ_i
- The expected response is

$$\begin{aligned} E\{Y_i\} &= E\{\beta_0 + \beta_1 X_i + \epsilon_i\} \\ &= \beta_0 + \beta_1 X_i + E\{\epsilon_i\} \\ &= \beta_0 + \beta_1 X_i \end{aligned}$$

Expectation Review

- Definition

$$E\{X\} = E\{X\} = \int XP(X)dX, \ X \in \mathcal{R}$$

- Linearity property

$$\begin{aligned} E\{aX\} &= aE\{X\} \\ E\{aX + bY\} &= aE\{X\} + bE\{Y\} \end{aligned}$$

- Obvious from definition

Example Expectation Derivation

$$P(X) = 2X, 0 \leq X \leq 1$$

Expectation

$$\begin{aligned} E\{X\} &= \int_0^1 XP(X)dX \\ &= \int_0^1 2X^2 dX \\ &= \frac{2X^3}{3}\Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

Expectation of a Product of Random Variables

If X,Y are random variables with joint distribution $P(X, Y)$ then the expectation of the product is given by

$$E\{XY\} = \int_{XY} XYP(X, Y)dXdY.$$

Expectation of a product of random variables

What if X and Y are independent? If X and Y are independent with density functions f and g respectively then

$$
\begin{aligned}
E\{XY\} &= \int_{XY} XYf(X)g(Y)dXdY \\
&= \int_X \int_Y XYf(X)g(Y)dXdY \\
&= \int_X Xf(X)[\int_Y Yg(Y)dY]dX \\
&= \int_X Xf(X)E\{Y\}dX \\
&= E\{X\}E\{Y\}
\end{aligned}
$$

Regression Function

- The response Y_i comes from a probability distribution with mean

$$E\{Y_i\} = \beta_0 + \beta_1 X_i$$

- This means the regression function is

$$E\{Y\} = \beta_0 + \beta_1 X$$

Since the regression function relates the means of the probability distributions of Y for a given X to the level of X

Error Terms

- The response Y_i in the i^{th} trial exceeds or falls short of the value of the regression function by the error term amount ϵ_i
- The error terms ϵ_i are assumed to have constant variance σ^2

Response Variance

Responses Y_i have the same constant variance

$$\begin{aligned} \sigma^2\{Y_i\} &= \sigma^2\{\beta_0 + \beta_1 X_i + \epsilon_i\} \\ &= \sigma^2\{\epsilon_i\} \\ &= \sigma^2 \end{aligned}$$

Variance (2^{nd} central moment) Review

- Continuous distribution

$$\sigma^2\{X\} = E\{(X - E\{X\})^2\} = \int (X - E\{X\})^2 P(X) dX,\ X \in \mathcal{R}$$

- Discrete distribution

$$\sigma^2\{X\} = E\{(X - E\{X\})^2\} = \sum_i (X_i - E\{X\})^2 P(X_i),\ X \in \mathcal{Z}$$

Alternative Form for Variance

$$\begin{aligned}
\sigma^2\{X\} &= E\{(X - E\{X\})^2\} \\
&= E\{(X^2 - 2XE\{X\} + E\{X\}^2)\} \\
&= E\{X^2\} - 2E\{X\}E\{X\} + E\{X\}^2 \\
&= E\{X^2\} - 2E\{X\}^2 + E\{X\}^2 \\
&= E\{X^2\} - E\{X\}^2.
\end{aligned}$$

Example Variance Derivation

$$P(X) = 2X, 0 \leq X \leq 1$$

$$\begin{aligned}
\sigma^2\{X\} &= E\{(X - E\{X\})^2\} = E\{X^2\} - E\{X\}^2 \\
&= \int_0^1 2XX^2 dX - (\frac{2}{3})^2 \\
&= \frac{2X^4}{4}|_0^1 - \frac{4}{9} \\
&= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}
\end{aligned}$$

Variance Properties

$$\begin{aligned}
\sigma^2\{aX\} &= a^2\,\sigma^2\{X\} \\
\sigma^2\{aX + bY\} &= a^2\,\sigma^2\{X\} + b^2\,\sigma^2\{Y\}\,\text{if} X \perp\!\!\!\perp Y \\
\sigma^2\{a + cX\} &= c^2\,\sigma^2\{X\}\ \text{if} a, c \text{ both constant}
\end{aligned}$$

More generally

$$\sigma^2\{\sum a_i X_i\} = \sum_i \sum_j a_i a_j \mathrm{Cov}(X_i, X_j)$$

Covariance

- The covariance between two real-valued random variables X and Y, with expected values $E\{X\} = \mu$ and $E\{Y\} = \nu$ is defined as

$$Cov(X, Y) = E\{(X - \mu)(Y - \nu)\}$$

- Which can be rewritten as

$$\begin{aligned} Cov(X, Y) &= E\{XY - \nu X - \mu Y + \mu\nu\}, \\ Cov(X, Y) &= E\{XY\} - \nu E\{X\} - \mu E\{Y\} + \mu\nu, \\ Cov(X, Y) &= E\{XY\} - \mu\nu. \end{aligned}$$

Covariance of Independent Variables

If X and Y are independent, then their covariance is zero. This follows because under independence

$$E\{XY\} = E\{X\}E\{Y\} = \mu\nu.$$

and then

$$Cov(XY) = \mu\nu - \mu\nu = 0.$$

Least Squares Linear Regression

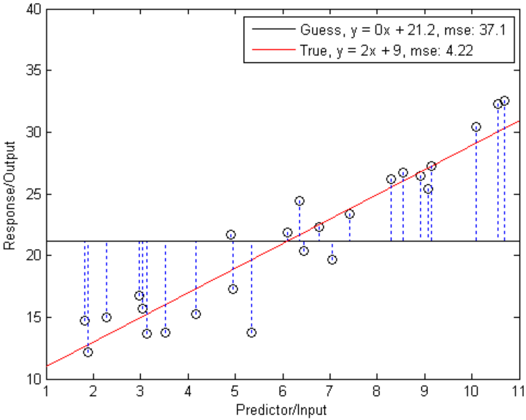
- Seek to minimize

$$Q = \sum_{i=1}^{n} (Y_i - (b_0 + b_1 X_i))^2$$

- By careful choice of b_0 and b_1 where b_0 is a point estimator for β_0 and b_1 is the same for β_1

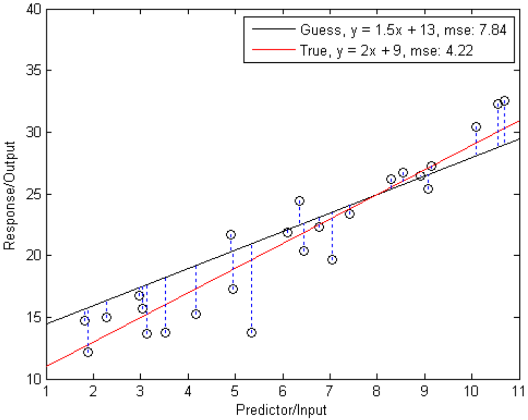
  How?

Guess #1



Guess, y = 0x + 21.2, mse: 37.1
True, y = 2x + 9, mse: 4.22
Response/Output
Predictor/Input
40
35
30
25
20
15
10
1
2
3
4
5
6
7
8
9
10
11

Guess #2


Guess, y = 1.5x + 13, mse: 7.84
True, y = 2x + 9, mse: 4.22
Response/Output
Predictor/Input
40
35
30
25
20
15
10
1
2
3
4
5
6
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8
9
10
11

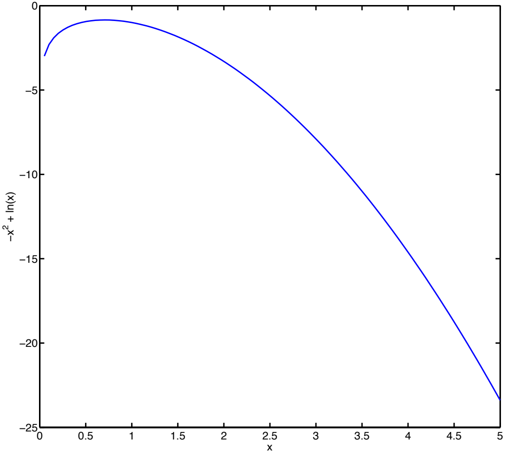
Function maximization

- Important technique to remember!
  - Take derivative
  - Set result equal to zero and solve
  - Test second derivative at that point
- Question: does this always give you the maximum?
- Going further: multiple variables, convex optimization

Function Maximization

Find

$$\underset{x}{\operatorname{argmax}} \; -x^2 + ln(x)$$



Least Squares Max(min)imization

- Function to minimize w.r.t. b_0 and b_1, b_0 and b_1 are called point estimators of β_0 and β_1 respectively

$$Q = \sum_{i=1}^{n} (Y_i - (b_0 + b_1 X_i))^2$$

- Minimize this by maximizing -Q
- Either way, find partials and set both equal to zero

$$\frac{dQ}{db_0} = 0$$
$$\frac{dQ}{db_1} = 0$$

Normal Equations

- The result of this maximization step are called the normal equations.

$$\begin{aligned}\sum Y_i &= nb_0 + b_1 \sum X_i \\ \sum X_i Y_i &= b_0 \sum X_i + b_1 \sum X_i^2\end{aligned}$$

- This is a system of two equations and two unknowns. The solution is given by...

Solution to Normal Equations

After a lot of algebra one arrives at

$$
\begin{aligned}
b_1 &= \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} \\
b_0 &= \bar{Y} - b_1\bar{X} \\
\bar{X} &= \frac{\sum X_i}{n} \\
\bar{Y} &= \frac{\sum Y_i}{n}
\end{aligned}
$$