LINEAR REGRESSION MODELS W4315

HOMEWORK 4 QUESTIONS

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1. (25 points) A is an $n \times p$ matrix (in typical multiple regression settings, n is the number of observations and p is the number of parameters, and $n \geq p$), prove that

(1) $A'A$ and AA' are symmetric matrices. (A' denotes the transpose of A)

(2) $A'A$ and AA' are semi-positive-definite matrices. (An $n \times n$ matrix M is semi-positive-definite if $\forall x \in \mathbb{R}^n$, $x'Mx \geq 0$.)

(3) If A has full column rank ($rank(A) = p$), then prove $A'A$ is a positive-definite matrix. (An $n \times n$ matrix M is positive-definite if $\forall$ nonzero $x \in \mathbb{R}^n$, $x'Mx > 0$.)

2. (25 points) A is an $n \times p$ matrix with full column rank. Let $P \equiv A(A'A)^{-1}A'$

(1) An $n \times n$ matrix M is a projection matrix if it is symmetric and idempotent (i.e. $A^2 = A$). Prove that P is a projection matrix.

(2) Give the rank of P and $I - P$. (I is the $n \times n$ identity matrix)

(3) Prove that the projection P is orthogonal. (i.e. $\forall x \in \mathbb{R}^n$, $(Px)'[(I - P)x] = 0$)

3. (25 points) $\vec{X}$ is a 3 dimensional Gaussian random vector, with distribution $N_3(\mu, \Sigma)$, in which

$$\mu = \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & -2 \\ 3 & -2 & 8 \end{pmatrix}$$

Let $Y_1 = X_1 + X_3$ and $Y_2 = 2X_2$, determine the distribution of $\vec{Y} = (Y_1, Y_2)'$ and the conditional distribution $Y_1 | Y_2 = 10$.

4. (10 points) Prove $\vec{Y} \sim N(0_{n\times 1}, I_{n\times n})$ implies that all Y_i i.i.d. follow $N(0, 1)$. (This problem may seem ridiculously easy to you. Just write out the joint density and see what it tells us about the marginal distributions.)

5. (15 points) Assume that an n dimensional Gaussian random vector X is distributed as $N(\mu, \Sigma)$

(1) Find a transformation of X, $Y = f(X)$, such that $Y \sim N(0_{n\times 1}, I_{n\times n})$.

(2) Prove that

$$(X - \mu)'\Sigma^{-1}(X - \mu) \sim \chi^2(n)$$