Neural Tangent Kernels (+ etc)

CPSC 532S: Modern Statistical Learning Theory

23 March 2022

cs.ubc.ca/~dsuth/532S/22/

Admin

- A3 Q1 is broken, to be replaced (with something as similar as possible) tonight
  - Whole assignment now due **Monday**
- A4 will be posted soon, due Friday the 8th (last day of term)
  - Yes, the same day as the project report; usual late policy

- Project scope:
  - I’m just looking for signs that you’ve read and understood the papers
  - If you’re doing a lit review: ~3-4 papers is the expected amount
    - Talk about how the settings / conclusions relate to each other, what they leave open, etc
  - An “extension” can absolutely just be poking at the assumptions and talking about when they hold / don’t

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  - Relationship to circuit complexity

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- What's the **optimization** error for SGD/similar?

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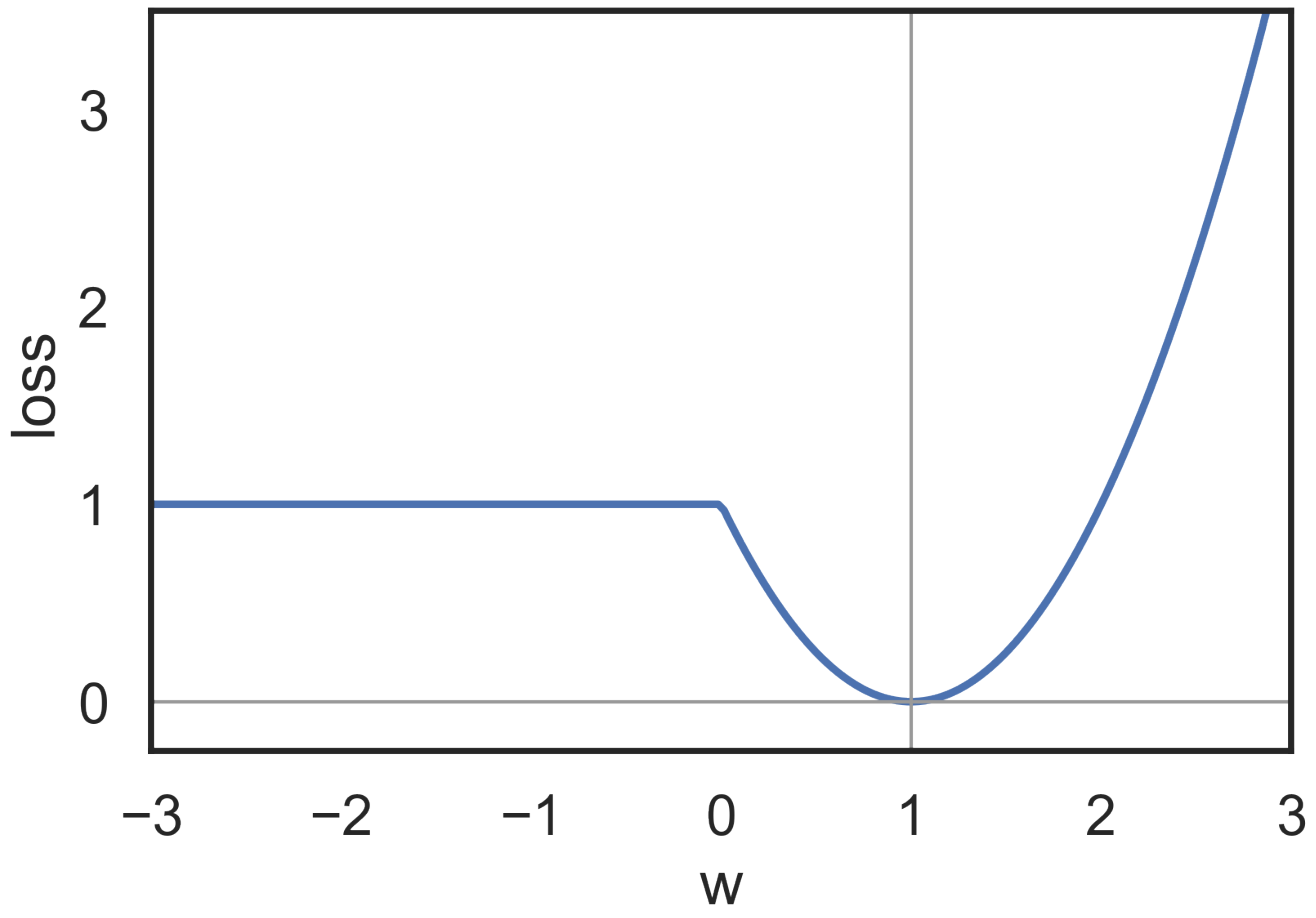
  - …but that doesn't happen on deep linear nets [under conditions] (Arora et al. 2019)

Bad local minima in ReLU nets

$h(x) = \text{ReLU}(wx)$ (reals to reals), square loss, $S = \big((1,1)\big)$:



Sub-Optimal Local Minima Exist for Neural Networks with Almost All Non-Linear Activations

Tian Ding*   Dawei Li †   Ruoyu Sun ‡

Nov 4, 2019

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- Implicit in these papers:
  - Behaviour of deep nets converges to kernel ridge regression with the **neural tangent kernel**

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  - We'll see that, for large m and random W_0, $f \approx f_{W_0}$ through training

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We'll see shortly that $f - f_0$ *shrinks* as m grows

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Linearization quality

- For a two-layer net with β-smooth hidden activations, second-layer weights $\leq 1/\sqrt{m}$ with linear activation, then for any $\|x\| \leq 1,$ $\quad |f(x; W) - f_0(x; W)| \leq \frac{\beta}{2\sqrt{m}} \|W - W_0\|_F^2$

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arccos kernel

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This kernel is universal on $\{x \in \mathbb{R}^{d+1} : \|x\| = 1, x_{d+1} = 1/\sqrt{2}\}$

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  - github.com/google/neural-tangents

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If $k_{SS}^{(w_t)} = k_{SS}$ is constant over time, exact same dynamics as kernel (ridgeless) regression

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- Scales network so that initialization has $f_S \approx 0$

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  - Proof actually needs infinite width but only really shows for finite time t

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- Telgarsky section 8 gives a simplified proof, but it’s a little bit WIP

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  - Can be practically useful in some settings

So, is deep learning just kernels?

- **No.**
  - Real neural net optimization isn't in "the NTK regime"
  - NTK regime *doesn't allow for feature learning* – the kernel doesn't change…
  - There are problems where NNs provably do better than *any* kernel method possibly could (next time!)

- But NTK is still useful:
  - AFAIK, the main (only?) proofs that GD optimizes deep networks reasonably
  - Can be practically useful in some settings
  - Probably a building block for whatever comes next