More on VC dimensions

CPSC 532S: Modern Statistical Learning Theory

24 January 2022

cs.ubc.ca/~dsuth/532S/22/

Admin

- A1 solutions are posted (just publicly on the course site)
  - Also see the post-mortem poll on Piazza
  - Grading: probably next week sometime (ICML deadline…)

- A2 will come probably next week + allow (encourage) groups

- No office hours this week (ICML…)
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- FYI, now presenting from a different app
  - Going to try out some live scribbles for part of this lecture
  - Will save stuff I write on slides and post after class
  - (Let me know if you hate it)

Last time: shattering / VC dimension

- **Restriction of $\mathscr{H}$ to C** is $\mathscr{H}_C = \left\{ \left(h(c_1), \ldots, h(c_{|C|}) \right) : h \in \mathscr{H} \right\}$
- $\mathscr{H}$ **shatters** $C \subseteq \mathscr{X}$ if $\mathscr{H}_C$ contains all functions from C to $\{0,1\}$: $|\mathscr{H}_C| = 2^{|C|}$
- $\mathrm{VCdim}(\mathscr{H})$ is size of the largest set that $\mathscr{H}$ can shatter, or ∞

- Doesn't need that *all* sets of size VCdim can be shattered – it's **worst-case**
  - There is a C with $|C| = \mathrm{VCdim}$ that can be shattered
  - There is **no** C with $|C| = \mathrm{VCdim} + 1$ that can be shattered

Last time: “Fundamental Theorem of Learning”

For binary classification with 0-1 loss:

These are all equivalent:

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$\sqrt{\log(2^{2n})} \to \sqrt{2n \log 2}$

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$\sqrt{\log \mathcal{O}(n^d)} / \sqrt{n} = \dfrac{\sqrt{d\, \mathcal{O}(\log n)}}{\sqrt{n}}$

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  We'll come back for a *much* better rate in δ – $\sqrt{\log \frac{1}{\delta}}$ instead of $\frac{1}{\delta}$ – pretty soon
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Sauer-Shelah lemma

- Independently proved by, at least:
  - Sauer 1972 (to solve a combinatorical problem posed by Erdős)
  - Shelah 1972 (with Perles) as a lemma about “stable models”
  - Perles, later on, in ergodic theory
  - Vapnik+Chervonenkis, also in the 70s, to make VC theory work

- **Sauer-Shelah lemma**: Let $\mathrm{VCdim}(\mathscr{H}) \leq d < \infty$. Then $\tau_{\mathscr{H}}(n) \leq \sum_{i=0}^{d} \binom{n}{i}$.
  - Corollary: for $n \geq d$, $\tau_{\mathscr{H}}(n) \leq \left(\frac{1}{d}en\right)^{d}$

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    - If $d \geq 1$: LHS is 2 ($\mathscr{H}_C = \{0,1\}$), RHS has empty set and C: 2

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    - $|\mathscr{H}'_{C'}| \leq \left| \{ B \subseteq C' : \mathscr{H}' \text{ shatters } B \} \right| \leq \left| \{ B \subseteq C : c_1 \in B, \mathscr{H} \text{ shatters } B \} \right|$
  - Have $|\mathscr{H}_C| = |Y_1| + |Y_2|$ since things in Y_2 "show up twice" in $\mathscr{H}_C$

- Supposing for all C with $|C| < n$, for all $\mathscr{H}$, $|\mathscr{H}_C| \leq \left| \left\{ B \subseteq C : \mathscr{H} \text{ shatters } B \right\} \right|$
  - Take $C = (x_1, \ldots, x_n)$, and let $C' = (x_2, \ldots, x_n)$
  - Let $Y_1 = \left\{ (y_2, \ldots, y_n) : (0, y_2, \ldots, y_n) \in \mathscr{H}_C \vee (1, y_2, \ldots, y_n) \in \mathscr{H}_C \right\} = \mathscr{H}_{C'}$
    - $|\mathscr{H}_{C'}| \leq \left| \{ B \subseteq C' : \mathscr{H} \text{ shatters } B \} \right| = \left| \{ B \subseteq C : c_1 \notin B, \mathscr{H} \text{ shatters } B \} \right|$
  - Let $\mathscr{H}' = \left\{ h \in \mathscr{H} : \exists h' \in \mathscr{H} \,.\, h \text{ and } h' \text{ agree on } C', \text{ disagree on } c_1 \right\}$
    - Let $Y_2 = \left\{ (y_2, \ldots, y_n) : (0, y_2, \ldots, y_n) \in \mathscr{H}_C \wedge (1, y_2, \ldots, y_n) \in \mathscr{H}_C \right\} = \mathscr{H}'_{C'}$
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  - Have $|\mathscr{H}_C| = |Y_1| + |Y_2|$ since things in Y_2 "show up twice" in $\mathscr{H}_C$
  - So $|\mathscr{H}_C| \leq \left| B \subseteq C : \mathscr{H} \text{ shatters } B \right|$

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$\leq \left(\frac{n}{d}\right)^{d} e^{d}$ e.g. from $1 - x \leq \exp(-x)$

(pause)

VC dimension of thresholds

$\mathcal{H} = \{h_a : a \in \mathbb{R}\}$

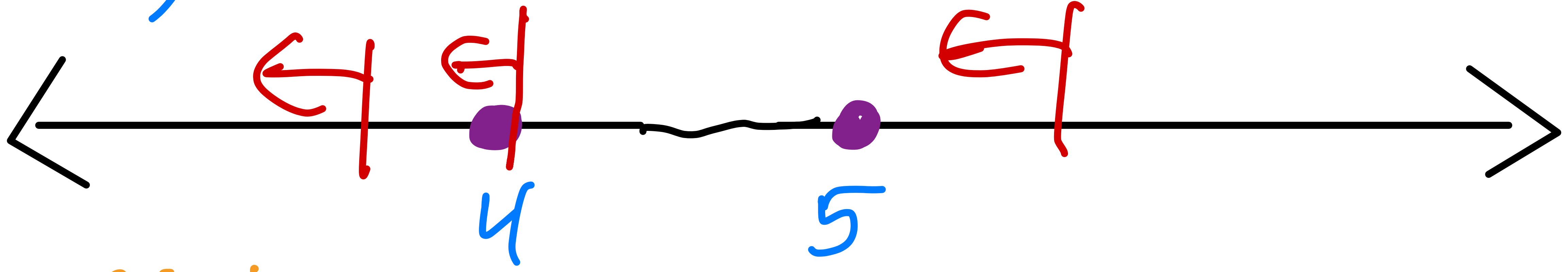
- Thresholds on $\mathbb{R}$: $h_a(x) = \mathbb{I}_{[x \le a]}$

$\exists\, C \subseteq \mathcal{X},\ |C| = d$, s.t. $\mathcal{H}$ shatters C

$\nexists\, C \subseteq \mathcal{X},\ |C| = d+1$ s.t. $\mathcal{H}$ shatters C

thresholds on $\mathbb{N}$

$\mathcal{H}' = \{h_z : z \in \mathbb{Z}\}$



$(0, 0) \leftarrow$ $h_3, h_2, h_{1.784}$

$(0, 1) \leftarrow$

$(1, 1) \leftarrow$

$\cancel{(1, 0)}$

$|\mathcal{H}_C|$

VCdim = 1

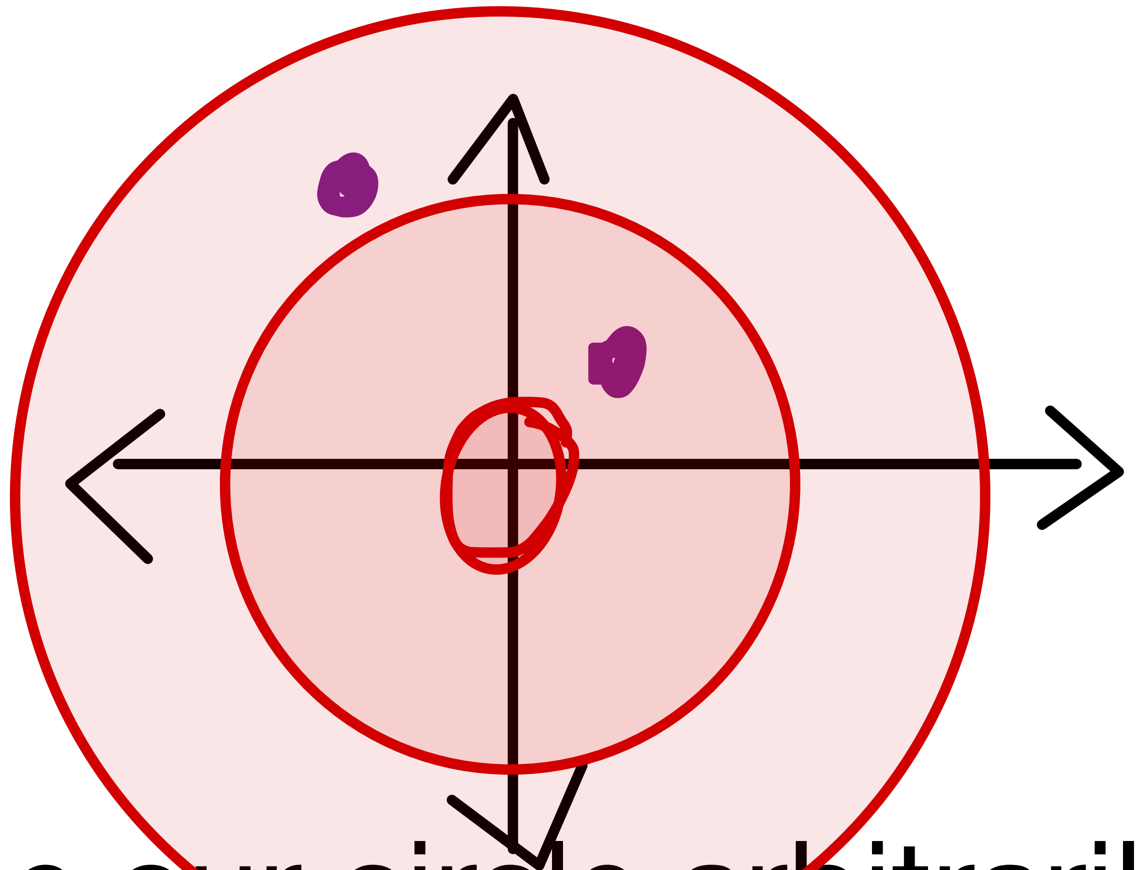
$\tau_{\mathcal{H}}(1) = 2$

$\tau_{\mathcal{H}}(2) = |\{(0,0), (0,1), (1,1)\}| = 3$

$\tau_{\mathcal{H}}(n) = n+1 \le \left(\frac{en}{\cancel{d}\ 1}\right)^{\cancel{d}\ 1} \le en$

VC dimension of circles

- From the homework: $h_r : \mathbb{R}^d \to \mathbb{R}$ given by $\mathbb{I}(\|x\| \leq r)$

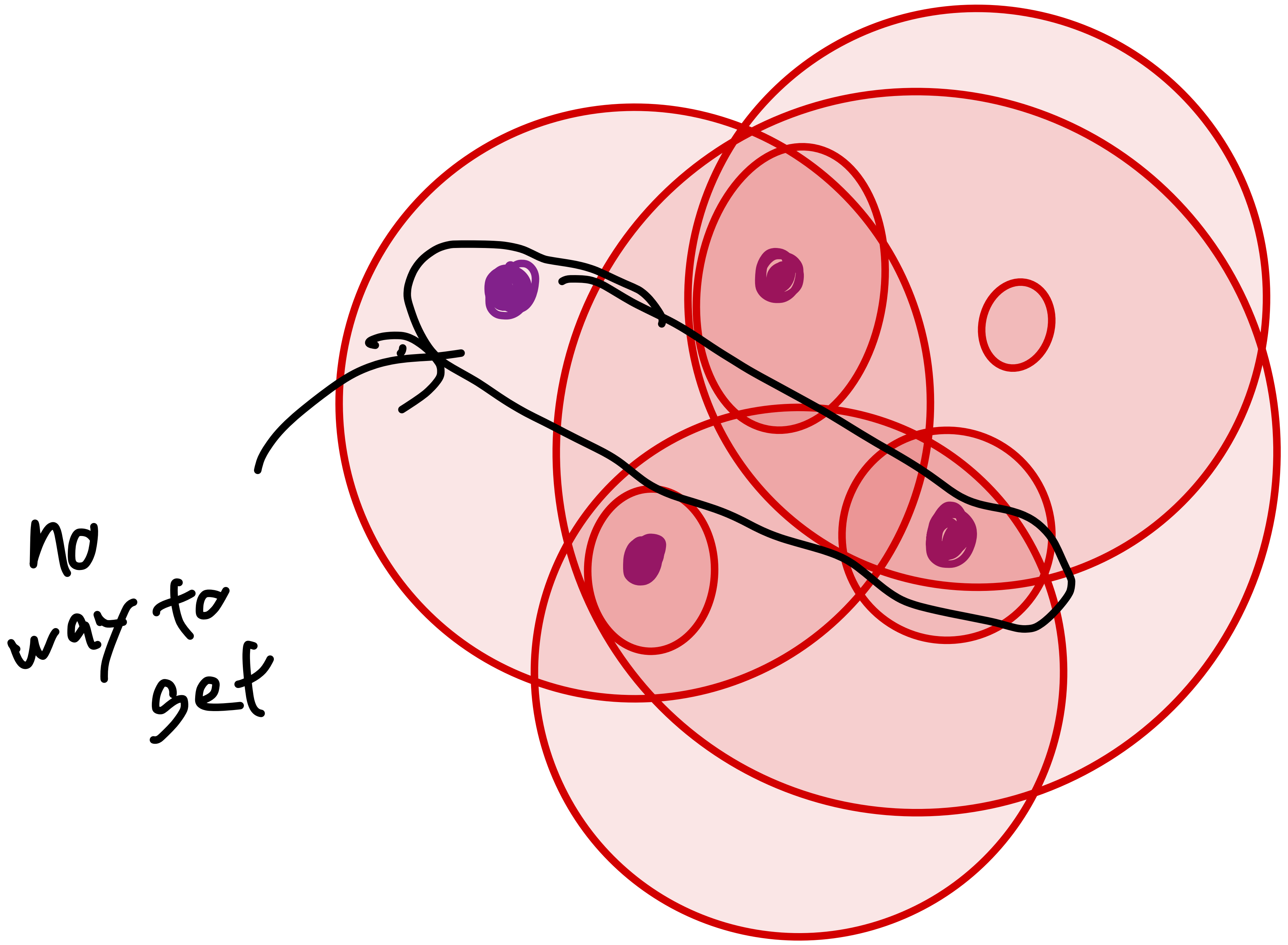


(0, 0)
(1, 0)
(1, 1)
$\cancel{(0, 1)}$

- What if we can place our circle arbitrarily? $h_{r,c} = \mathbb{I}(\|x - c\| \leq r)$

VC dimension of arbitrary circles

- What if we can place our circle arbitrarily? $h_{r,c} = \mathbb{I}(\|x - c\| \leq r)$ in $\mathbb{R}^2$



0, 0, 1
0, 1, 1
1, 0, 1
1, 1, 1

can shatter 3
can't shatter 4

VC dimension of finite classes

- Any $\mathcal{H}$ with $|\mathcal{H}| < \infty$

$$|\mathcal{H}_C| \leq |\mathcal{H}|$$

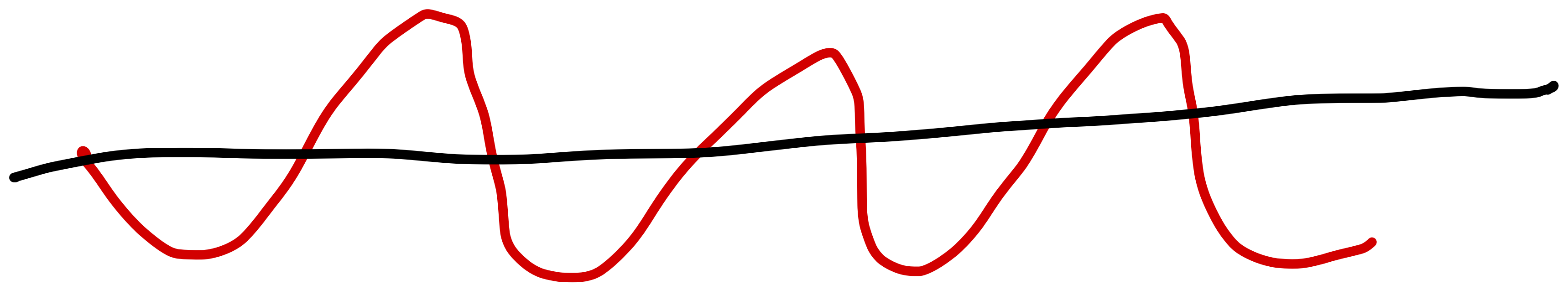
to shatter of size n, $|\mathcal{H}_C| = 2^n$

$$2^d \leq |\mathcal{H}_C| \leq |\mathcal{H}|$$

$$d \leq \log_2 |\mathcal{H}|$$

VC dimension of sign(sine)

- Let $h_w(x) = \mathbb{I}(\sin(wx) \geq 0)$ on $\mathbb{R}$



$$C = \{2^1, 2^2, \dots, 2^n\}$$
$$x_1, x_2, \dots, x_n$$

$$w = -\pi(0.y_1 y_2 y_3 y_4) = -\pi\left(\sum_{j=1}^{n} y_i 2^{-j}\right)$$

$$wx_i = -\pi\left(\sum_{j=1}^{n} y_j \underbrace{2^{i-j}}\right)$$

$$\sin(wx_i) = \sin\Big(\underbrace{-2k\pi} - \underbrace{\pi y_i} \underbrace{- \pi\sum_{j=i+1}^{n} y_j 2^{i-j}}_{<1}\Big)$$

$$\sin(-\pi - s(n)) \Rightarrow 0$$
$$\sin(0 - s(n)) \Rightarrow 1$$

$\therefore$ VC dim $= \infty$

Infinite VC dimension but can barely shatter anything

- Let $h_w(x) = \mathbb{I}(\sin(wx_1) \geq 0)$ on $\mathbb{R}^d$

$(0, \cdots)$

$(0, \cdots)$

$C_n = \{ (2^i, 0, \ldots, 0) : i \in [n] \}$

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- Saw a bunch of VC calculations
  - Linear classifiers are d without intercept, $d+1$ with
  - Even stupid function classes can have infinite VC dim