Grab bag:
Failures of uniform convergence
PAC-Bayes
Online learning

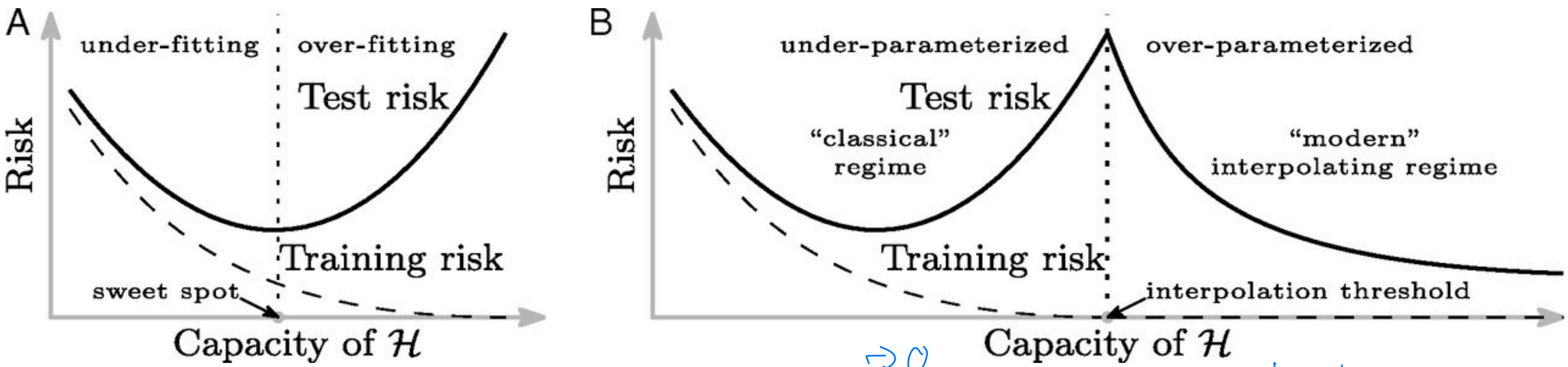
CPSC 532D: Modern Statistical Learning Theory

7 Dec 2022

cs.ubc.ca/~dsuth/532D/22w1/

Admin

- Topics that *won't* be on the final:
  - "Kernels IV", the stuff about operators / etc
  - The last couple lectures:
    - Implicit regularization
    - Neural tangent kernels
    - Universality
    - Rademacher complexity of deep nets
  - Details of any proof
- Stuff that could:
  - Working with basic definitions, etc
  - The homework question about monotonicity of VC/Rademacher is a decent example



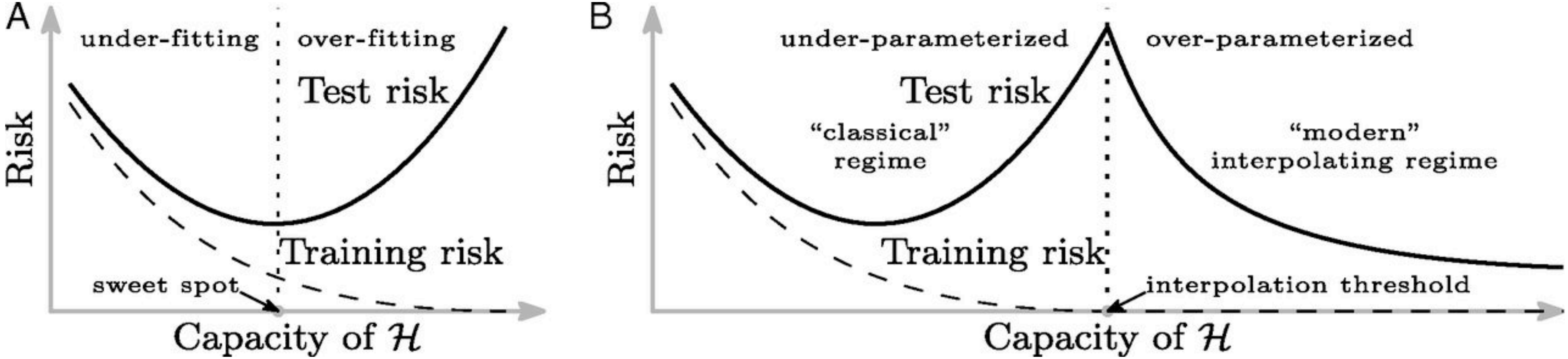
non-realizable classically:

$$\overbrace{\underbrace{L_D(h)}_{\ge \text{ Bayes error}}}^{0} \le \overbrace{\underbrace{L_S(h)}_{\to \approx \text{ Bayes error},\ \to \inf_{h\in\mathcal{H}} L_D(h)}}^{0} + \overbrace{\underbrace{\underbrace{\sup_{h\in\mathcal{H}} L_D(h)-L_S(h)}_{\text{bound}}}_{\to 0}}^{\to 0 \text{ if realizable}}$$

$$\sup_{\substack{h\in\mathcal{H}\\ \text{s.t. } L_S(h)=0}} L_D(h) - \cancel{L_S(h)}$$

interpolating non-realizable:

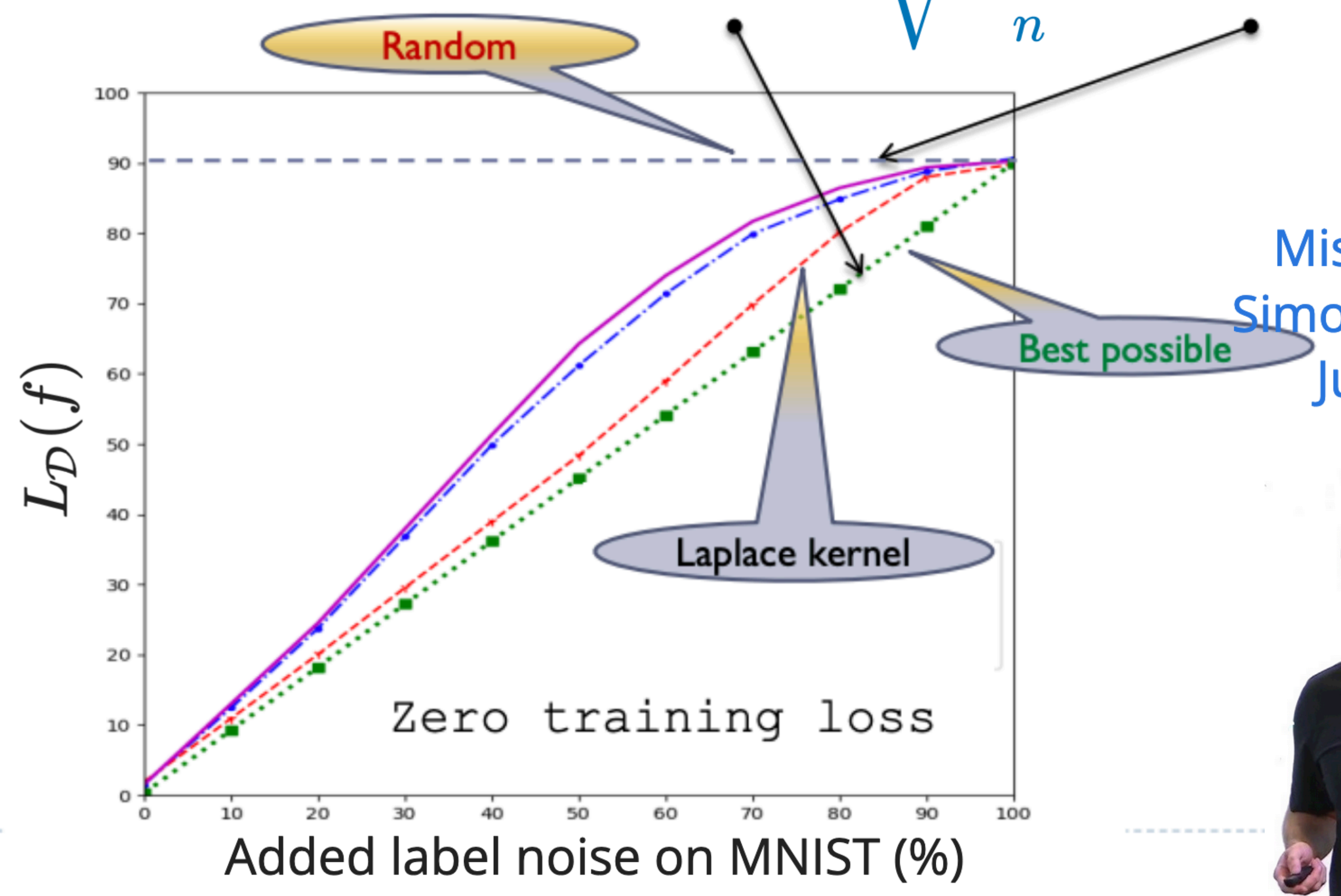
$$\underbrace{L_D(h)}_{\ge \text{ Bayes error}} \le \underbrace{L_S(h)}_{0} + \underbrace{\sup_{h\in\mathcal{H}} L_D(h)-L_S(h)}_{\text{bound} \xrightarrow{\text{want}} \text{Bayes error}}$$



Bounds? $L_{\mathcal{D}}(\hat{f}) \leq \underbrace{L_{\mathbf{S}}(\hat{f})}_{0} + \sqrt{\frac{C_{\mathcal{F},\delta}}{n}}$

What kind of generalization bound could work here?

$0.7 < \sqrt{\frac{C_{\mathcal{F},\delta}}{n}} < 0.9$



Misha Belkin
Simons Institute
July 2019

Belkin/Ma/Mandal, ICML 2018

Uniform convergence may be unable to explain generalization in deep learning

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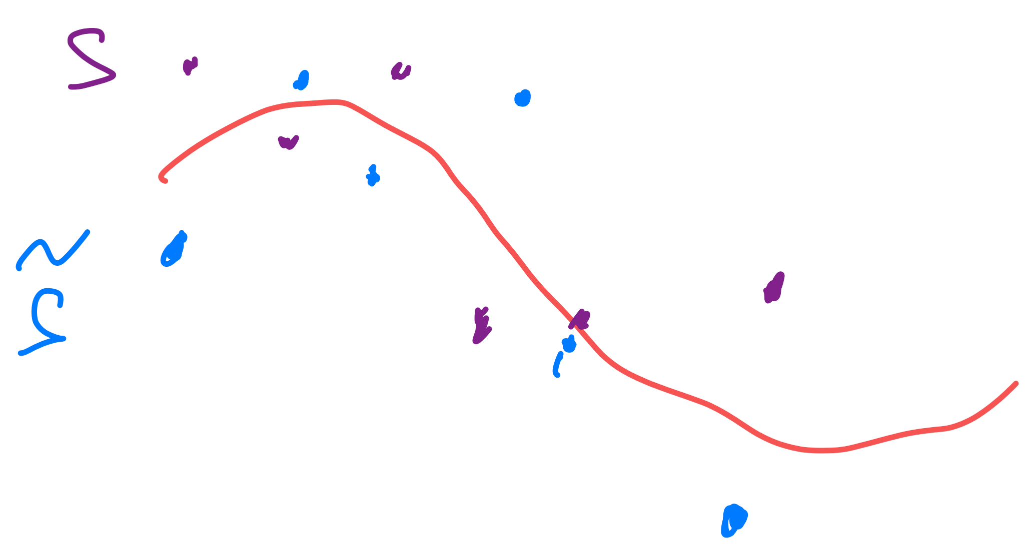
$$\lim_{n\to\infty} \mathbb{E}_{S\sim D^n} \sup_{h\in\mathcal{H}} \Big| \overset{\sigma^2}{L_D(h)} - \overset{4\sigma^2}{L_S(h)} \Big| \ge 3\sigma^2$$

[and $L_S(h)=0$]

$L_D(A(S)) \xrightarrow{n\to\infty} \sigma^2 \leftarrow$ Bayes error



$X \sim \mathcal{N}(0,\Sigma) \in \mathbb{R}^d$ $\quad$ $d = \omega(n)$

$y = Xw^* + \varepsilon$

$\varepsilon \sim \mathcal{N}(0,\sigma^2)$

$A(S) = X^\dagger y$

$L_S(A(S)) = 0$

$L_S(A(\tilde{S}))$

$= \frac{1}{n}\sum_{i=1}^{n} \left(w^*\cdot x_i - \varepsilon_i - (w^*\cdot x_i + \varepsilon_i) \right)^2$

$= \frac{1}{n}\sum_i (2\varepsilon_i)^2$

$= 4 \cdot \underbrace{\frac{1}{n}\sum_i \varepsilon_i^2} \to 4\sigma^2$

If Σ satisfies some conditions,

$L_D(A(S)) \to \sigma^2$

Bartlett, Long, Lugosi, Tsigler (2020) "Benign overfitting in ..."

$L_D(A(\tilde{S})) \to \sigma^2$

choose e.g. $\mathcal{H} = \{x \mapsto w^T x : \|w\| \le B_n\}$:(

smallest possible $\mathcal{H}$: $\mathcal{H}_n = \{A(S) : S \in \mathcal{S}_n\}$, $\Pr_{S\sim D^n}(S \in \mathcal{S}_n) \ge 1-\delta$

must be at least one pair with $S \in \mathcal{S}_n$, $\tilde{S} \in \mathcal{S}_n$

S has $\underset{n\times d}{X}, \underset{n\times 1}{y}$

$\tilde{S}$ with X, $2Xw^* - y = 2Xw^* - (Xw^* + \varepsilon) = Xw^* - \varepsilon$

(pause)

A road to PAC-Bayes

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- PAC-Bayes: analyzes *any* prior-posterior pair (potentially even totally unrelated)

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- McAllester-style bound (SSBD theorem 31.1):
    - If $\ell(h, z) \in [0,1]$, with probability at least $1 - \delta$ over $S \sim \mathcal{D}^n$,

$$L_{\mathcal{D}}(\rho) - L_S(\rho) \leq \sqrt{\frac{\mathrm{KL}(\rho \| \pi) + \log \frac{n}{\delta}}{2(n-1)}}$$

where $\mathrm{KL}(\rho \| \pi) = \mathbb{E}_{h\sim\rho} \log \frac{\rho(h)}{\pi(h)}$ (the usual KL divergence)

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  - Proved in SSBD chapter 31 (not bad at all)

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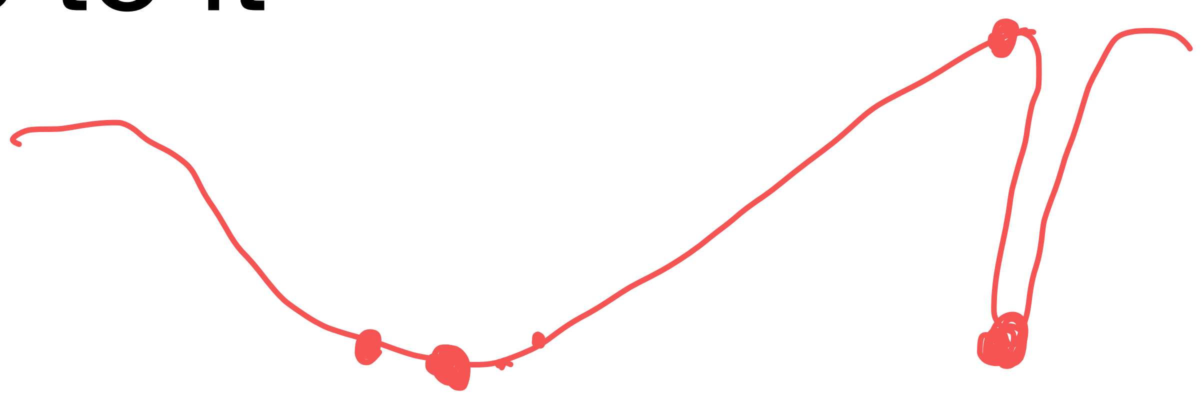

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  - For instance: could learn a $\hat{h}$ with (S)GD and then add noise to it
  - If $\hat{h}$ is in a **flat minimum**, then $\hat{h} +$ noise will still be good
  - But note that if $\rho \to$ point mass and π continuous, $\mathrm{KL}(\rho \| \pi) \to \infty$

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  - …as long as we can still bound $\mathrm{KL}(\rho \| \pi)$

Other forms of PAC-Bayes bounds

- Linear bound: $L_{\mathscr{D}}(\rho) \leq \frac{1}{\beta} L_S(\rho) + \frac{\mathrm{KL}(\rho \| \pi) + \log \frac{1}{\delta}}{2\beta(1-\beta)n}$ for any $\beta \in (0,1)$

- Catoni bound: for $\alpha > 1$, $\Phi_\gamma^{-1}(x) = (1 - \exp(-\gamma x))/(1 - \exp(-\gamma))$,
$$L_{\mathscr{D}}(\rho) \leq \inf_{\lambda > 1} \Phi_{\lambda/n}^{-1}\left(L_S(\rho) + \frac{\alpha}{\lambda}\left[\mathrm{KL}(\rho \| \pi) - \log \varepsilon + 2 \log \frac{\log(\alpha^2 \lambda)}{\log \alpha}\right]\right)$$
  - Can be much tighter (unfortunately) if $\mathrm{KL}(\rho \| \pi)/n$ is big

- Also variants based on general f-divergences, Wasserstein, …

Non-vacuous Generalization Bounds at the ImageNet Scale: a PAC-Bayesian Compression Approach

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- Pre-pick a coding scheme to represent networks (e.g. compress the weights)
- Train a network with SGD, sparsify it/etc to $\hat{h}$, then add a little noise to weights

NON-VACUOUS GENERALIZATION BOUNDS AT THE IMAGENET SCALE: A PAC-BAYESIAN COMPRESSION APPROACH

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Table 1: Summary of bounds obtained from compression

Dataset	Orig. size	Comp. size	Robust. Adj.	Eff. Size	Error Bound Top 1	Error Bound Top 5
MNIST	168.4 KiB	8.1 KiB	1.88 KiB	6.23 KiB	$< 46\,\%$	NA
ImageNet	5.93 MiB	452 KiB	102 KiB	350 KiB	$< 96.5\,\%$	$< 89\,\%$

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- But…these then become "two-sided" bounds
  - Subject to the Nagarajan/Kolter failure mode (their Appendix J)

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(pause)

Online learning

- Class so far has been in the **(passive) batch setting**:
  - Observe training set $S \sim \mathscr{D}^n$, pick h, test on new examples from $\mathscr{D}$

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Hello Danica,

I am incredibly sorry about this! It looks like the earlier CMT emails went to my spam folder. I can do this review within the next 12 hours (i.e. by midnight

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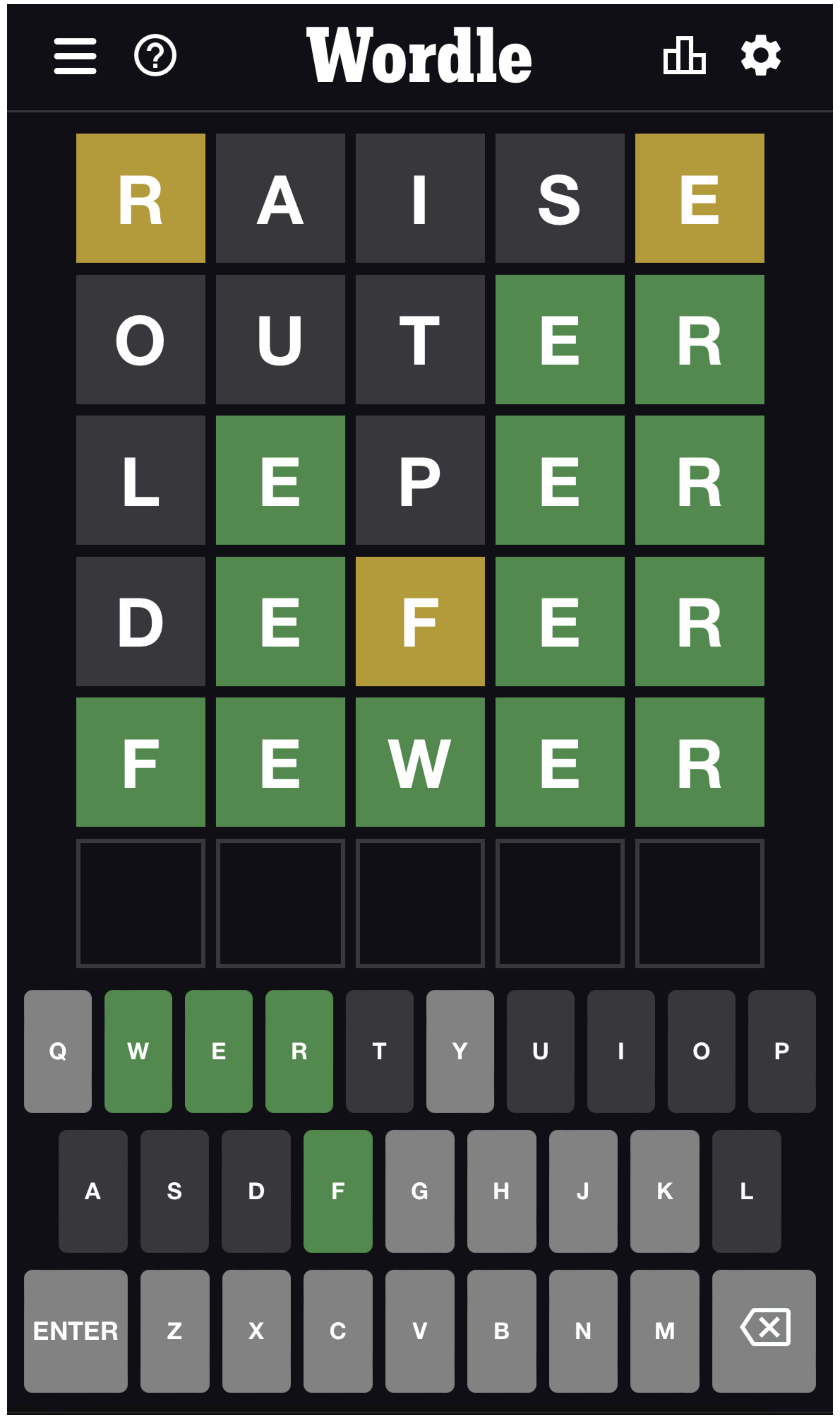

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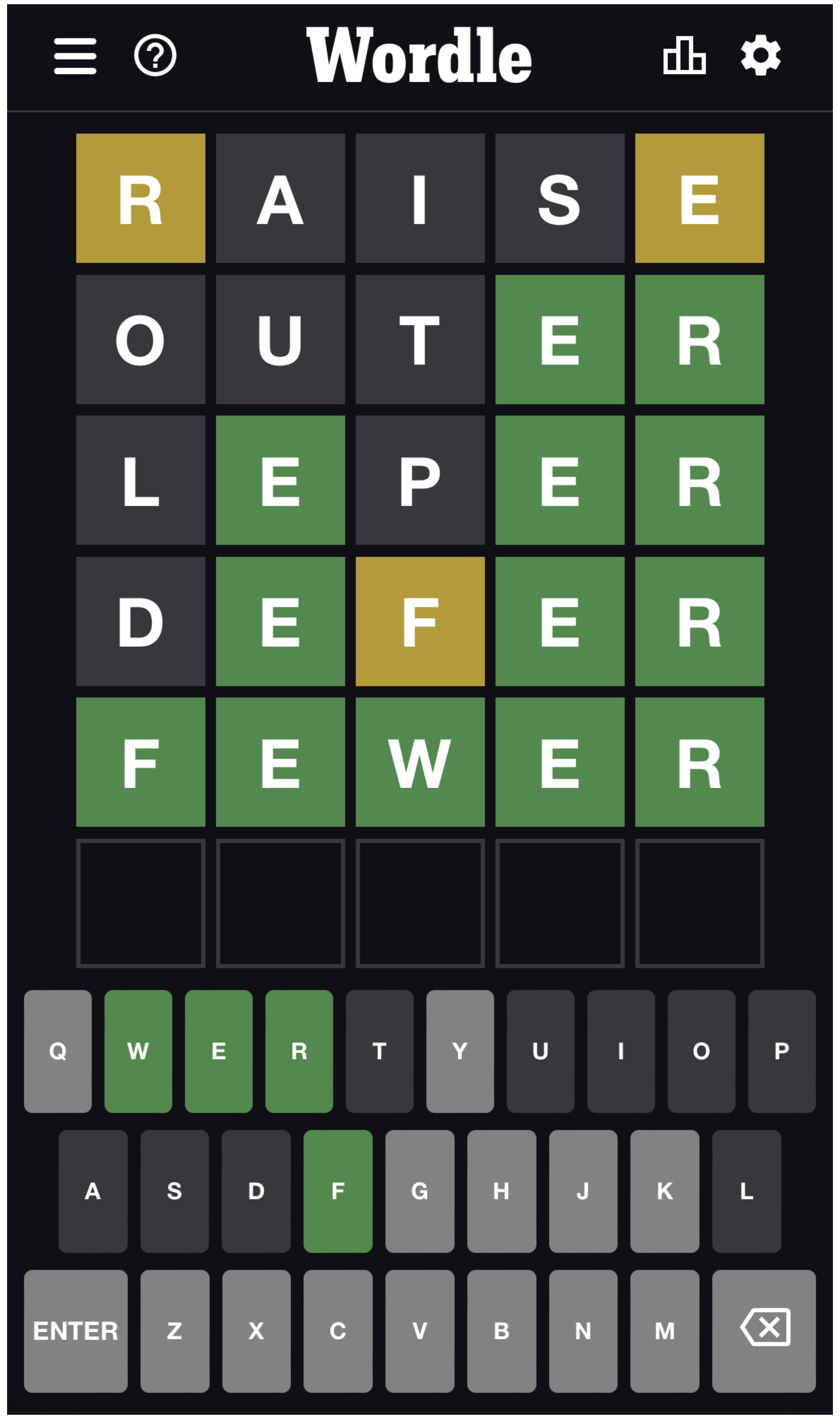

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- Have mistake bound $M_{\texttt{Consistent}}(\mathcal{H}) \leq |\mathcal{H}| - 1$

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- $M_{\texttt{Halving}}(\mathscr{H}) \leq \log_2 |\mathscr{H}|$ – *way* better bound

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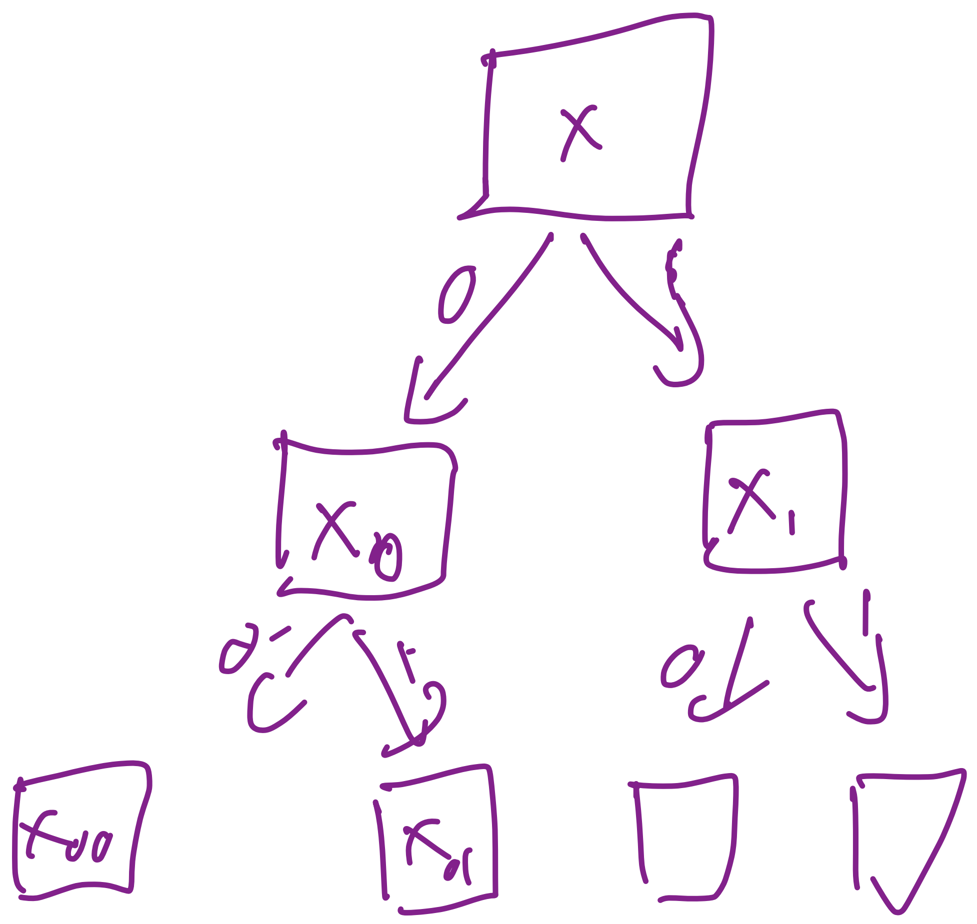
  - Put points $x_t \in \mathcal{X}$ into a full binary tree
  - Start at the root, move left if learner predicts 0, right if it predicts 1



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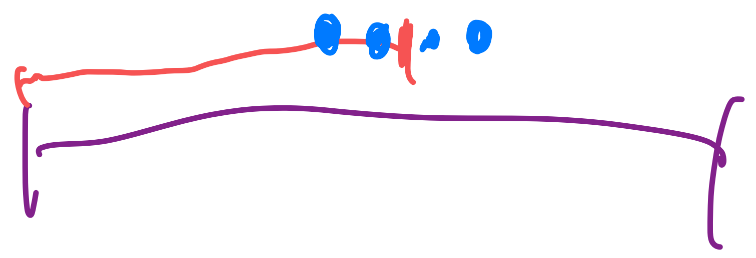
- If $\mathscr{X} = [0,1]$ and $\mathscr{H} = \{x \mapsto \mathbb{1}(x \leq a) : a \in [0,1]\}$, have $\mathrm{Ldim}(\mathscr{H}) = \infty$ (!)



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- But SOA is not necessarily easy to compute!

(pause)

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  - Realizable PAC assumes labels come from $h^* \in \mathscr{H}$
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- Ideally, we want **sublinear regret**: $\frac{1}{T} \mathrm{Regret}_A(\mathcal{H}, T) \xrightarrow{T \to \infty} 0$

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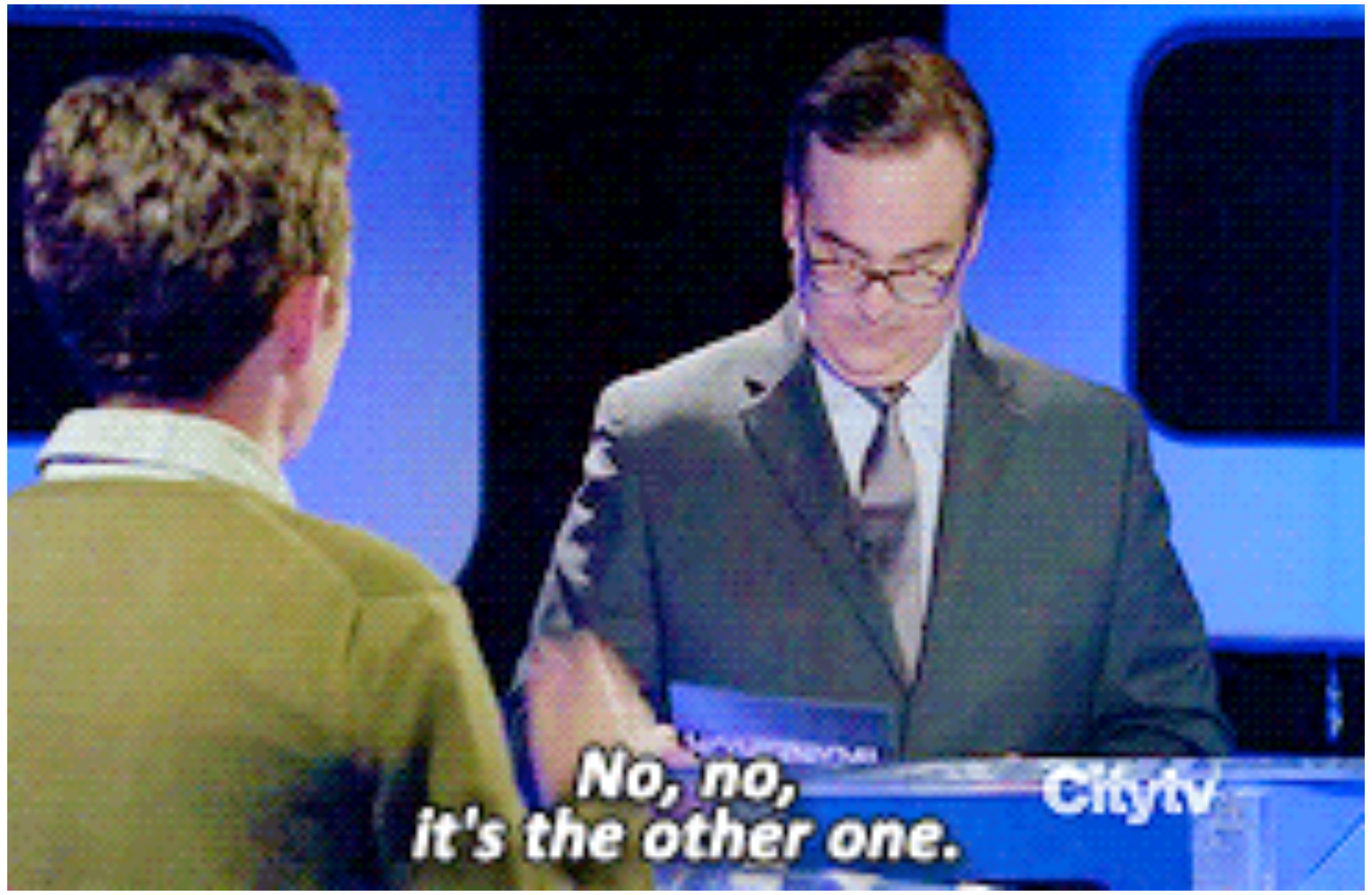

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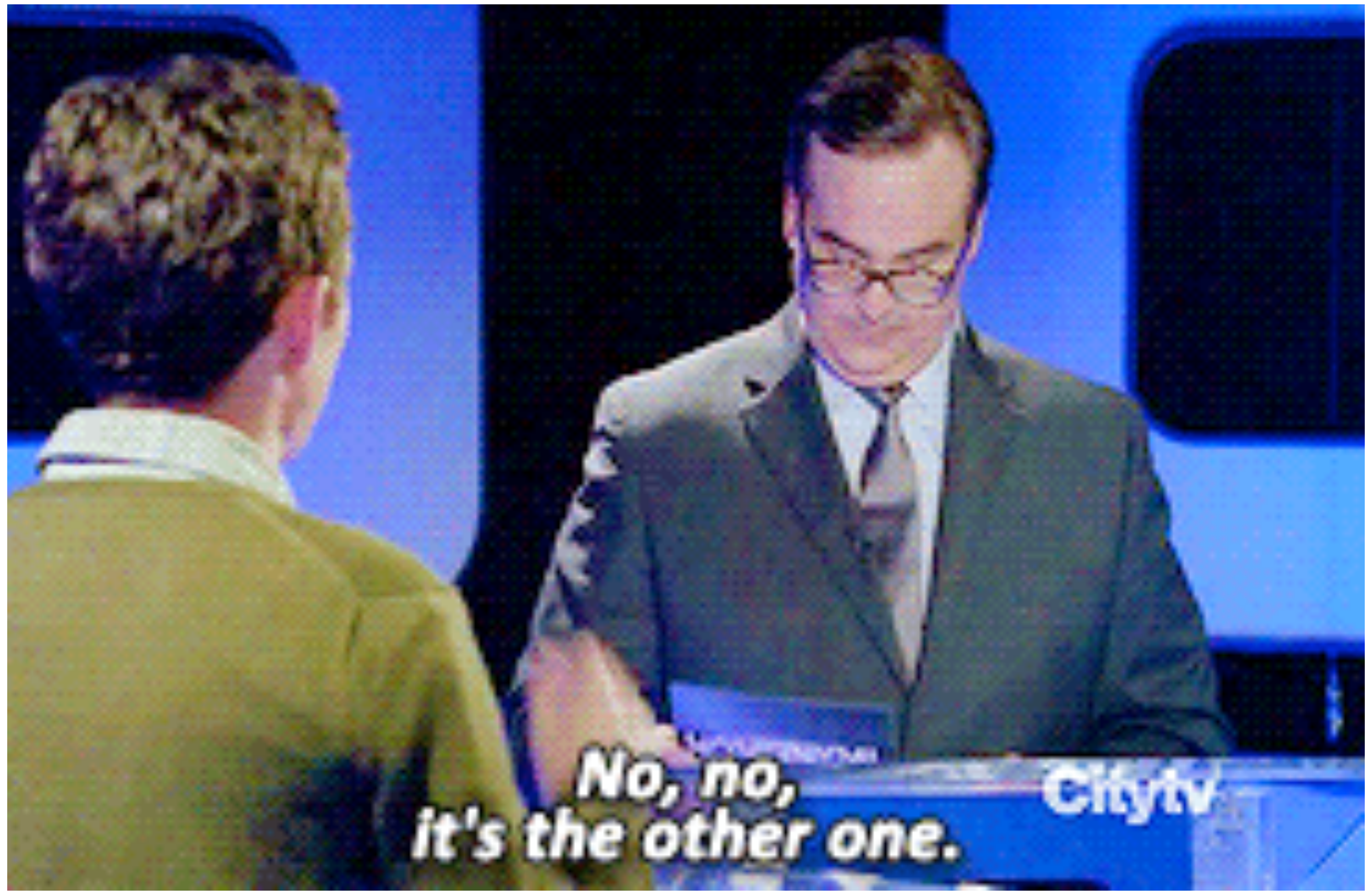

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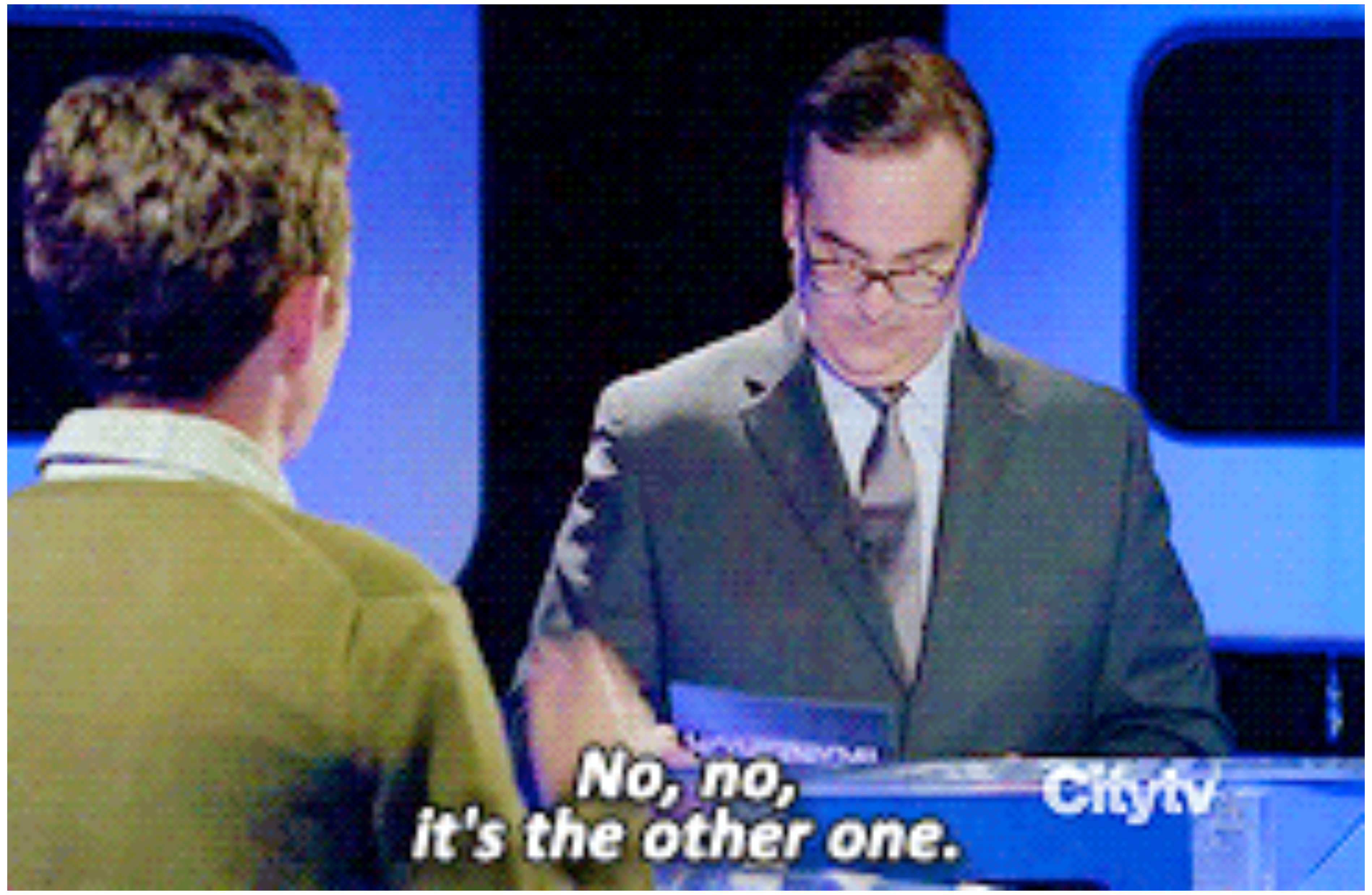

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  - Measure **expected** loss $\Pr(\hat{y}_t \neq y_t) = |p_t - y_t|$

Low regret for online classification

- For every $\mathcal{H}$, there's an algorithm with
  $\text{Regret}_A(\mathcal{H}, T) \leq \sqrt{2 \min\left(\log|\mathcal{H}|, (1 + \log T)\ \text{Ldim}(\mathcal{H})\right) T}$
- Also a lower bound of $\Omega\left(\sqrt{\text{Ldim}(\mathcal{H})\ T}\right)$
- Based on Weighted-Majority algorithm for **learning with expert advice**

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  - Can show (21.13-14) that one expert is as good as the best $h \in \mathscr{H}$,
    and there aren't too many of them,
    giving $\mathrm{Regret}_A(\mathscr{H}, T) \leq \sqrt{2(1 + \log T)\ \mathrm{Ldim}(\mathscr{H})\ T}$

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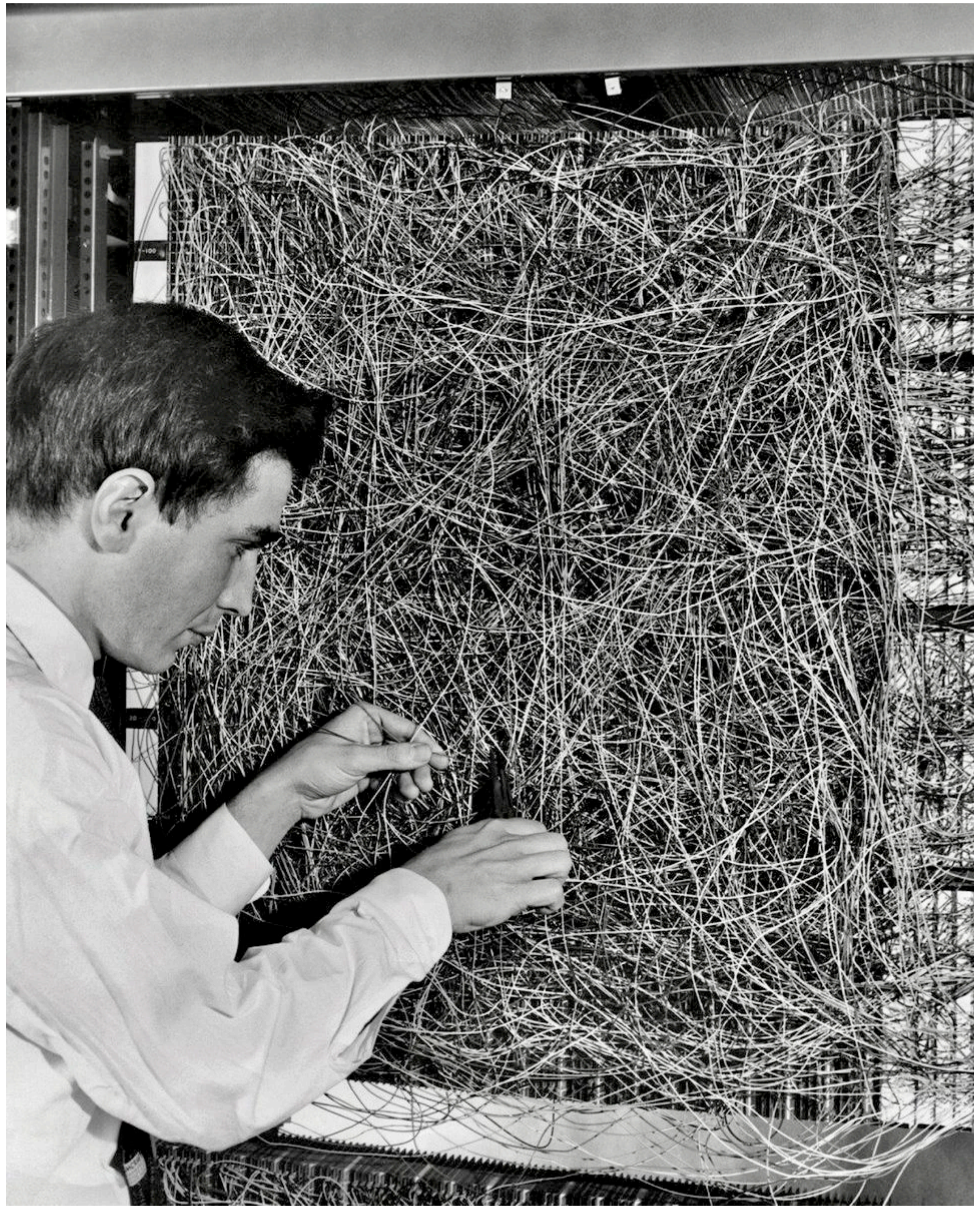
  - $\mathrm{Regret}(w^*, T) \le B\rho\sqrt{T}$ if ℓ_t are ρ-Lipschitz, $\mathscr{H}$ is B-bounded, $\eta = B/\left(\rho\sqrt{T}\right)$

Online Perceptron

- You learned about Batch Perceptron in HW3
- Original algorithm is online
- Essentially identical, just only update on mistake
- Corresponds to online gradient descent on hinge loss
- Get same $(R/\gamma)^2$ margin-based mistake bound
  - Ldim = ∞ without the margin condition



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- MRT **Theorem** 8.15: if $\ell(\,\cdot\,, z)$ is also convex,

$$L_{\mathscr{D}}\left(\frac{1}{T}\sum_{t=1}^{T} h_t\right) \leq \inf_{h \in \mathscr{H}} L_{\mathscr{D}}(h) + \frac{1}{T}\mathrm{Regret}_A(\mathscr{H}, T) + 2M\sqrt{\frac{2}{T}\log\frac{2}{\delta}}$$

(pause)

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- Can be thought of as a particular **form of stability**

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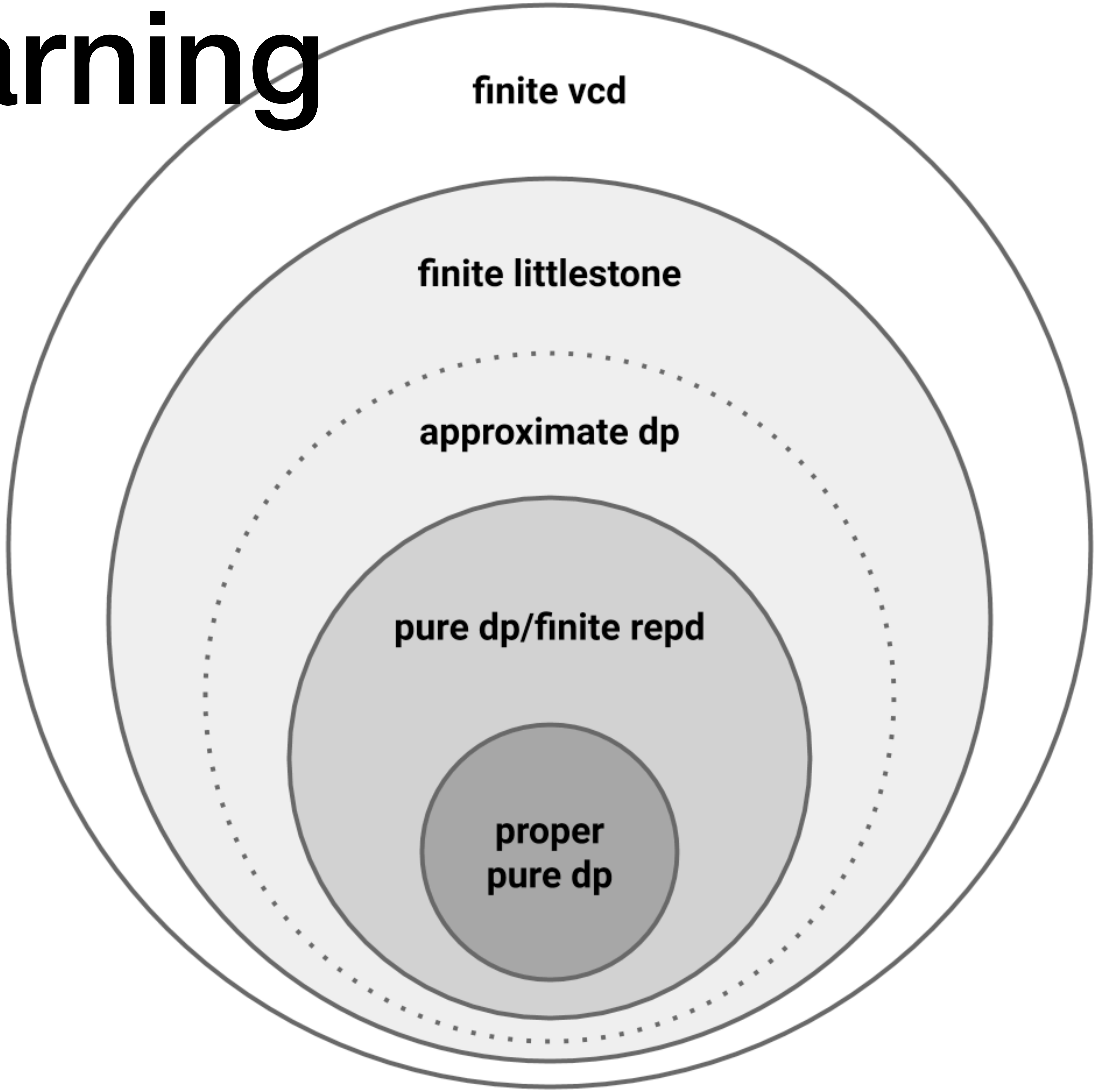

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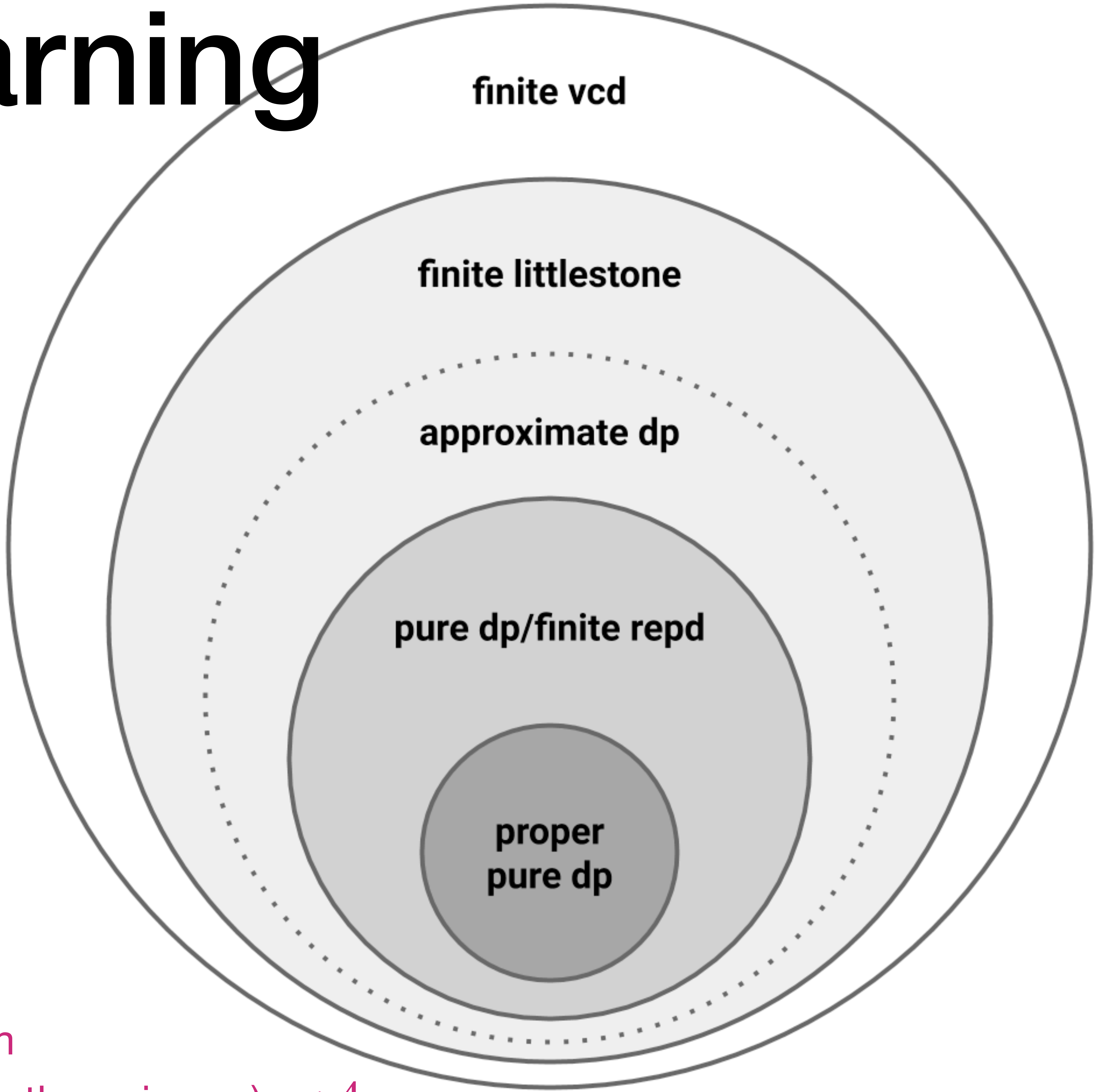

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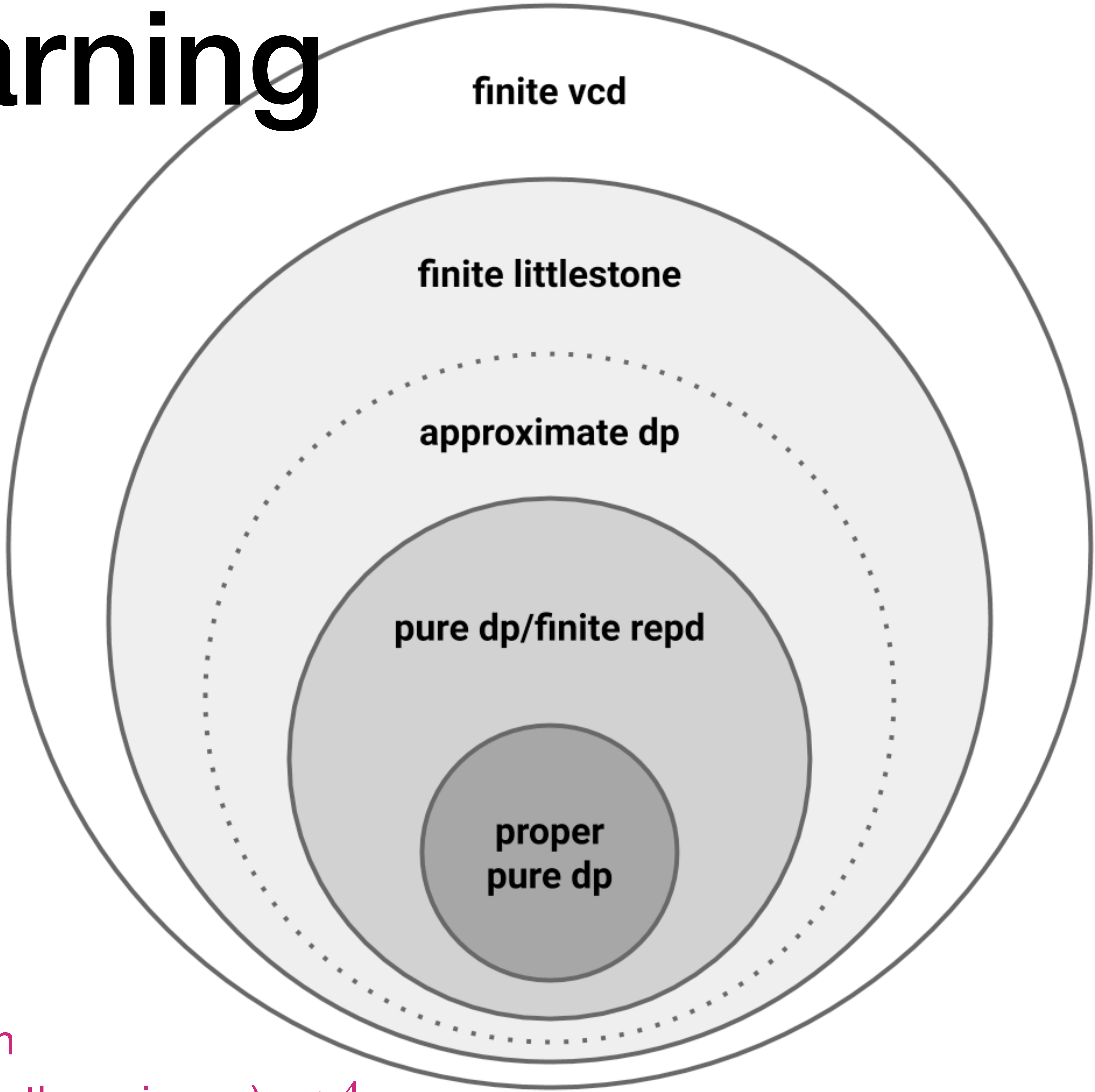

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DP and online learning

- Can learn differentially privately iff can learn online
  - Close connections via stability
  - But huge gap in sample and time complexity
  - Indications (Bun 2020) that converting one to the other isn't possible with polynomial time + sample complexity
  - Still a lot to understand here

Some of the stuff we didn't cover

- **Multiclass learning**: can use same techniques, need right loss
- **Ranking**: which search results are most relevant?
- **Boosting**: combine "weak learners" to a strong one (kind of like A3 Q3 b)
- **Transfer learning** / **out-of-domain generalization** / …: train on $\mathscr{D}$, test on $\mathscr{D}'$
- Do ImageNet Classifiers Generalize to ImageNet? / The Ladder mechanism
- **Robustness**: what if we have some adversarially-corrupted training data?
- **Unsupervised learning** (just the PCA question on A1)
  "How well can we 'understand' a data distribution?"
- **Semi-supervised learning** (just the algorithm from A4)
- **Active learning**: if xs are available but labeling them is expensive,
  can we choose which to label?
- **Multi-armed bandits**: which action should I take?
- **Reinforcement learning**: interacting with an environment with hidden state
- …