Bidding in First-Price Auctions

Game Theory Course:
Jackson, Leyton-Brown & Shoham
Theorem

First-Price (sealed bid) and Dutch auctions are strategically equivalent.

- In both, a bidder must decide on the amount s/he’s willing to pay, conditional on it being the highest bid.
  - Dutch auctions are extensive-form games, but the only thing a winning bidder knows is that all others have not to bid higher.
  - Same as a bidder in a first-price auction.
Equivalence of First-Price and Dutch Auctions

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- So, why are both auction types used?
  - First-price auctions can be held asynchronously.
  - Dutch auctions are fast, and require minimal communication: only one bit needs to be transmitted from the bidders to the auctioneer.
Discussion

- How should bidders bid in these auctions?
  - Bid less than valuation.
  - There’s a tradeoff between:
    - probability of winning
    - amount paid upon winning
  - Bidders don’t have a dominant strategy.
Theorem

In a first-price auction with two risk-neutral bidders whose valuations are IID and drawn from $U(0, 1)$, $(\frac{1}{2} v_1, \frac{1}{2} v_2)$ is a Bayes-Nash equilibrium strategy profile.

Proof.

Assume that bidder 2 bids $v_2$, and bidder 1 bids $s_1$. 1 wins when $v_2 < s_1$, and gains utility $v_1 s_1$, but loses when $v_2 > s_1$ and then gets utility 0: (we can ignore the case where the agents have the same valuation, because this occurs with probability zero).

$$E[u_1] = \int_0^{s_1} (v_1 s_1) dv_2 + \int_{s_1}^{\frac{1}{2} v_2} (0) dv_2 = \frac{1}{2} v_1 s_1 \frac{1}{2} v_2$$

(1)
**Analysis**

**Theorem**

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**Proof.**

Assume that bidder 2 bids $\frac{1}{2} v_2$, and bidder 1 bids $s_1$.

$1$ wins when $v_2 < 2s_1$, and gains utility $v_1 - s_1$, but loses when $v_2 > 2s_1$ and then gets utility 0: (we can ignore the case where the agents have the same valuation, because this occurs with probability zero).

\[
E[u_1] = \int_0^{2s_1} (v_1 - s_1) dv_2 + \int_{2s_1}^1 (0) dv_2
\]

\[
= (v_1 - s_1) v_2 \bigg|_{0}^{2s_1}
\]

\[
= 2v_1 s_1 - 2s_1^2.
\]
Analysis

Theorem

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Proof Continued.

We can find bidder 1’s best response to bidder 2’s strategy by taking the derivative of (I) and setting it equal to zero:

$$\frac{\partial}{\partial s_1} (2v_1 s_1 - 2s_1^2) = 0$$

$$2v_1 - 4s_1 = 0$$

$$s_1 = \frac{1}{2}v_1$$

Thus when player 2 is bidding half her valuation, player 1’s best reply is to bid half his valuation. The calculation of the optimal bid for player 2 is analogous, given the symmetry of the game.
More than two bidders

- Narrow result: two bidders, uniform valuations.
- Still, first-price auctions are not incentive compatible as direct mechanisms.
  - Need to solve for equilibrium.

Theorem.

In a first-price sealed bid auction with \( n \) risk-neutral agents whose valuations are independently drawn from a uniform distribution on \([0,1]\), the (unique) symmetric equilibrium is given by the strategy profile \((v_1; \ldots; v_n)\).
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**Theorem**

In a first-price sealed bid auction with $n$ risk-neutral agents whose valuations are independently drawn from a uniform distribution on $[0, 1]$, the (unique) symmetric equilibrium is given by the strategy profile $(\frac{n-1}{n} v_1, \ldots, \frac{n-1}{n} v_n)$. 
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**Theorem**

*In a first-price sealed bid auction with $n$ risk-neutral agents whose valuations are independently drawn from a uniform distribution on $[0, 1]$, the (unique) symmetric equilibrium is given by the strategy profile $(\frac{n-1}{n} v_1, \ldots, \frac{n-1}{n} v_n)$.*

- proven using a similar argument.
- A broader problem: the proof only verified an equilibrium strategy.
  - How do we find the equilibrium?