Vickrey-Clarke-Groves Mechanisms: Definitions

Game Theory Course:
Jackson, Leyton-Brown & Shoham

A positive result

- Recall that in the quasilinear utility setting, a direct mechanism consists of a choice rule and a payment rule.
- A VCG mechanism:
  - has truth as a dominant strategy (satisfies truthfulness, is strategy-proof)
  - makes efficient choices (*not including payments*)

A positive result

- Recall that in the quasilinear utility setting, a direct mechanism consists of a choice rule and a payment rule.
- A VCG mechanism:
  - has truth as a dominant strategy (satisfies truthfulness, is strategy-proof)
  - makes efficient choices (*not including payments*)
- And, under additional assumptions about the setting, can satisfy:
  - weak budget balance
  - *interim* individual rationality

Groves Mechanisms

Definition (Groves mechanisms)

Direct mechanisms, (χ, p), such that

$$\chi(\hat{v}) \in \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i\left(\hat{v}_{-i}\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Some people refer to these as VCG mechanisms, although that name has more recently started to be used to refer to a specific mechanism within this class.

The Vickrey-Clarke-Groves Mechanism

Definition (A Vickrey-Clarke-Groves (VCG) mechanism, a.k.a. a Pivotal mechanism)

A Vickrey-Clarke-Groves mechanism or a pivotal mechanism is a Groves mechanism (χ, p), such that

$$\chi(\hat{v}) \in \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \max_x \sum_{j \neq i} \hat{v}_j(x) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

VCG discussion

$$\chi(\hat{v}) \in \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \max_x \sum_{j \neq i} \hat{v}_j(x) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- You get paid everyone's utility under the allocation that is actually chosen
  - except your own, but you get that directly as utility
- Then you get charged everyone's utility in the world where you don't participate
- Thus you pay your social cost

VCG discussion

$$\chi(\hat{v}) = \arg\max_{x} \sum_{i} \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?

VCG discussion

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don’t affect the outcome

VCG discussion

$$\chi(\hat{v}) = \arg\max_{x} \sum_{i} \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?

VCG discussion

$$\chi(\hat{v}) = \arg\max_{x} \sum_{i} \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing

VCG discussion

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing
- who gets paid?

VCG discussion

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don’t affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing
- who gets paid?
  - (pivotal) agents who make things better for others by existing

VCG and Groves Mechanisms: Truthfulness

Theorem

Truth telling is a dominant strategy under any Groves mechanism including the pivotal mechanism (a VCG mechanism).

Consider agent i's problem of choosing the best strategy $\hat{v}_i$. A best strategy for i is solves

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v}_i, \hat{v}_{-i})) - p(\hat{v}_i, \hat{v}_{-i}) \right)$$

Substituting in the payment function for a Groves mechanism this becomes:

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) - h_i\left(\hat{v}_{-i}\right) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right)$$

Since $h_i\left(\hat{v}_{-i}\right)$ does not depend on $\hat{v}_i$, it is sufficient to solve

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).$$

VCG Truthfulness

So, i would like to pick a declaration $\hat{v}_i$ that will lead the mechanism to pick an $x \in X$ which solves

$$\max_x \left(v_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right). \tag{1}$$

Under a Groves mechanism,

$$\chi(\hat{v}) \in \arg\max_x \left(\hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).$$

A Groves mechanism will choose x in a way that solves the maximization problem in Equation (1) when $\hat{v}_i = v_i$. Thus, truth-telling is a dominant strategy for agent i. $\square$

Groves Uniqueness

Theorem (Green–Laffont)

Suppose that for all agents any $v_i : X \mapsto \mathbb{R}$ is a feasible preference. Then an "efficient" mechanism (χ, p) (such that $\chi(\hat{v}) \in \arg\max_x \sum_i \hat{v}_i(x)$) has truthful reporting as a dominant strategy for all agents and preferences only if it is Groves mechanism: $p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(\chi(v))$.

A proof can be found at
`http://www.stanford.edu/~jacksonm/mechtheo.pdf`

Summary

- Groves mechanisms, and VCG mechanisms in particular, have nice dominant strategy properties
- Agents' payments include the impact of their announcements on other agents
- Internalize the externalities and lead to efficient decisions (x's)
- But may burn payments to do so!