Lecture Overview

• SAT: example
• First Order Logics
  • Language and Semantics
  • Inference
Satisfiability problems (SAT)

Consider a CNF sentence, e.g.,

\((\neg D \lor \neg B \lor C) \land (B \lor \neg A \lor \neg C) \land (\neg C \lor \neg B \lor E) \land (E \lor \neg D \lor B) \land (B \lor E \lor \neg C)\)

*Is there an interpretation in which this sentence is true (i.e., that is a model of this sentence)?*

Many combinatorial problems can be reduced to checking the satisfiability of propositional sentences ......and returning a model
In *combinatorics* and in experimental design, a **Latin square** is

- an *\( n \times n \) array*
- filled with *\( n \) different symbols*,
- each occurring exactly once in each row and exactly once in each column.

Here is an example:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

Here is another one:
Encoding Latin Square in Propositional Logic:

**Propositions**

Variables must be binary! (They must be propositions)

Each variables represents a color assigned to a cell $i j$.

Assume colors are encoded as an integer $k$

$$x_{ijk} \in \{0,1\}$$

Assuming colors are encoded as follows

(black, 1) (red, 2) (blue, 3) (green, 4) (purple, 5)

$$x_{233} = 0$$

True or false, ie. 1 or 0 with respect to the interpretation represented by the picture?

How many vars/propositions overall? $n^3$
Encoding Latin Square in Propositional Logic

Variables must be binary! (They must be propositions)

Each variables represents a color assigned to a cell.
Assume colors are encoded as integers

\[ x_{ijk} \in \{0,1\} \]

Assuming colors are encoded as follows
(black, 1) (red, 2) (blue, 3) (green, 4) (purple, 5)

\[ x_{233} = 0 \]

True or false, ie. 0 or 1 with respect to the interpretation represented by the picture?

How many vars/propositions overall? \( n^3 \)
Encoding Latin Square in Propositional Logic: Clauses

- Some color must be assigned to each cell (clause of length n);

\[ \forall_{ij} (x_{ij1} \lor x_{ij2} \ldots x_{ijn}) \quad \forall_{ik} (x_{ilk} \lor x_{i2k} \ldots x_{ink}) \]

- No color is repeated in the same row (sets of negative binary clauses);

\[ \forall_{ik} (\neg x_{ilk} \lor \neg x_{i2k}) \land (\neg x_{ilk} \lor \neg x_{i3k}) \ldots (\neg x_{ilk} \lor \neg x_{ink}) \ldots (\neg x_{ilk} \lor \neg x_{i(n-1)k}) \lor \neg x_{ink} \]

How many clauses?
Encoding Latin Square in Propositional Logic: Clauses

- Some color must be assigned to each cell (clause of length n);
  \[ \forall_{ij} (x_{ij1} \lor x_{ij2} \cdots x_{ijn}) \]

- No color repeated in the same row (sets of negative binary clauses);
  \[ \forall_{ik} (\neg x_{i1k} \lor \neg x_{i2k}) \land (\neg x_{i1k} \lor \neg x_{i3k}) \cdots (\neg x_{i1k} \lor \neg x_{i2k}) \cdots (\neg x_{i(n-1)k} \lor \neg x_{i2k}) \]

\[ \frac{n*(n-1)}{2} \]

\[ \frac{n^2*n*(n-1)}{2} \]

\[ O(n^4) \]

How many clauses?
Encoding Latin Square Problems in Propositional Logic: FULL MODEL

Variables: $x_{ijk}$ cell $i,j$ has color $k$; $i,j,k=1,2,...,n$. $x_{ijk} \in \{0,1\}$

Each variables represents a color assigned to a cell.

Clauses: $O(n^4)$

- Some color must be assigned to each cell (clause of length n);
  $\forall ij (x_{ij1} \lor x_{ij2} \ldots x_{ijn})$

- No color repeated in the same row (sets of negative binary clauses);
  $\forall ik (\neg x_{ilk} \lor \neg x_{i2k}) \land (\neg x_{ilk} \lor \neg x_{i3k}) \ldots (\neg x_{ilk} \lor \neg x_{ink}) \ldots (\neg x_{ilk} \lor \neg x_{in(n-1)k} \lor \neg x_{ink})$

- No color repeated in the same column (sets of negative binary clauses);
  $\forall jk (\neg x_{1jk} \lor \neg x_{2jk}) \land (\neg x_{1jk} \lor \neg x_{3jk}) \ldots (\neg x_{1jk} \lor \neg x_{njk}) \ldots (\neg x_{1jk} \lor \neg x_{n(n-1)jk} \lor \neg x_{njk})$
Logics in AI: Similar slide to the one for planning

- Propositional Definite Clause Logics
- Semantics and Proof Theory
- Satisfiability Testing (SAT)
- Hardware Verification
- Product Configuration
- Cognitive Architectures
- Production Systems
- Information Extraction
- Summarization
- Tutoring Systems
- Semantic Web
- Video Games
- Ontologies
- Description Logics
- First-Order Logics
- Datalog
- Cognitive Architectures
Relationships between different Logics

First Order Logic

\[ \forall X \exists Y p(X, Y) \iff \neg q(Y) \]
\[ p(a_1, a_2) \]
\[ \neg q(a_5) \]

Propositional Logic

\[ \neg (p \lor q) \rightarrow (r \lor s \lor t) \]
\[ p, r \]

Datalog

\[ p(X) \iff q(X) \land r(X, Y) \]
\[ r(x, y) \iff s(y) \]
\[ s(a_1), q(a_2) \]

PDCL

\[ p \iff s \lor t \]
\[ r \iff s \land q \land p \]
\[ r, p \]
Lecture Overview

- Finish SAT (example)
- **First Order Logics**
  - Language and Semantics
  - Inference
Representation and Reasoning in Complex domains (from 322)

• In complex domains expressing knowledge with propositions can be quite limiting.

\[
\begin{align*}
&\text{up}(s_2) \\
&\text{up}(s_3) \\
&\text{ok}(cb_1) \\
&\text{ok}(cb_2) \\
&\text{live}(w_1) \\
&\text{connected}(w_1, w_2)
\end{align*}
\]

• It is often natural to consider individuals and their properties.

\[
\begin{align*}
&\text{up}(s_2) \\
&\text{up}(s_3) \\
&\text{ok}(cb_1) \\
&\text{ok}(cb_2) \\
&\text{live}(w_1) \\
&\text{connected}(w_1, w_2)
\end{align*}
\]

There is no notion that the system can reason about the same property of the same individual.

\[
\begin{align*}
&\text{up}(s_2) \\
&\text{up}(s_3) \\
&\text{live}(w_1) \\
&\text{connected}(w_1, w_2)
\end{align*}
\]
(from 322) What do we gain....

By breaking propositions into relations applied to individuals?

- Express **knowledge** that **holds for set of individuals** (by introducing **variables**)

  \[
  \text{live}(W) \leftarrow \text{connected}_\to(W,W1) \land \text{live}(W1) \land \\
  \text{wire}(W) \land \text{wire}(W1).
  \]

- We can **ask generic queries** (i.e., containing **variables**)

  \[
  ? \text{connected}_\to(W, w_1)
  \]
“Full” First Order Logics (FOL)

LIKE DATALOG: Whereas propositional logic assumes the world contains facts, FOL (like natural language) assumes the world contains

- **Objects**: people, houses, numbers, colors, baseball games, wars, …
- **Relations**: red, round, prime, brother of, bigger than, part of, comes between, …
- **Functions**: father of, best friend, one more than, plus, …

FURTHERMORE WE HAVE

- **More Logical Operators**:…..
- **Equality**: coreference (two terms refer to the same object)
- **Quantifiers**
  - Statements about unknown objects
  - Statements about classes of objects
Syntax of FOL

- **Constants**
  - KingJohn, 2, ...

- **Predicates**
  - Brother, >, ...

- **Functions**
  - Sqrt, LeftLegOf, ...

- **Variables**
  - x, y, a, b, ...

- **Connectives**
  - ¬, ⇒, ∧, ∨, ⇔

- **Equality**
  - =

- **Quantifiers**
  - ∀, ∃
Atomic sentences

Term is a \textit{function} \((\text{term}_1, \ldots, \text{term}_n)\) or constant or variable

Atomic sentence is \textit{predicate} \((\text{term}_1, \ldots, \text{term}_n)\) or \(\text{term}_1 = \text{term}_2\)

E.g.,

\begin{itemize}
  \item \textit{Brother}(\text{KingJohn}, \text{RichardTheLionheart})
  \item \textit{predicate}(\text{function}(\text{function}(\text{constant})), \text{function}(\text{function}(\text{function}(\text{constant}))))
  \item > (\text{Length}(\text{LeftLegOf}(\text{Richard})), \text{Length}(\text{LeftLegOf}(\text{KingJohn})))
\end{itemize}
Complex sentences

Complex sentences are made from atomic sentences using connectives

\[ \neg S, \quad S_1 \land S_2, \quad S_1 \lor S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2, \]

E.g.

\[ \text{Sibling}(\text{KingJohn}, \text{Richard}) \Rightarrow \text{Sibling}(\text{Richard}, \text{KingJohn}) \]

\[ \forall x \ P(x) \text{ is true in an interpretation } I \text{ iff } P \text{ is true with } x \text{ being each possible object in } I \]

\[ \exists x \ P(x) \text{ is true in an interpretation } I \text{ iff } P \text{ is true with } x \text{ being some possible object in } I \]
Truth in first-order logic

Like in Prop. Logic a sentences is true with respect to an interpretation

\[ \neg A \land (B \Rightarrow C), \]

\[
\begin{array}{ccc}
A & B & C \\
T & F & F \\
T & T & T \\
\end{array}
\]

In FOL interpretations are **much more complex** but still same idea:

possible configuration of the world

2 objects \( \Delta \) \( \Box \)

2 constant symbols \( \{c_1, c_2\} \) \( \rightarrow \) \( \Delta \) \( \Box \)

1 unary predicate \( P \) \( \rightarrow \) \( \{\Delta\} \)

1 binary predicate \( Q \) \( \rightarrow \) \( \{\{\Delta, \Delta\}\} \)

Is \( \forall x \ P(x) \) True?

A. yes
B. no
Truth in first-order logic

Like in Prop. Logic a sentences is true with respect to an interpretation

\[ \neg A \land B \Rightarrow C, \]

\[
\begin{array}{c|c|c|c}
A & B & C & \Rightarrow \\
T & F & F & x \\
T & T & T & v \\
\end{array}
\]

In FOL interpretations are much more complex but still same idea: possible configuration of the world

- \( \Delta \) \quad \square
- \( |C_1, C_2| \) \quad \( \Rightarrow \) \quad \{ \Delta \} \quad \text{but if } P \Rightarrow \{ \Delta, \square \}
- \{ \Delta, \Delta \} \quad \{ \Delta, \Delta, \square \}

Is \( \forall x P(x) \) true? No Yes!
Interpretations for FOL: Example

5 objects

binary relations 2
unary relations 3
functions 1

person

C

R

J

left leg

left leg

brother

on head

crown

king

person

CONSTANT SYMBOLS 5

RLL

JLL
Same interpretation with sets

Since we have a one to one mapping between symbols and object we can use symbols to refer to objects

- \{R, J, RLL, JLL, C\}

**Property Predicates**
- Person = \{R, J\}
- Crown = \{C\}
- King = \{J\}

**Relational Predicates**
- Brother = \{<R,J>, <J,R>\}
- OnHead = \{<C,J>\}

**Functions**
- LeftLeg = \{<R, RLL>, <J, JLL>\}
How many Interpretations with....

5 Objects and 5 symbols
- \{R, J, RLL, JLL, C\}

3 Property Predicates (Unary Relations)
- Person
- Crown
- King

2 Relational Predicates
- Brother
- OnHead

1 Function
- LeftLeg

Assuming unique names

\[ 5! \quad J \quad R \quad C \quad RLL \quad JLL \]

25 possibilities of each one can be \( \frac{5!}{5} \) so \( 2^{25} \)

A. \( 2^5 \)  
B. \( 2^{25} \)  
C. \( 25^2 \)  

Total
\[ 5^5 \]

\[ 5! \times \left(2^5\right)^3 \times \left(2^{25}\right)^2 \times 5^5 \]
To summarize: Truth in first-order logic

- Sentences are true with respect to an **interpretation**
- World contains objects (**domain elements**)
- Interpretation specifies referents for
  - constant symbols $\rightarrow$ objects
  - predicate symbols $\rightarrow$ relations
  - function symbols $\rightarrow$ functional relations

- An atomic sentence $\text{predicate}(\text{term}_1, \ldots, \text{term}_n)$ is true
  iff the **objects** referred to by $\text{term}_1, \ldots, \text{term}_n$
  are in the **relation** referred to by $\text{predicate}$
Quantifiers

Allows us to express

- **Properties of collections of objects** instead of enumerating objects by name
- **Properties of an unspecified object**

Universal: “for all” \( \forall \)
Existential: “there exists” \( \exists \)
Universal quantification

\( \forall \text{<variables> <sentence>} \)

Everyone at UBC is smart:
\( \forall x \text{ At}(x, \text{UBC}) \Rightarrow \text{Smart}(x) \)

\( \forall x \text{ } P \text{ is true in an interpretation } I \text{ iff } P \text{ is true with } x \text{ being each possible object in } I \)

Equivalent to the conjunction of instantiations of \( P \)

\[ \text{At(KingJohn, UBC) } \Rightarrow \text{Smart(KingJohn)} \]
\[ \land \text{At(Richard, UBC) } \Rightarrow \text{Smart(Richard)} \]
\[ \land \text{At(Ralphie, UBC) } \Rightarrow \text{Smart(Ralphie)} \]
\[ \land \ldots \]
Existential quantification

\[ \exists \langle \text{variables} \rangle \langle \text{sentence} \rangle \]

Someone at UBC is smart:
\[ \exists x \ \text{At}(x, \text{UBC}) \land \text{Smart}(x) \]

\[ \exists x \ P \] is true in an interpretation \( I \) iff \( P \) is true with \( x \) being some possible object in \( I \)

Equivalent to the disjunction of instantiations of \( P \)
\[ \text{At}(\text{KingJohn}, \text{UBC}) \land \text{Smart}(\text{KingJohn}) \]
\[ \lor \text{At}(\text{Richard}, \text{UBC}) \land \text{Smart}(\text{Richard}) \]
\[ \lor \text{At}(\text{Ralphie}, \text{UBC}) \land \text{Smart}(\text{Ralphie}) \]
\[ \lor \ldots \]
Properties of quantifiers

\( \exists x \ \forall y \) is \textbf{not} the same as \( \forall y \ \exists x \)

\( \exists x \ \forall y \) Loves\((x,y)\)
- “There is a person who loves everyone in the world”

\( \forall y \ \exists x \) Loves\((x,y)\)
- “Everyone in the world is loved by at least one person”

\textbf{Quantifier duality}: each can be expressed using the other

\( \forall x \) Likes\((x,\text{IceCream})\) \quad \neg \exists x \ \neg \text{Likes}(x,\text{IceCream})

\( \exists x \) Likes\((x,\text{Broccoli})\) \quad \neg \forall x \ \neg \text{Likes}(x,\text{Broccoli})
Lecture Overview

• Finish SAT (example)
• **First Order Logics**
  • Language and Semantics
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FOL: Inference

Resolution Procedure can be generalized to FOL

- Every formula can be rewritten in logically equivalent CNF
  - Additional rewriting rules for quantifiers
- Similar Resolution step, but variables need to be unified (like in DATALOG)

\[
\{ \neg \text{In}(x,y) \lor \neg \text{Charged}(x) \} \\
\{ \neg \text{In}(z,y) \lor \text{Connected}(z) \} \\
\rightarrow \neg \text{Charged}(x) \lor \text{Connected}(x)
\]

\[\Theta = \{ z/x, y/y \}\]
NLP Practical Goal for FOL: the ultimate Web question-answering system?

Map NL queries into FOPC so that answers can be effectively computed

**What African countries are not on the Mediterranean Sea?**

$$\exists c \text{ Country}(c) \land \neg \text{Borders}(c, \text{Med.Sea}) \land \text{In}(c, \text{Africa})$$

- **Was 2007 the first El Nino year after 2001?**

  $$\text{ElNino}(2007) \land \neg \exists y \text{ Year}(y) \land \text{After}(y, 2001) \land \text{Before}(y, 2007) \land \text{ElNino}(y)$$
Learning Goals for today’s class

You can:

• Explain differences between Proposition Logic and First Order Logic
• Compute number of interpretations for FOL
• Explain the meaning of quantifiers
• Describe application of FOL to NLP: Web question answering
Next class Wed

- Ontologies (e.g., Wordnet, Probase), Description Logics…
- Midterm will be returned Fri or next Mon

Assignment-3 will be out tonight