# Reasoning Under Uncertainty: Bnet Inference

# (Variable elimination)

**Computer Science cpsc322, Lecture 29**

***(Textbook Chpt 6.4)***

**June, 15, 2017**

# Lecture Overview

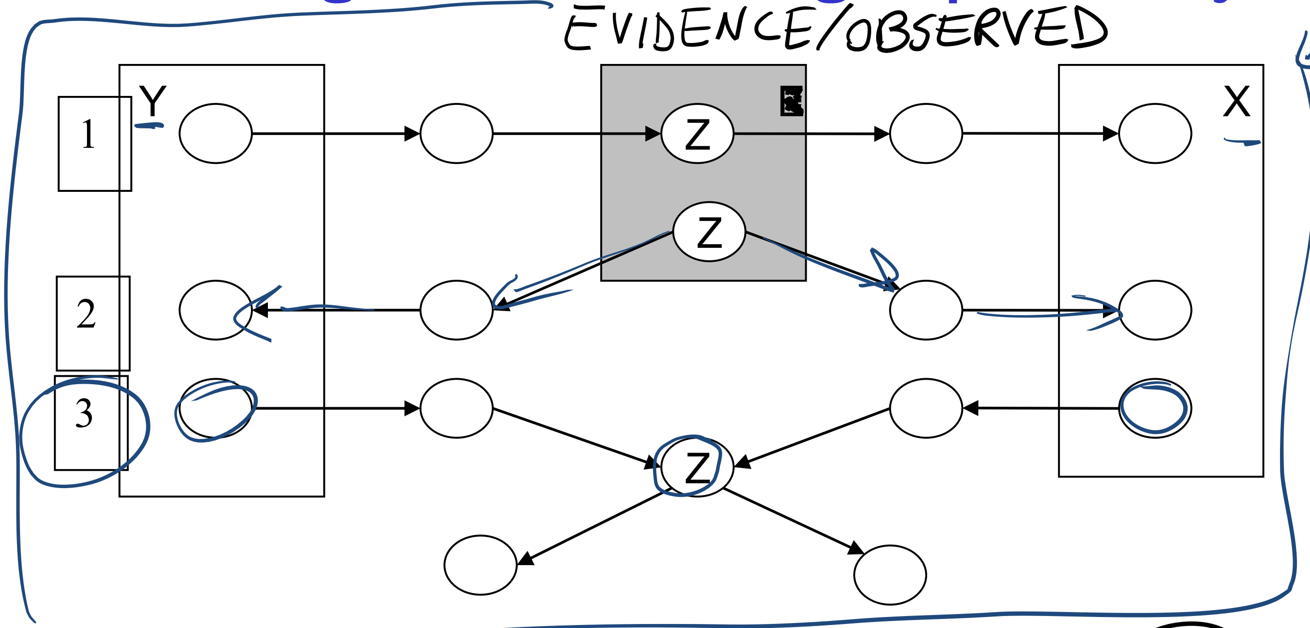
- **Recap Learning Goals previous lecture**
- Bnets Inference
  - Intro
  - Factors
  - Variable elimination Intro

# Learning Goals for previous class

**You can:**

- In a Belief Net, determine whether one variable is independent of another variable, given a set of observations.
- Define and use **Noisy-OR** distributions. Explain assumptions and benefit.
- Implement and use a **naïve Bayesian classifier**. Explain assumptions and benefit.

# 3 Configuration blocking dependency (belief propagation)




true $\boxed{2}$

Indep( , {E})

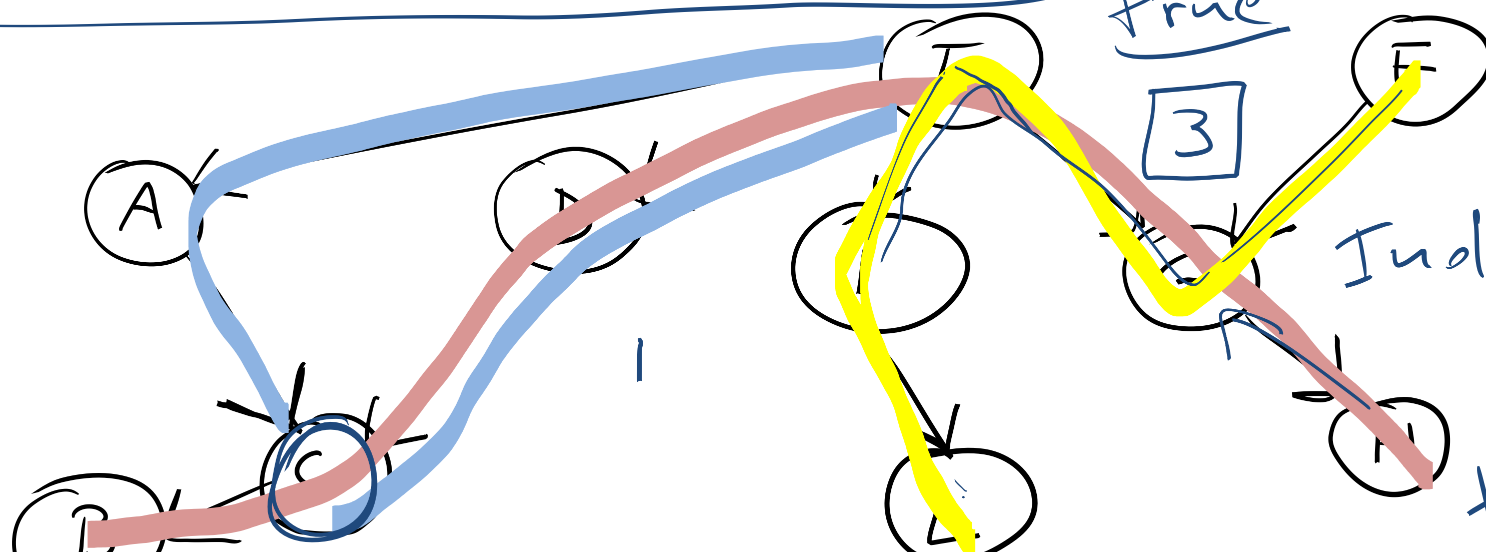
false

Indep( , {D})

Indep( , {C})

true

$\boxed{3}$

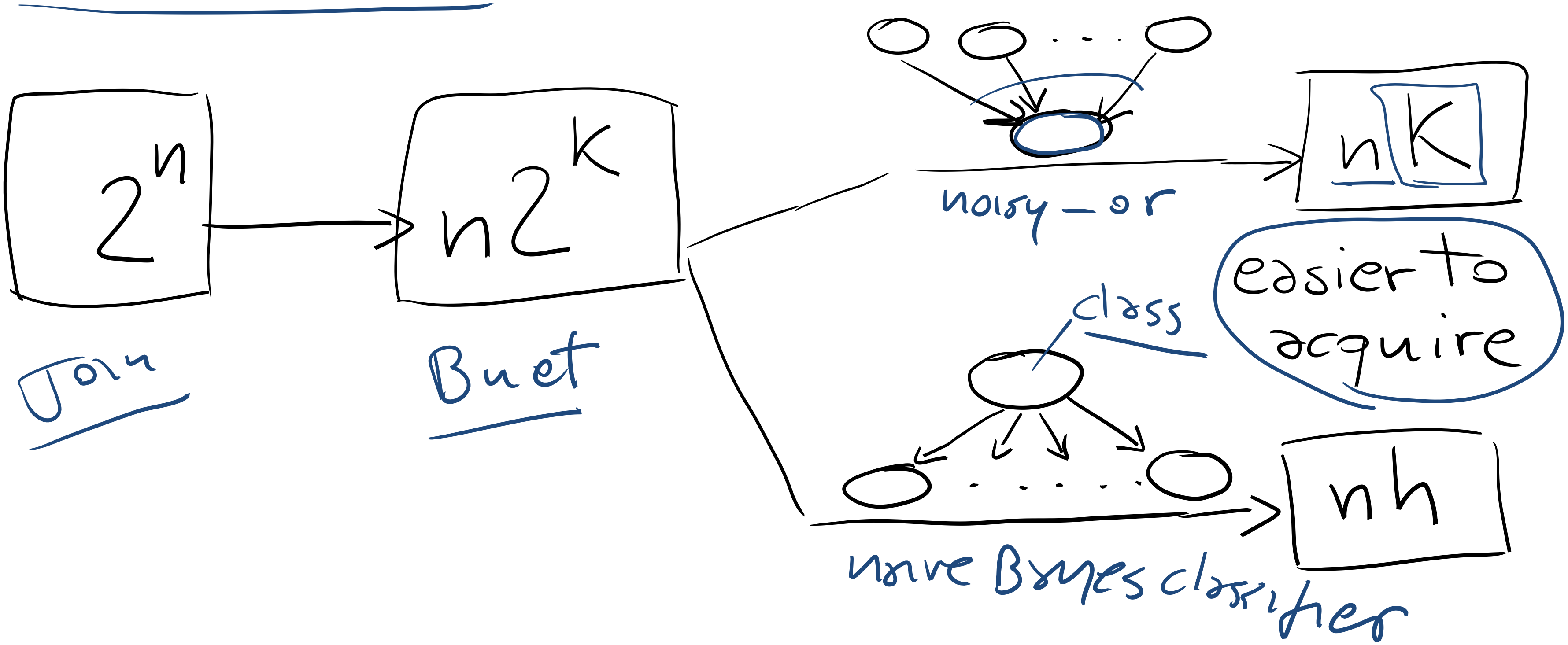



Ind(L, F {H})

false

# Bnets: Compact Representations

n Boolean variables, k max. number of parents




Only one parent with h possible values

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  - Intro
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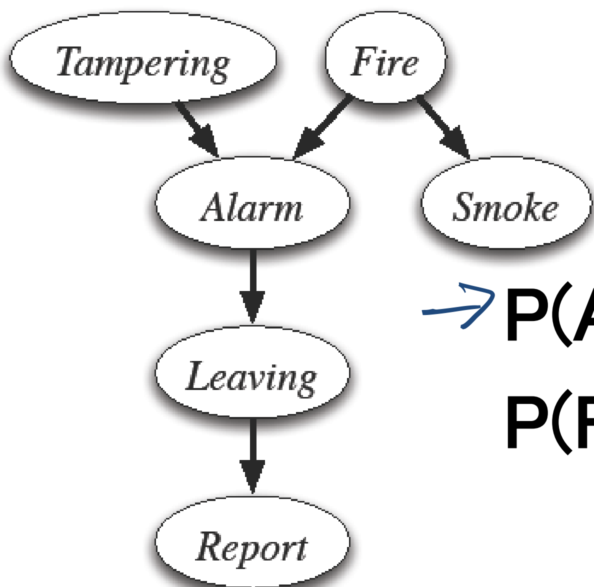
# Bnet Inference

- **Our goal:** compute probabilities of variables in a belief network

  What is the posterior distribution over one or ~~more~~ variables, ~~conditioned~~ on one or more observed variables?



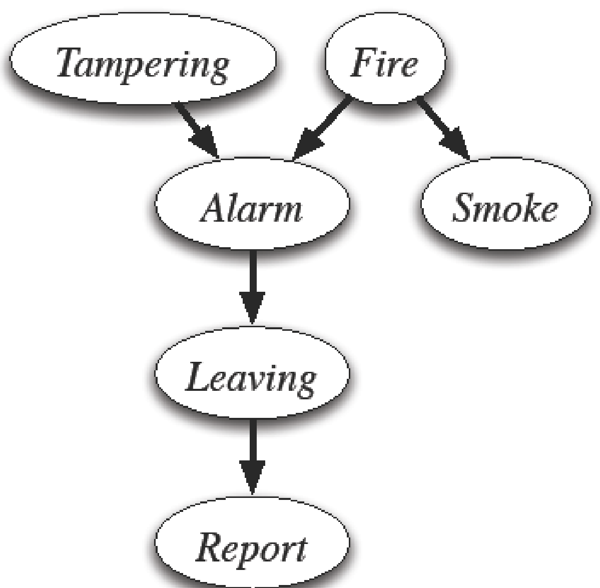

examples

→ **P(Alarm| Smoke =f )**

**P(Fire | Smoke =t ,Leaving =f )**

# Bnet Inference: General

- Suppose the variables of the belief network are $X_1, \cdots, X_n$.
- Z is the query variable
- *$Y_1=v_1, \cdots, Y_j=v_j$ are the observed variables (with their values)*
- $Z_1, \cdots, Z_k$ are the remaining variables
- What we want to compute: $\boxed{P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j)}$




*Example:*

**P(L | S = t , R = f)**

$Z \leftrightarrow L$

$Y_1 Y_2 \leftrightarrow S, R$

$Z_1 Z_2 Z_3 \leftrightarrow T, F, A$

# What do we need to compute?

Remember conditioning and marginalization…

$$P(L \mid S = t, R = f) = \frac{P(L, S=t, R=f)}{P(S=t, R=f)}$$

① ② ③

| L | S | R | P(L, S=t, R=f) |
|---|---|---|---|
| t | t | f | .3 |
| f | t | f | .2 |

Do they have to sum up to one?

A. yes

B. no

② = .5

| L | S | R | P(L \| S=t, R=f) |
|---|---|---|---|
| t | t | f | .6 |
| f | t | f | .4 |

# In general…..

$$P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j) = \boxed{\frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)}} = \frac{\boxed{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}}{\sum_Z P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}$$

1

2

- We only need to **compute the** numerator and then **normalize**
- This can be framed in terms of **operations between factors** (that satisfy the semantics of probability)

# Lecture Overview

- Recap Bnets
- Bnets Inference
  - Intro
  - **Factors**
  - Variable elimination Algo

# Factors

- A **factor** is a representation of a function from a tuple of random variables into a number. [0, 1]
- We will write factor $f$ on variables $X_1, \cdots, X_j$ as $f(X_1 \ldots X_j)$

- A **factor** can denote:
  - One distribution
  - One *partial* distribution
  - Several distributions
  - Several *partial* distributions

  over the given tuple of variables

# Factor: Examples

$P(X_1, X_2)$ is a factor $f(X_1, X_2)$

*Distribution*

| $X_1$ | $X_2$ | $f(X_1, X_2)$ |
|---|---|---|
| T | T | .12 |
| T | F | .08 |
| F | T | .08 |
| F | F | .72 |

*Partial distribution*

$P(X_1, X_2 = v_2)$ is a factor $f(X_1)_{X2=v2}$

| $X_1$ | $X_2$ | $f(X_1)_{X2=F}$ |
|---|---|---|
| T | F | .08 |
| F | F | .72 |

# Factors: More Examples

- A factor denotes one or more (possibly partial) distributions over the given tuple of variables
  - e.g., $P(X_1, X_2)$ is a factor $f(X_1, X_2)$ *Distribution*
  - e.g., $P(X_1, X_2, X_3 = v_3)$ is a factor $f(X_1, X_2)_{X3 = v3}$ *Partial distribution*
  - e.g., $P(X \mid Z,Y)$ is a factor $f(X,Z,Y)$ *Set of Distributions*
  - e.g., $P(X_1, X_3 = v_3 \mid X_2)$ is a factor $f(X_1, X_2)_{X3 = v3}$ *Set of partial Distributions*

iclicker.

| X | Y | Z | val |
|---|---|---|---|
| t | t | t | 0.1 |
| t | t | f | 0.9 |
| t | f | t | 0.2 |
| t | f | f | 0.8 |
| f | t | t | 0.4 |
| f | t | f | 0.6 |
| f | f | t | 0.3 |
| f | f | f | 0.7 |

f(X,Y,Z) ??

A. P(X,Y,Z)

B. P(Y|Z,X)

C. P(Z|X,Y)

D. None of the above

# Factors

- A **factor** is a representation of a function from a tuple of random variables into a number.

$[0, 1]$

- We will write factor $f$ on variables $X_1, \cdots, X_j$ as

$f(X_1 \cdots X_j)$

- A factor denotes one or more (possibly partial) distributions over the given tuple of variables

P(Z|X,Y)

  - e.g., $P(X_1, X_2)$ is a factor $f(X_1, X_2)$ — *Distribution*
  - e.g., $P(X_1, X_2, X_3 = v_3)$ is a factor $f(X_1, X_2)_{X3 = v3}$ — *Partial distribution*
  - *e.g., $P(X \mid Z, Y)$ is a factor $f(X, Z, Y)$* — *Set of Distributions*
  - e.g., $P(X_1, X_3 = v_3 \mid X_2)$ is a factor $f(X_1, X_2)_{X3 = v3}$ — *Set of partial Distributions*

f(X,Y,Z)

| X | Y | Z | val |
|---|---|---|---|
| t | t | t | 0.1 |
| t | t | f | 0.9 |
| t | f | t | 0.2 |
| t | f | f | 0.8 |
| f | t | t | 0.4 |
| f | t | f | 0.6 |
| f | f | t | 0.3 |
| f | f | f | 0.7 |

# Manipulating Factors:

We can make new factors out of an existing factor

- Our first operation: we can *assign* some or all of the variables of a factor.

f(X,Y,Z):

| X | Y | Z | val |
|---|---|---|---|
| t | t | t | 0.1 |
| t | t | f | 0.9 |
| t | f | t | 0.2 |
| t | f | f | 0.8 |
| ~~f~~ | ~~t~~ | ~~t~~ | ~~0.4~~ |
| ~~f~~ | ~~t~~ | ~~f~~ | ~~0.6~~ |
| ~~f~~ | ~~f~~ | ~~t~~ | ~~0.3~~ |
| ~~f~~ | ~~f~~ | ~~f~~ | ~~0.7~~ |

*What is the result of assigning X= t ?*

f(X=t,Y,Z)

$f(X, Y, Z)_{X = t}$

# More examples of assignment

r(X,Y,Z):

| X | Y | Z | val |
|---|---|---|---|
| t | t | t | 0.1 |
| t | t | f | 0.9 |
| t | f | t | 0.2 |
| t | f | f | 0.8 |
| f | t | t | 0.4 |
| f | t | f | 0.6 |
| f | f | t | 0.3 |
| f | f | f | 0.7 |

r(X=t,Y,Z):

| Y | Z | val |
|---|---|---|
| ~~t~~ | ~~t~~ | ~~0.1~~ |
| t | f | 0.9 |
| ~~f~~ | ~~t~~ | ~~0.2~~ |
| f | f | 0.8 |

r(X=t,Y,Z=f):

| Y | val |
|---|---|
| ~~t~~ | ~~.9~~ |
| f | .8 |

r(X=t,Y=f,Z=f):

| val |
|---|
| .8 |

# Summing out a variable example

Our second operation: we can ***sum out*** a variable, say $X_1$ with domain $\{v_1, \cdots, v_k\}$, from factor $f(X_1, \cdots, X_j)$, resulting in a factor on $X_2, \cdots, X_j$ defined by:

$f_3(A,B,C)$:

| B | A | C | val |
|---|---|---|---|
| t | t | t | 0.03 |
| t | t | f | 0.07 |
| f | t | t | 0.54 |
| f | t | f | 0.36 |
| t | f | t | 0.06 |
| t | f | f | 0.14 |
| f | f | t | 0.48 |
| f | f | f | 0.32 |

$\sum_B f_3(A,B,C)$:

| A | C | val |
|---|---|---|
| t | t | .57 |
| t | f | .43 |
| f | t | |
| f | f | |

$$\left(\sum_{X_1} f\right)(X_2, \ldots, X_j) = f(X_1 = v_1, X_2, \ldots, X_j) + \ldots + f(X_1 = v_k, X_2, \ldots, X_j)$$

# Multiplying factors

- Our third operation: factors can be ***multiplied*** together.

$f_1(A,B)$:

| A | B | Val |
|---|---|---|
| t | t | 0.1 |
| t | f | 0.9 |
| f | t | 0.2 |
| f | f | 0.8 |

$f_2(B,C)$:

| B | C | Val |
|---|---|---|
| t | t | 0.3 |
| t | f | 0.7 |
| f | t | 0.6 |
| f | f | 0.4 |

$f_1(A,B) \times f_2(B,C)$:

| A | B | C | val |
|---|---|---|---|
| t | t | t | .03 |
| t | t | f | .07 |
| t | f | t | .054 |
| t | f | f | |
| f | t | t | |
| f | t | f | |
| f | f | t | |
| f | f | f | |

# Multiplying factors

- Our third operation: factors can be ***multiplied*** together.

$f_1(A,B)$:

| A | B | Val |
|---|---|---|
| t | t | 0.1 |
| t | f | 0.9 |
| f | t | 0.2 |
| f | f | 0.8 |

$f_2(B,C)$:

| B | C | Val |
|---|---|---|
| t | t | 0.3 |
| t | f | 0.7 |
| f | t | 0.6 |
| f | f | 0.4 |

$f_1(A,B) \times f_2(B,C)$:

| A | B | C | val |
|---|---|---|---|
| t | t | t | |
| t | t | f | |
| **t** | **f** | **t** | **??** |
| t | f | f | |
| f | t | t | |
| f | t | f | |
| f | f | t | |
| f | f | f | |

A. 0.32

B. 0.54

C. 0.24

D. 0.06

# Multiplying factors: Formal

·The **product** of factor $f_1(A, B)$ and $f_2(B, C)$, where $B$ is the variable in common, is the factor $(f_1 \times f_2)(A, B, C)$ defined by:

$$f_1(A,B)\,f_2(B,C) = (f_1 \times f_2)(A,B,C)$$

t f     t f

A B     B C

t f f

**Note1:** it's defined on all *A, B, C* triples, obtained by multiplying together the appropriate pair of entries from $f_1$ and $f_2$.

**Note2:** *A, B, C* can be sets of variables

# Factors Summary

- A factor is a representation of a function from a tuple of random variables into a number.
  - $f(X_1, \cdots, X_j)$.
- We have defined three operations on factors:
  1. Assigning one or more variables
     - $f(X_1=v_1, X_2, \cdots, X_j)$ is a factor on $X_2, \cdots, X_j$, also written as $f(X_1, \cdots, X_j)_{X_1=v_1}$
  2. Summing out variables is a factor on $X_2, \cdots, X_j$
     - $\sum_{X_1} f(X_1, X_2, .., X_j) = f(X_1=v_1, X_2, , X_j) + \cdots + f(X_1=v_k, X_2, , X_j)$
  3. Multiplying factors
     - $f_1(A, B)\ f_2\ (B, C) = (f_1 \times f_2)(A, B, C)$

# Lecture Overview

- Recap Bnets
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  - Factors
  - Intro Variable elimination Algo

# Variable Elimination Intro

- Suppose the variables of the belief network are $X_1, \cdots, X_n$.
- Z is the query variable
- $Y_1=v_1, \cdots, Y_j=v_j$ *are the observed variables (with their values)*
- $Z_1, \cdots, Z_k$ are the remaining variables
- What we want to compute: $\boxed{P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j)}$
- We showed before that what we actually need to compute is

$$\boxed{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}$$

This can be computed in terms of **operations between factors** (that satisfy the semantics of probability)

# Variable Elimination Intro

- If we express the joint as a factor,

observed

sum out

$$f(Z, \overbrace{Y_1 \cdots, Y_j}^{\text{observed}}, Z_1 \cdots, Z_j)$$

assign

- We can compute $P(Z, Y_1 = v_1, \cdots, Y_j = v_j)$ by ??
  - **assigning** $Y_1 = v_1, \cdots, Y_j = v_j$
  - and **summing out** the variables $Z_1, \cdots, Z_k$

$$P(Z, Y_1 = v_1, \ldots, Y_j = v_j) = \boxed{\sum_{Z_k} \cdots \sum_{Z_1}} f(Z, Y_1, \ldots, Y_j, Z_1, \ldots, Z_k)_{Y_1 = v_1, \ldots, Y_j = v_j}$$

this is the joint TOO BIG!

*Are we done?* no

# Learning Goals for today' s class

**You can:**

- Define **factors**. Derive new factors from existing factors. Apply operations to factors, including assigning, summing out and multiplying factors.

- (*Minimally*) Carry out **variable elimination** by using factor representation and using the factor operations. Use techniques to simplify variable elimination.

# Next Class

**Variable Elimination**

- The algorithm
- An example

**Temporal models**

# Course Elements

- Work on Practice Exercises 6A and 6B
- Assignment 3 is due on Tue the $20^{th}$ !
- Assignment 4 will be available on Tue.