# Bottom Up: Soundness and Completeness

Computer Science cpsc322, Lecture 21

*(Textbook Chpt 5.2)*

Oct, 24, 2012

# Lecture Overview

- Recap
- Soundness of Bottom-up Proofs
- Completeness of Bottom-up Proofs

# (Propositional) Logic: Key ideas

Given a domain that can be represented with **n propositions** you have …… interpretations (possible worlds) $2^n$

If you do not know anything you can be in any of those

If you know that some **logical formulas** are true (your KB ……..). You know that you can be only in ……… interpretations
in which the KB is true (i.e. the models of KB)

It would be nice to know what else is true in all those…
models what is logically entailed

# PDCL syntax / semantics / proofs

Domain can be represented by three propositions: *p, q, r*

$$
\underline{KB} = \begin{cases} q. \leftarrow \\ r. \leftarrow \\ p \leftarrow \underline{q \wedge r}. \end{cases}
$$

Models?

Interpretations?

| $r$ | $q$ | $p$ |
|---|---|---|
| T | T | T |
| ~~T~~ | ~~T~~ | ~~F~~ |
| ~~T~~ | ~~F~~ | ~~T~~ |
| ~~T~~ | ~~F~~ | ~~F~~ |
| ~~F~~ | ~~T~~ | ~~T~~ |
| ~~F~~ | ~~T~~ | ~~F~~ |
| ~~F~~ | ~~F~~ | ~~T~~ |
| ~~F~~ | ~~F~~ | ~~F~~ |

What is logically entailed ? r, q, p

$G \subseteq C$

Prove $\underline{G = \boxed{(q \wedge p)}}$

$C = \{q, r, p\}$ $KB \vdash_{BU} G$

# PDCL syntax / semantics / proofs

$$KB = \begin{cases} \boxed{p \leftarrow q \wedge r.} \\ q. \end{cases}$$

Interpretations

| $r$ | $q$ | $p$ |
|---|---|---|
| T | T | T |
| ~~T~~ | ~~T~~ | ~~F~~ |
| ~~T~~ | ~~F~~ | ~~T~~ |
| ~~T~~ | ~~F~~ | ~~F~~ |
| F | T | T |
| F | T | F |
| ~~F~~ | ~~F~~ | ~~T~~ |
| ~~F~~ | ~~F~~ | ~~F~~ |

Models

What is logically entailed?

$C = \{q\}$

Prove $\underline{G = (q \wedge p)}$

$G \notin C$

$KB \not\vdash_{BU} G$

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- Completeness of Bottom-up Proofs

# Soundness of bottom-up proof procedure

**Generic Soundness of proof procedure**:

**If** G can be proved by the procedure (KB $\vdash$ G)
**then** G is logically entailed by the KB (KB $\models$ G)

**For Bottom-Up proof**

**if** $G \subseteq C$ at the end of procedure
**then** G is logically entailed by the KB

So BU is sound, if all the atoms in…C…

are logically entailed by the KB

# Soundness of bottom-up proof procedure

**Suppose this is not the case.**

1. Let $h$ be the first atom added to $C$ that is not entailed by KB (i.e., that's not true in every model of $KB$)

2. Suppose $h$ isn't true in model $M$ of $KB$.

3. Since $h$ was added to C, there must be a clause in KB of form: $h \leftarrow b_1 \wedge \ldots \wedge b_m$

4. Each $b_i$ is true in $M$ *(because of 1.)*. $h$ is false in $M$. So…… the clause is false in M

5. Therefore M is not a model

6. Contradiction! thus no such $h$ exists.

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# Completeness of Bottom Up

**Generic Completeness of proof procedure**:

**If** G is logically entailed by the KB ($KB \models G$)

**then** G can be proved by the procedure ($KB \vdash G$)

$G \subseteq C$

**Sketch of our proof:**

1. Suppose $KB \models G$. Then G is true in all models of $KB$.
2. Thus $G$ is true in any particular model of KB
3. We will define a model so that if G is true in that model, G is proved by the bottom up algorithm. $G \subseteq C$
4. Thus $KB \vdash_{BU} G$.

# Let's work on step 3

**3. We will define a model so that if G is true in that model, G is proved by the bottom up algorithm.**

$G \subseteq C$

3.1 We will define an interpretation $\mathcal{I}$ so that if G is true in $\mathcal{I}$, G is proved by the bottom up algorithm.

$G \subseteq C$

3.2 We will then show that ……… $\mathcal{I}$ is a model

# Let's work on step 3.1

3.1 Define interpretation $\mathcal{I}$ so that if G is true in $\mathcal{I}$, Then $G \subseteq C$.

> Let $I$ be the interpretation in which every element of $C$ is true and every other atom is false .

$a \leftarrow e \wedge g.$
$b \leftarrow f \wedge g.$
$c \leftarrow e.$
$f \leftarrow c \wedge e.$
$e.$
$d.$

| F | F | T | T | T | T | F |
|---|---|---|---|---|---|---|
| { $a$, | $b$, | $c$, | $d$, | $e$, | $f$, | $g$ } |

C

$\{ \}$
$\{ e \}$
$\{ e, d \}$
$\{ e, d, c \}$
$\{ e, d, c, f \}$

# Let's work on step 3.2

**Claim:** $I$ is a model of *KB* *(we'll call it the* ***minimal model****)*.

**Proof:** **Assume** that $I$ is not a model of *KB*.

- **Then** there must exist some clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in *KB* (having zero or more $b_i$'s) which is false in $I$.
- The only way this can occur is if $b_1 \ldots b_m$ are true in $I$ (i.e., are in C) and $h$ is false in $I$ (i.e., is not in C)
- **But** if each $b_i$ belonged to $C$, Bottom Up would have added $h$ to $C$ as well.
- So, there can be no clause in the KB that is false in interpretation $I$ (which implies the claim :-)

# Completeness of Bottom Up
## (proof summary)

$\boxed{\text{If } KB \models G \text{ then } KB \vdash_{BU} G}$

- Suppose $KB \models G$.
- Then $G$ is true in all the models
- Thus $G$ is true in the minimal model
- Thus $G \subseteq C$
- Thus $G$ is proved by…. BU
- ~~Thus~~ $KB \vdash_{BU} G$

i.e.

$\boxed{KB \models_{BU} G}$ $\xrightarrow{\text{Completeness}}$ $\boxed{KB \vdash_{BU} G}$

$\boxed{KB \models_{BU} G}$ $\xleftarrow{\text{Soundness}}$ $\boxed{KB \vdash_{BU} G}$

RELATION BETWEEN SOUNDNESS & COMPLETENESS

# An exercise for you $BU\ C = \{d, e, c, f\}$

# Let's consider these two alternative proof procedures for PDCL

**KB**

$a \leftarrow e \wedge g.$

$b \leftarrow f \wedge g.$

$c \leftarrow e.$

$f \leftarrow c$

$e.$

$d.$

A. $C_A$ = {All clauses in KB with empty bodies} $= \{e, d\}$

B. $C_B$ = {All atoms in the knowledge base} $\{e\ d\ f\ c\ \overset{*}{g}\ \overset{*}{a}\}$

Both A and B are sound and complete

Both A and B are neither sound nor complete

A is sound only and B is complete only $\Leftarrow$

A is complete only and B is sound only

# An exercise for you

BU $C = \{d, e, c, f\}$

## Let's consider these two alternative proof procedures for PDCL

**KB**

$a \leftarrow e \wedge g.$

$b \leftarrow f \wedge g.$

$c \leftarrow e.$

$f \leftarrow c$

$e.$

$d.$

A. $C_A$ = {All clauses in KB with empty bodies} $= \{e, d\}$

B. $C_B$ = {All atoms in the knowledge base}

$\{e\ d\ f\ c\ g^*\ a^*\}$

**A is sound only and B is complete only**

# Learning Goals for today's class

**You can:**

- Prove that BU proof procedure is sound
- Prove that BU proof procedure is complete

# Next class

(still section 5.2)

- Using PDC Logic to model the electrical domain
- Reasoning in the electrical domain
- Top-down proof procedure (as Search!)

**Assignment-1 marked**