Arc Consistency and Domain Splitting in CSPs

CPSC 322 – CSP 3

Textbook Poole and Mackworth: § 4.5 and 4.6

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Lecture Overview

- Solving Constraint Satisfaction Problems (CSPs)
  - Recap: Generate & Test
  - Recap: Graph search
- Arc consistency
  - GAC algorithm
  - Complexity analysis
  - Domain splitting

Constraint Satisfaction Problems (CSPs): Definition

Definition:
A constraint satisfaction problem (CSP) consists of:
- a set of variables $\mathcal{V}$
- a domain dom(V) for each variable $V \in \mathcal{V}$
- a set of constraints $\mathcal{C}$

Definition:
A possible world of a CSP is an assignment of values to all of its variables.

Definition:
A model of a CSP is a possible world that satisfies all constraints.

An example CSP:
- $\mathcal{V} = \{V_1, V_2\}$
  - $dom(V_1) = \{1,2,3\}$
  - $dom(V_2) = \{1,2\}$
- $\mathcal{C} = \{C_1, C_2, C_3\}$
  - C_1: $V_2 \neq 2$
  - C_2: $V_1 + V_2 < 5$
  - C_3: $V_1 > V_2$

Possible worlds for this CSP:

$\{V_1=1, V_2=1\}$
$\{V_1=1, V_2=2\}$
$\{V_1=2, V_2=1\}$ (one model)
$\{V_1=2, V_2=2\}$
$\{V_1=3, V_2=1\}$ (another model)
$\{V_1=3, V_2=2\}$

Generate and Test (G&T) Algorithms

- Generate and Test:
  - Generate possible worlds one at a time.
  - Test constraints for each one.

Example: 3 variables A,B,C

```
For a in dom(A)
    For b in dom(B)
        For c in dom(C)
            if {A=a, B=b, C=c} satisfies all constraints
                return {A=a, B=b, C=c}
fail
```

- Simple, but slow:
  - k variables, each domain size d, c constraints: $O(cd^k)$

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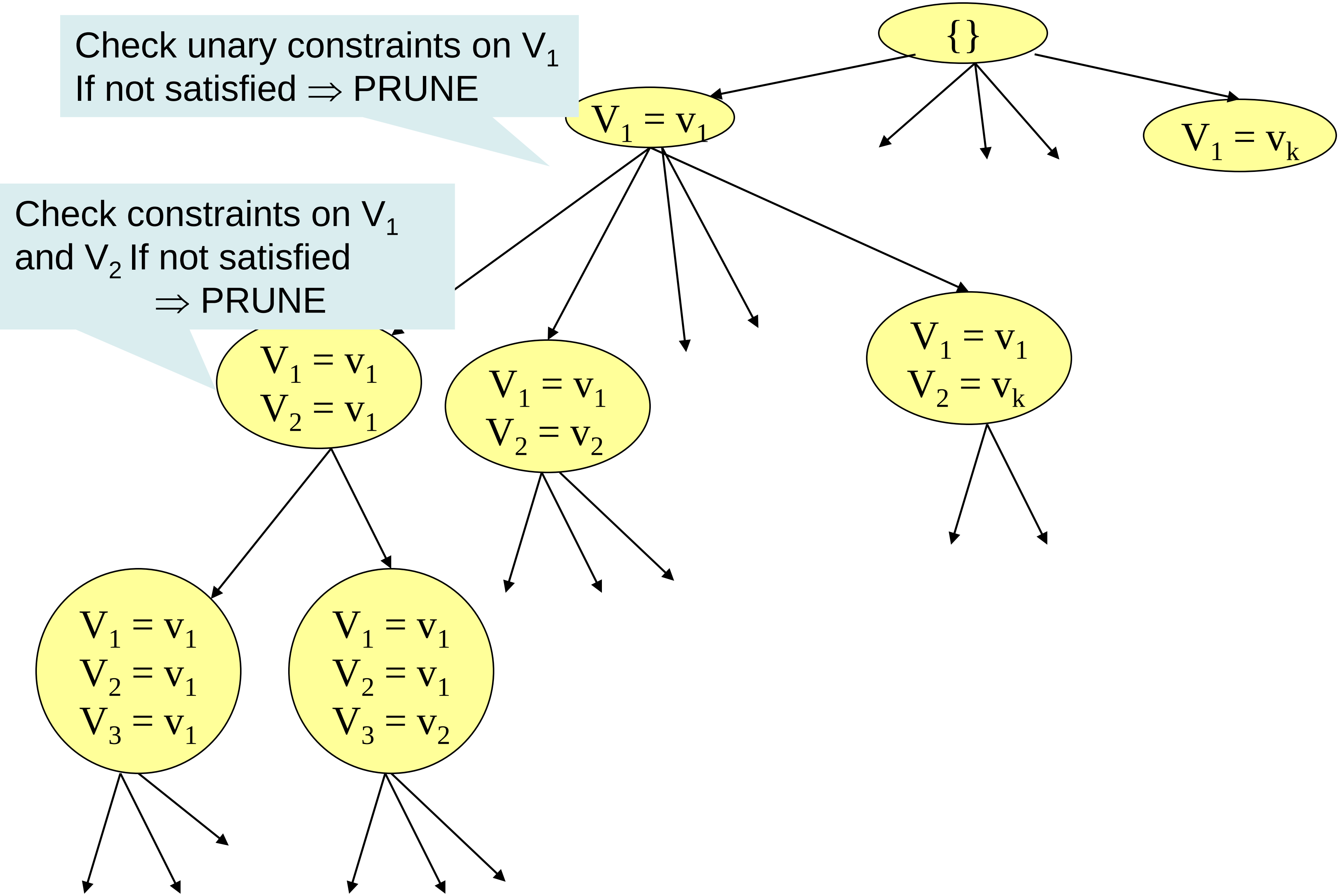
Backtracking algorithms

- Explore search space via DFS but evaluate each constraint as soon as all its variables are bound.
- Any partial assignment that doesn't satisfy the constraint can be pruned.
- Example:
  - 3 variables A, B,C, each with domain {1,2,3,4}
  - {A = 1, B = 1} is inconsistent with constraint $A \neq B$ regardless of the value of the other variables

    $\Rightarrow$ Fail. Prune!

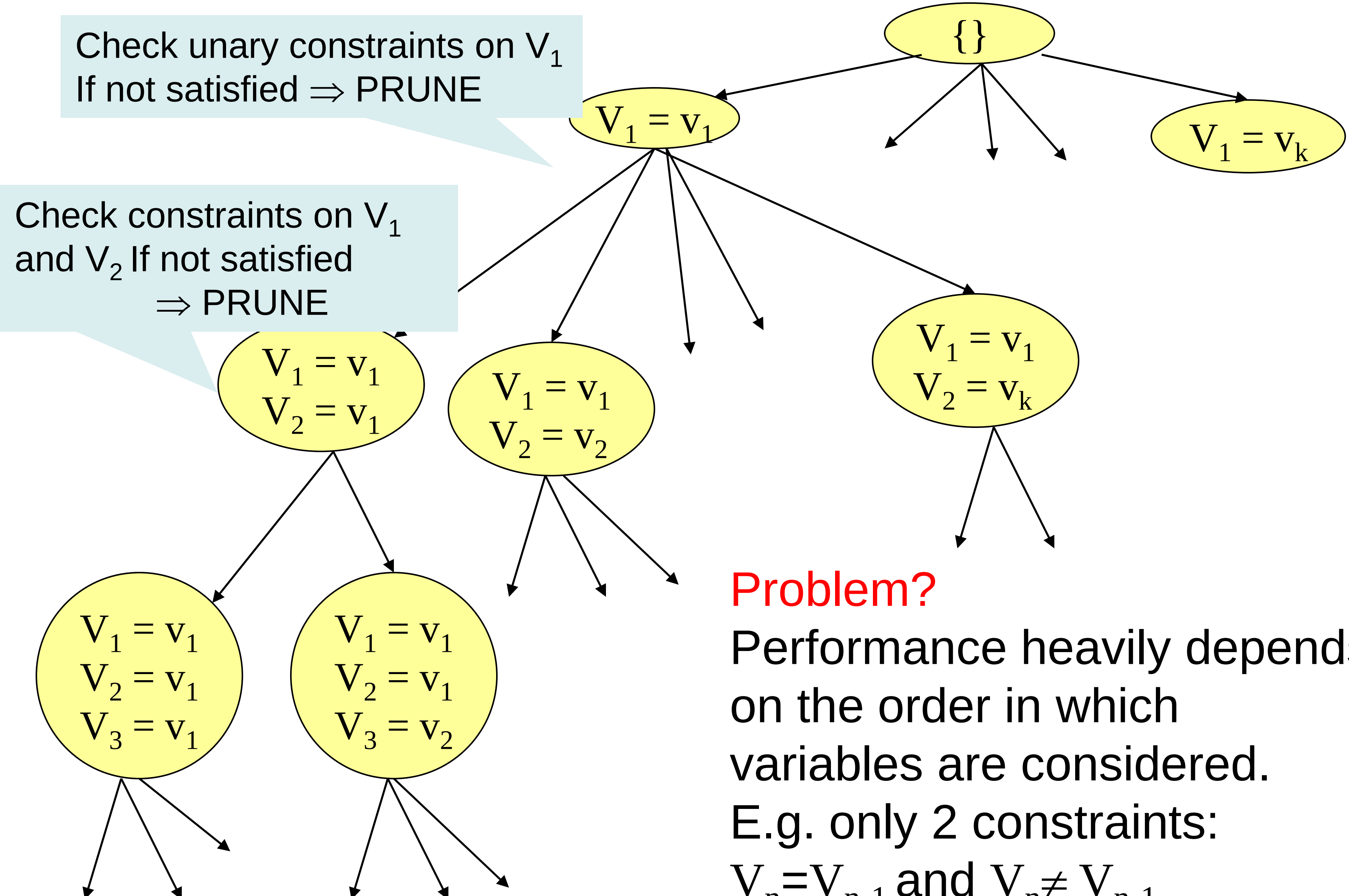
CSP as Graph Searching



Standard Search vs. Specific R&R systems

- Constraint Satisfaction (Problems):
  - State: assignments of values to a subset of the variables
  - Successor function: assign values to a ‘free’ variable
  - Goal test: all variables assigned a value and all constraints satisfied?
  - Solution: possible world that satisfies the constraints
  - Heuristic function: none (all solutions at the same distance from start)
- Planning :
  - State
  - Successor function
  - Goal test
  - Solution
  - Heuristic function
- Inference
  - State
  - Successor function
  - Goal test
  - Solution
  - Heuristic function

CSP as Graph Searching



Problem?

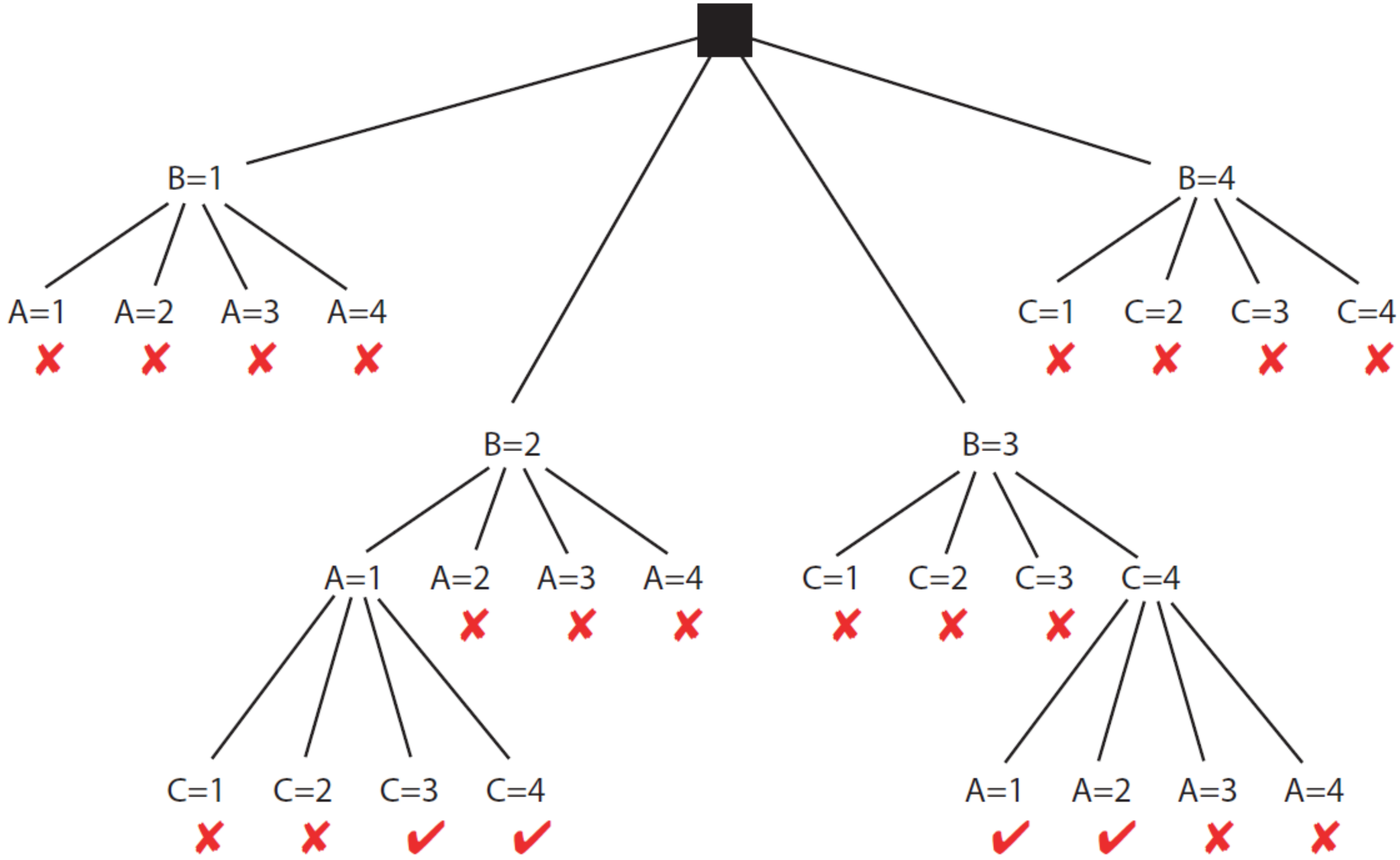
Performance heavily depends on the order in which variables are considered. E.g. only 2 constraints: $V_n = V_{n-1}$ and $V_n \neq V_{n-1}$

CSP as a Search Problem: another formulation

- States: partial assignment of values to variables
- Start state: empty assignment
- Successor function: states with the next variable assigned
  - Assign any previously unassigned variable
  - A state assigns values to some subset of variables:
    - E.g. $\{V_7 = v_{1,} V_2 = v_1, V_{15} = v_1\}$
    - Neighbors of node $\{V_7 = v_{1,} V_2 = v_1, V_{15} = v_1\}$:
      nodes $\{V_7 = v_{1,} V_2 = v_1, V_{15} = v_1, V_x = y\}$
      for some variable $V_x \in \mathcal{V} \setminus \{V_7, V_2, V_{15}\}$ and all values $y \in dom(V_x)$
- Goal state: complete assignments of values to variables that satisfy all constraints
  - That is, models
- Solution: assignment (the path doesn't matter)

CSP as Graph Searching

- 3 Variables: A,B,C. All with domains = {1,2,3,4}
- Constraints: A<B, B<C



Selecting variables in a smart way

- Backtracking relies on one or more heuristics to select which variables to consider next.
  - E.g. variable involved in the largest number of constraints: “If you are going to fail on this branch, fail early!”
  - Can also be smart about which values to consider first
- This is a different use of the word ‘heuristic’!
  - Still true in this context
    - Can be computed cheaply during the search
    - Provides guidance to the search algorithm
  - But not true anymore in this context
    - ‘Estimate of the distance to the goal’

- Both meanings are used frequently in the AI literature.
- ‘heuristic’ means ‘serves to discover’: goal-oriented.
- Does not mean ‘unreliable’!

Learning Goals for solving CSPs so far

- Verify whether a possible world satisfies a set of constraints
  i.e. whether it is a model - a solution.

- Implement the Generate-and-Test Algorithm.
  Explain its disadvantages.

- Solve a CSP by search (specify neighbors, states, start state, goal state). Compare strategies for CSP search. Implement pruning for DFS search in a CSP.

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Can we do better than Search?

Key idea

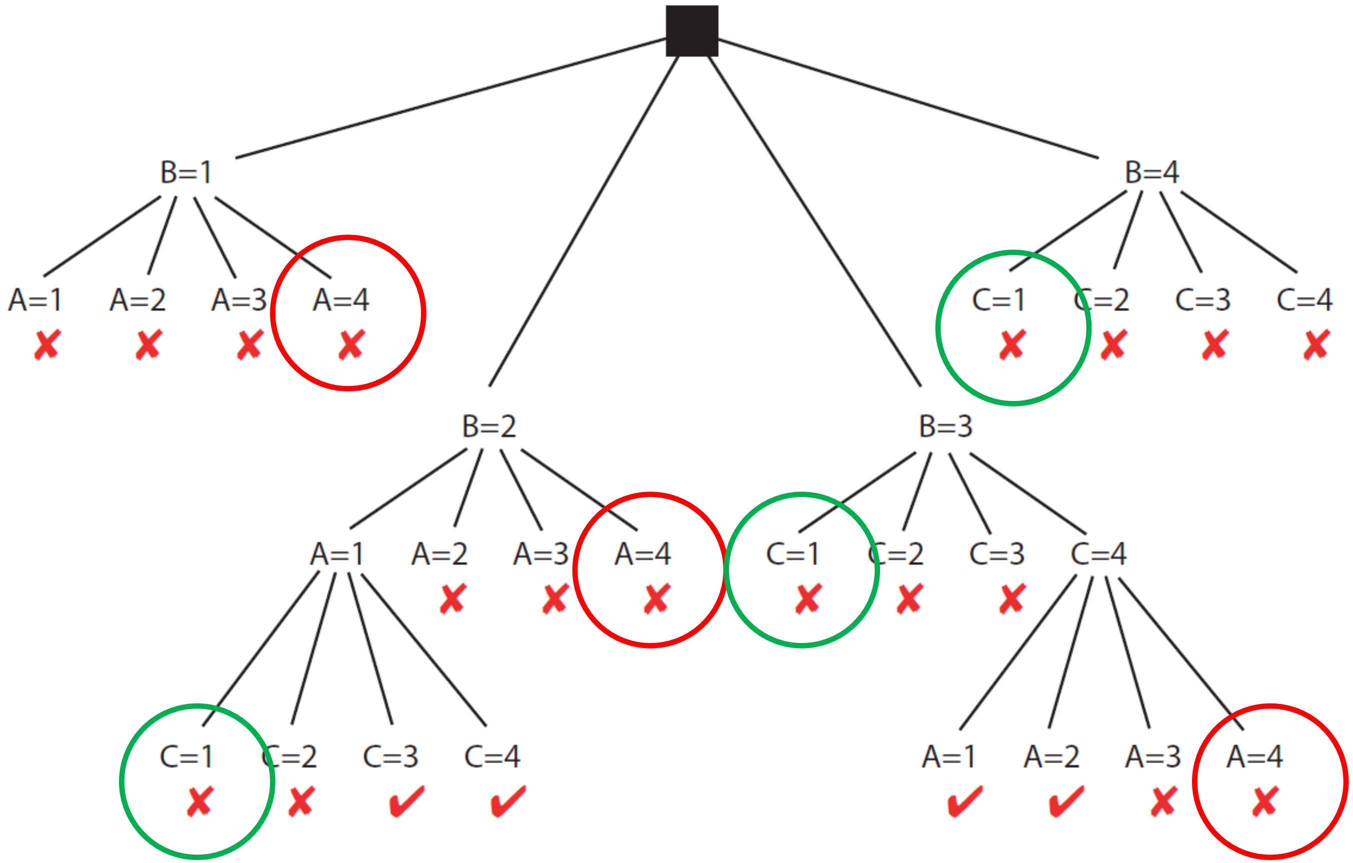
- prune the domains as much as possible before searching for a solution.

Def.: A variable is domain consistent if no value of its domain is ruled impossible by any unary constraints.

- Example: $dom(V_2) = \{1, 2, 3, 4\}$. $V_2 \neq 2$
- Variable V_2 is not domain consistent.
  - It is domain consistent once we remove 2 from its domain.
- Trivial for unary constraints. Trickier for k-ary ones.

Graph Searching Repeats Work

- 3 Variables: A,B,C. All with domains = {1,2,3,4}
- Constraints: A<B, B<C
- A ≠ 4 is rediscovered 3 times. So is C ≠ 1
  - Solution: remove values from A's domain and C's, once and for all

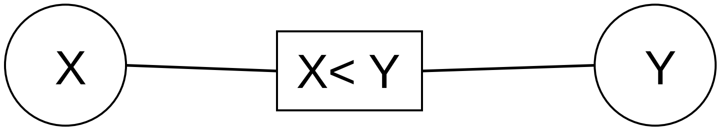


Constraint network: definition

Def. A constraint network is defined by a graph, with

- one node for every variable (drawn as circle)
- one node for every constraint (drawn as rectangle)
- undirected edges running between variable nodes and constraint nodes whenever a given variable is involved in a given constraint.

- Example:
  - Two variables X and Y
  - One constraint: X<Y



Constraint network: definition

Def. A constraint network is defined by a graph, with

- one node for every variable (drawn as circle)
- one node for every constraint (drawn as rectangle)
- Edges/arcs running between variable nodes and constraint nodes whenever a given variable is involved in a given constraint.

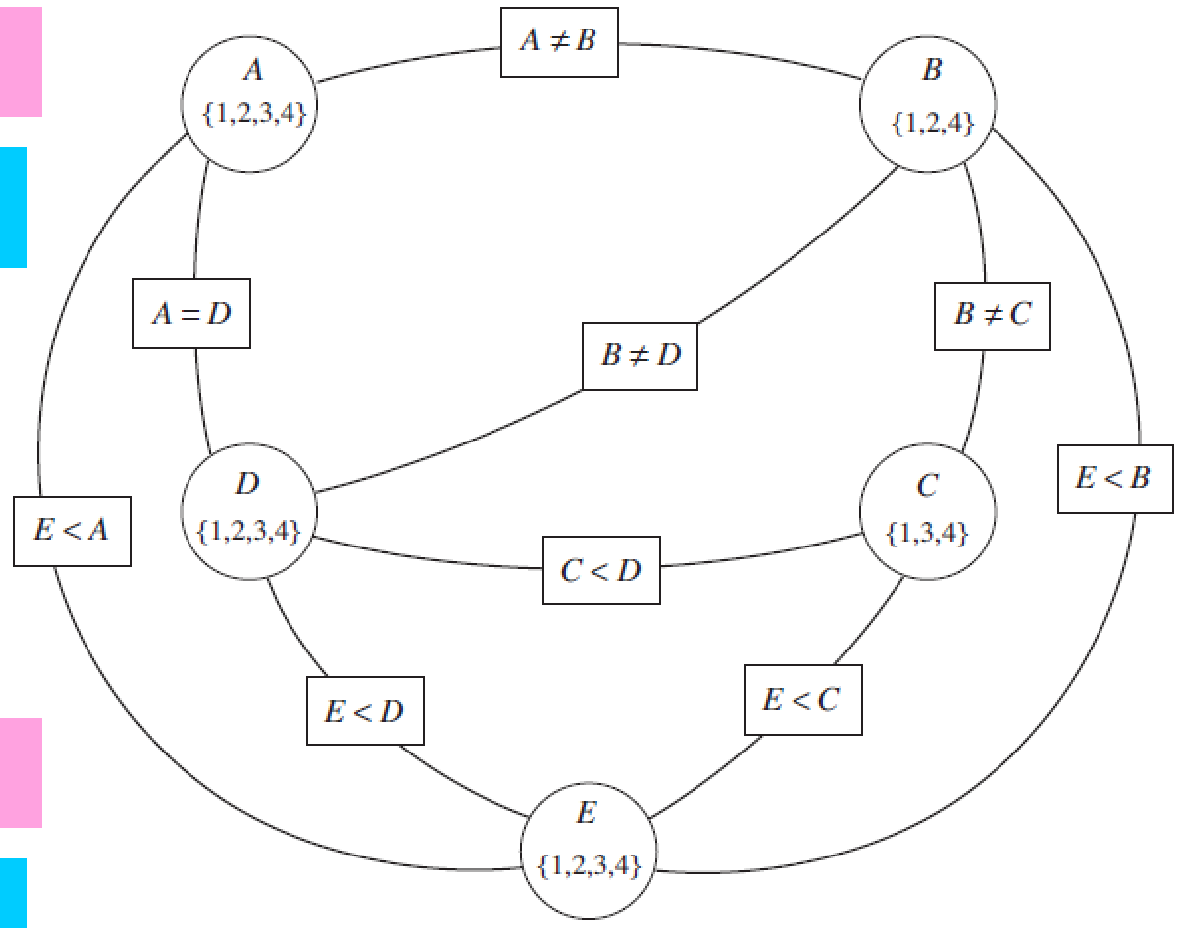
- Whiteboard example: 3 Variables A,B,C
  - 3 Constraints: A<B, B<C, A+3=C
  - 6 edges/arcs in the constraint network:
    - ⟨A,A<B⟩ , ⟨B,A<B⟩
    - ⟨B,B<C⟩ , ⟨C,B<C⟩
    - ⟨A, A+3=C⟩ , ⟨C,A+3=C⟩

A more complicated example

- How many variables are there in this constraint network?

5	6
9	14

  – Variables are drawn as circles



- How many constraints are there?

5	6
9	14

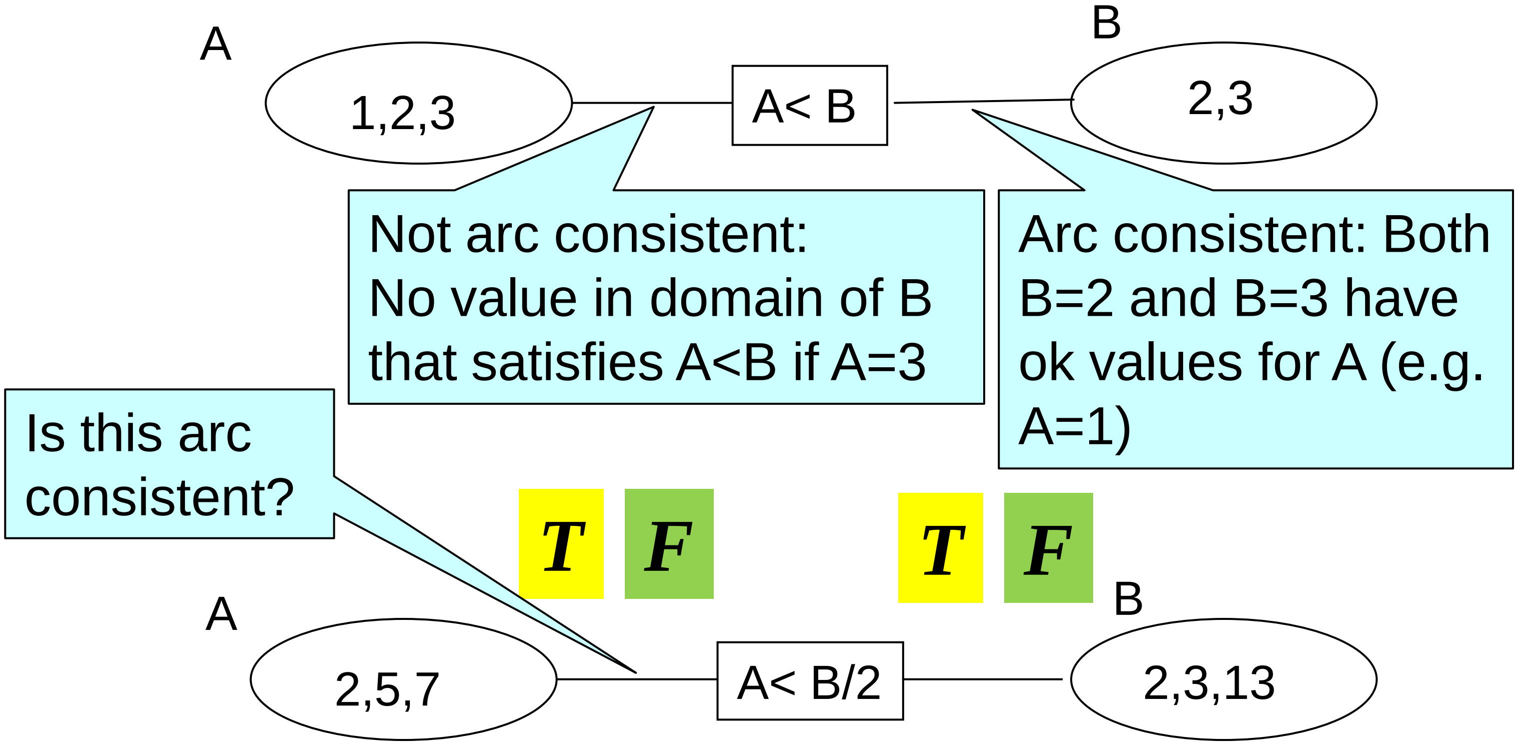
  – Constraints are drawn as rectangles

Arc Consistency

Definition:

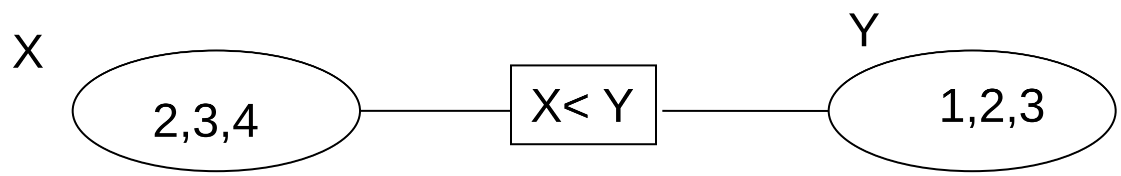
An arc <x, r(x,y)> is arc consistent if for each value x in dom(X) there is some value y in dom(Y) such that r(x,y) is satisfied.

A network is arc consistent if all its arcs are arc consistent.



How can we enforce Arc Consistency?

- If an arc <*X, r(X,Y)*> is not arc consistent
  - Delete all values *x* in *dom(X)* for which there is no corresponding value in *dom(Y)*
  - This deletion makes the arc <*X, r(X,Y)*> arc consistent.
  - This removal can never rule out any models/solutions
    - Why?



Run this example: http://cs.ubc.ca/~mack/CS322/AIspace/simple-network.xml

in AIspace (Save to a local file and open file.)

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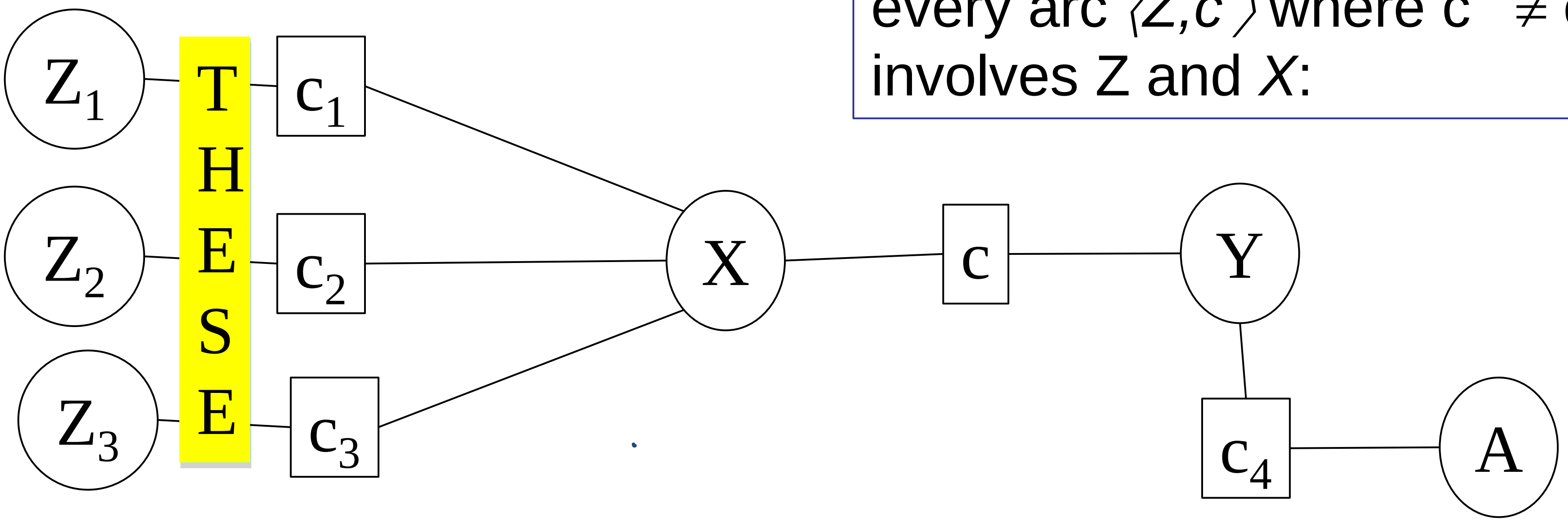
Arc Consistency Algorithm: high level strategy

- Consider the arcs in turn, making each arc consistent
- Reconsider arcs that could be made inconsistent again by this pruning of the domains
- Eventually reach a ‘fixed point’: all arcs consistent
- Run ‘simple problem 1’ in AIspace for an example:

Which arcs need to be reconsidered?

- When we reduce the domain of a variable X to make an arc ⟨X,c⟩ arc consistent, which arcs do we need to reconsider?

every arc ⟨*Z,c'*⟩ where c' ≠ c involves Z and *X*:



- You do not need to reconsider other arcs
  - If arc ⟨Y,c⟩ was arc consistent before, it will still be arc consistent
  - If an arc ⟨X,c'⟩ was arc consistent before, it will still be arc consistent
  - Nothing changes for arcs of constraints not involving X

Which arcs need to be reconsidered?

- Consider the arcs in turn, making each arc consistent
- Reconsider arcs that could be made inconsistent again by this pruning

- DO trace on 'simple problem 1' and on
  'scheduling problem 1', trying to predict

- which arcs are not consistent and
- which arcs need to be reconsidered after each removal

in AIspace

Arc consistency algorithm (for binary constraints)

```
Procedure GAC(V,dom,C)
    Inputs
        V: a set of variables
        dom: a function such that dom(X) is the domain of variable X
        C: set of constraints to be satisfied
    Output
        arc-consistent domains for each variable
    Local
        D_X is a set of values for each variable X
        TDA is a set of arcs

    for each variable X do
        D_X ←dom(X)
    TDA ←{⟨X,c⟩| X ∈ V, c ∈ C  and X ∈ scope(c)}

    while (TDA ≠ {})
        select ⟨X,c⟩ ∈TDA
        TDA ←TDA \ {⟨X,c⟩}
        ND_X ←{x| x ∈ D_X and ∃ y ∈ D_Y s.t. (x, y) satisfies c}
        if (ND_X ≠ D_X) then
            TDA ←TDA ∪ { ⟨Z,c'⟩ | X ∈ scope(c'), c' ≠ c, Z ∈ scope(c') \ {X} }
            D_X ←ND_X

    return {D_X| X is a variable}
```

TDA: ToDoArcs, blue arcs in AIspace

Scope of constraint c is the set of variables involved in that constraint

ND_X: values x for X for which there a value for y supporting x

X's domain changed: ⇒ arcs (Z,c') for variables Z sharing a constraint c' with X could become inconsistent

Arc Consistency Algorithm: Interpreting Outcomes

- Three possible outcomes
  (when all arcs are arc consistent):
  - Each domain has a single value, e.g.
    http://www.cs.ubc.ca/~mack/CS322/AIspace/simple-network.xml
    (Download the file and load it as a local file in AIspace)
    - We have a (unique) solution.
  - At least one domain is empty, e.g.
    http://www.cs.ubc.ca/~mack/CS322/AIspace/simple-infeasible.xml
    - No solution! All values are ruled out for this variable.
  - Some domains have more than one value, e.g.
    built-in example "simple problem 2"
    - There may be a solution, multiple ones, or none
    - Need to solve this new CSP (usually simpler) problem:
      same constraints, domains have been reduced

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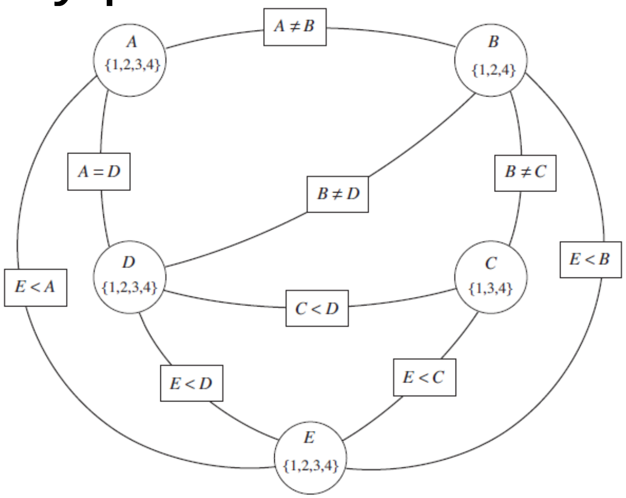
Arc Consistency Algorithm: Complexity

- Worst-case complexity of arc consistency procedure on a problem with N variables
  - let ***d*** be the max size of a variable domain
  - let ***c*** be the number of constraints



- How often will we prune the domain of variable V? O(d) times
- How many arcs will be put on the ToDoArc list when pruning domain of variable V?
  - O(degree of variable V)
  - In total, across all variables: sum of degrees of all variables = 2*number of constraints, i.e. 2*c
- Together: we will only put O(dc) arcs on the ToDoArc list
- Checking consistency is $O(d^2)$ for each of them

- Overall complexity: $O(cd^3)$
- Compare to $O(d^N)$ of DFS!! Arc consistency is MUCH faster

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Can we have an arc consistent network with non-empty domains that has no solution?

YES

NO

- Example: vars A, B, C with domain {1, 2} and constraints $A \neq B$, $B \neq C$, $A \neq C$
- Or see AIspace CSP applet Simple Problem 2

Domain splitting (or case analysis)

- Arc consistency ends: Some domains have more than one value $\rightarrow$ may or may not have a solution
  A. Apply Depth-First Search with Pruning or
  B. Split the problem in a number of disjoint cases:

     CSP with dom(X) = $\{x_1, x_2, x_3, x_4\}$ becomes

     CSP_1 with dom(X) = $\{x_1, x_2\}$ and
     CSP_2 with dom(X) = $\{x_3, x_4\}$

- Solution to CSP is the union of solutions to CSP_i

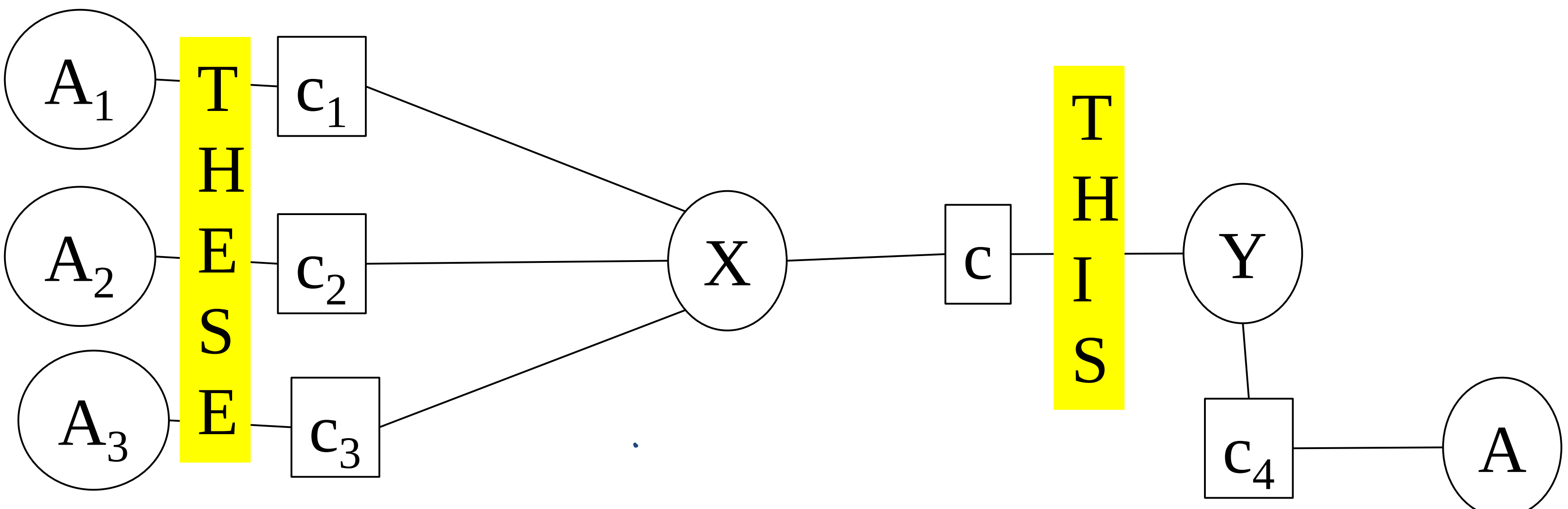
Domain splitting

- Each smaller CSP is easier to solve
  - Arc consistency might already solve it
- For each subCSP, which arcs have to be on the ToDoArcs list when we get the subCSP by splitting the domain of X?

arcs <Z, r(Z,X)>

arcs <Z, r(Z,X)> and <X, r(Z,X)>

All arcs



Domain splitting in action

- Trace it on “simple problem 2”

Searching by domain splitting

CSP, apply AC

If domains with multiple values

Split on one

CSP$_1$, apply AC

If domains with multiple values
Split on one

CSP$_2$, apply AC

If domains with multiple values…..Split on one

How many CSPs do we need to keep around at a time?
With depth m and 2 children at each split: O(2m). It’s a DFS.

Learning Goals for today's class

- Define/read/write/trace/debug the arc consistency algorithm. Compute its complexity and assess its possible outcomes
- Define/read/write/trace/debug domain splitting and its integration with arc consistency

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- Coming up: local search, Section 4.8