Constraint Satisfaction Problems (CSPs)

CPSC 322 – CSP 1

Poole & Mackworth textbook: Sections §4.0-4.2

Lecturer: Alan Mackworth

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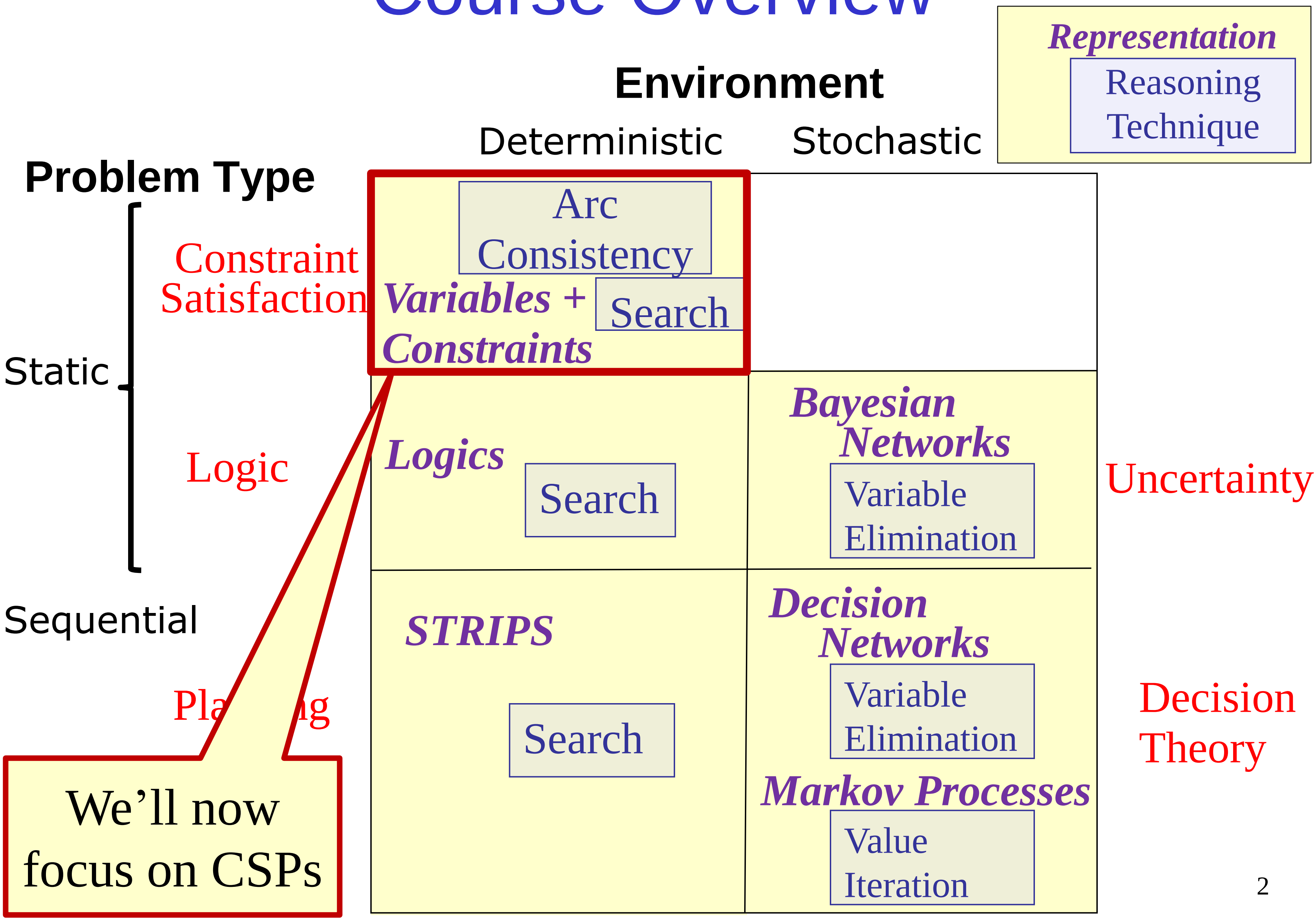
Course Overview

Course Module



Lecture Overview

- Representations: States vs. Features (Variables)
- Variables and Possible Worlds
- Constraints
- Constraint Satisfaction Problems (CSPs)

Main Representational Dimensions (Lecture 2)

Domains can be classified by the following dimensions:

- 1. Uncertainty
  - Deterministic vs. stochastic domains
- 2. How many actions does the agent need to perform?
  - Static vs. sequential domains

An important design choice is:

- 3. Representation scheme
  - Explicit states vs. features (vs. relations)

Explicit State vs. Features (Lecture 2)

How do we model the environment?

- You can enumerate the possible states of the world
- A state can be described in terms of features
  - Assignment to (one or more) variables
  - Often the more natural description
  - 30 binary features can represent 2^{30} =1,073,741,824 states

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Variables/Features and Possible Worlds

- Variable: a synonym for feature
  - We denote variables using capital letters
  - Each variable V has a domain dom(V) of possible values
- Variables can be of several main kinds:
  - Boolean: |dom(V)| = 2 e.g. {true, false}
  - Finite: |dom(V)| is finite e.g. {1, 2, 3, 4, 5}
  - Infinite but discrete: the domain is countably infinite e.g. the positive integers, {1, 2, 3, …}
  - Continuous: e.g., real numbers from 0 to 1, [0,1]
- Possible world
  - Complete assignment of values to each variable
  - In contrast, states also include partial assignments

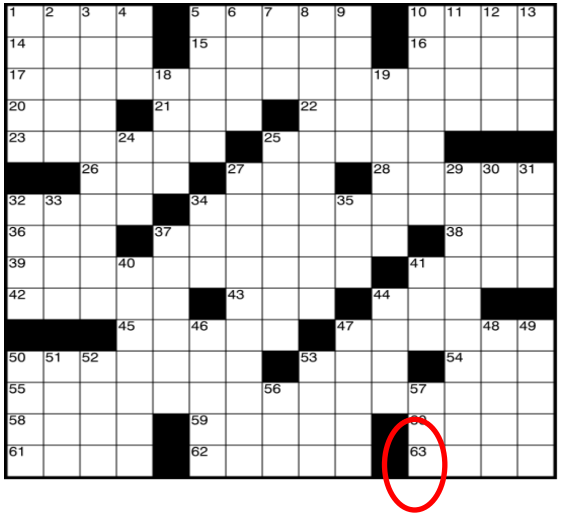
Examples: variables, domains, possible worlds

- Crossword Puzzle:
  - variables are words that have to be filled in
  - domains are English words of correct length
  - possible worlds: all ways of assigning words

- Crossword 2:
  - variables are cells (individual squares)
  - domains are letters of the alphabet
  - possible worlds: all ways of assigning letters to cells

How many possible worlds?

- Crossword Puzzle:
  - variables are words that have to be filled in
  - domains are English words of correct length
  - possible worlds: all ways of assigning words



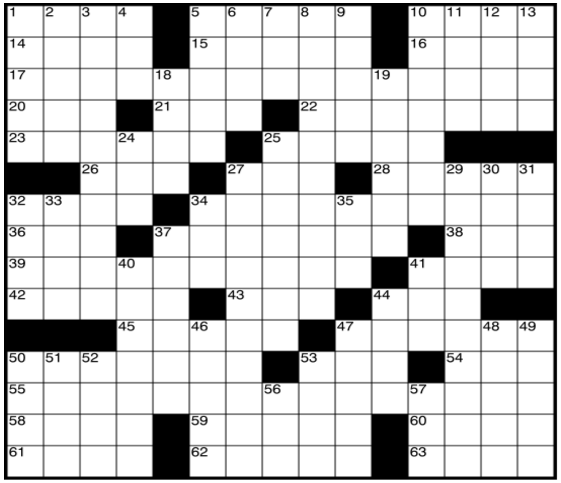
- Number of English words? Let's say 150,000
  - Of the right length? Assume for simplicity: 15,000 for each word
- Number of words to be filled in? 63

- How many possible worlds? (assume any combination is ok)

$15000*63$ | 15000^{63} | 63^{15000}

How many possible worlds?

- Crossword 2:
  - variables are cells (individual squares)
  - domains are letters of the alphabet
  - possible worlds: all ways of assigning letters to cells



- Number of empty cells? 15*15 – 32 = 193
- Number of letters in the alphabet? 26

- How many possible worlds? (assume any combination is ok)

  $193*26$ 193^{26} 26^{193}

- In general: $(\text{domain size})^{\#\text{variables}}$ (only an upper bound)

Examples: variables, domains, possible worlds

Sudoku rules are extremely easy: Fill all empty squares so that the numbers 1 to 9 appear once in each row, column and 3x3 box.

Sudoku Puzzle

	9	3	6	2	8	1	4	
	6						5	
	3			1			9	
	5		8		2		7	
	4			7			6	
	8						3	
	1	7	5	9	3	4	2	

Sudoku Solution

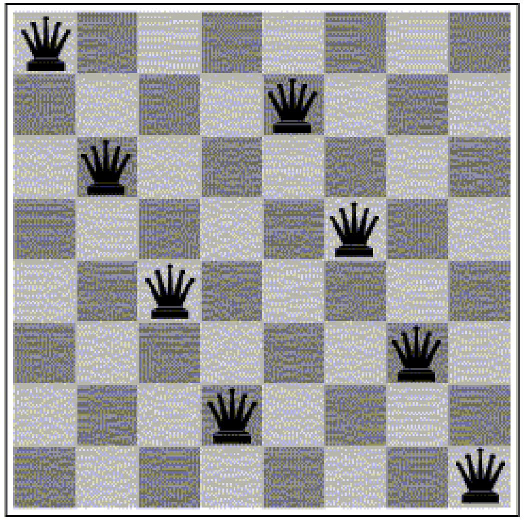
2	7	1	9	5	4	6	8	3
5	9	3	6	2	8	1	4	7
4	6	8	1	3	7	2	5	9
7	3	6	4	1	5	8	9	2
1	5	9	8	6	2	3	7	4
8	4	2	3	7	9	5	6	1
9	8	5	2	4	1	7	3	6
6	1	7	5	9	3	4	2	8
3	2	4	7	8	6	9	1	5

- Sudoku
  - variables are cells
  - domains are numbers between 1 and 9
  - possible worlds: all ways of assigning numbers to cells

Examples: variables, domains, possible worlds

- Scheduling Problem:
  - variables are different tasks that need to be scheduled (e.g., course in a university; job in a machine shop)
  - domains are the different combinations of times and locations for each task (e.g., time/room for course; time/machine for job)
  - possible worlds: time/location assignments for each task

- n-Queens problem
  - variable: location of a queen on a chess board
    - there are n of them in total, hence the name
  - domains: grid coordinates
  - possible worlds: locations of all queens



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- Representations: States vs. Features (Variables)
- Variables and Possible Worlds
- ➡ Constraints
- Constraint Satisfaction Problems (CSPs)

Constraints 1

- Constraints are restrictions on the values that one or more variables can take
  - Unary constraint: restriction involving a single variable
    - of course, we could also achieve the same thing by using a smaller domain in the first place
  - k-ary constraint: restriction involving k different variables
    - We will mostly deal with binary constraints (k=2 variables)
  - Constraints can be specified by
    1. listing all combinations of valid domain values for the variables participating in the constraint (or listing all invalid combinations)
    2. giving a function (predicate) that returns true when given values for each variable which satisfy the constraint

Constraints 2

- A possible world satisfies a set of constraints
  - If, in that possible world, the values for the variables involved in each constraint are consistent with that constraint:

    either:

    Elements of the list of valid domain values

    or:

    Function returns true for those values

Examples: variables, domains, constraints

- Crossword Puzzle:
  - variables are words that have to be filled in
  - domains are English words of correct length
  - (binary) constraints: two words have the same character where they intersect

- Crossword 2:
  - variables are cells (individual squares)
  - domains are letters of the alphabet
  - (k-ary) constraints: sequences of letters form valid English words

Examples: variables, domains, constraints

Sudoku rules are extremely easy: Fill all empty squares so that the numbers 1 to 9 appear once in each row, column and 3x3 box.

Sudoku Puzzle

	9	3	6	2	8	1	4	
	6						5	
	3			1			9	
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	4			7			6	
	8						3	
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Sudoku Solution

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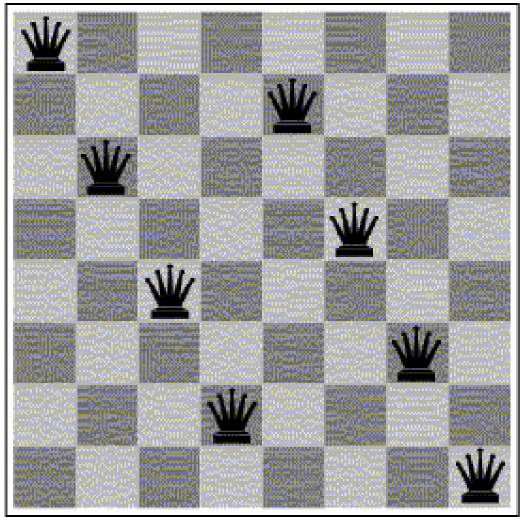
9	8	5	2	4	1	7	3	6
6	1	7	5	9	3	4	2	8
3	2	4	7	8	6	9	1	5

- Sudoku
  - variables are cells
  - domains are numbers between 1 and 9
  - constraints: rows, columns, boxes contain all different numbers: they are each a permutation of {1, 2, 3, 4, 5, 6, 7, 8, 9}

Examples: variables, domains, constraints

- Scheduling Problem:
  - variables are different tasks that need to be scheduled (e.g., course in a university; job in a machine shop)
  - domains are the different combinations of times and locations for each task (e.g., time/room for course; time/machine for job)
  - constraints: tasks can't be scheduled in the same location at the same time; certain tasks can't be scheduled in different locations at the same time; some tasks must come earlier than others; etc.

- n-Queens problem
  - variable: location of a queen on a chess board
    - there are n of them in total, hence the name
  - domains: grid coordinates
  - constraints: no queen can attack another



Lecture Overview

- Representations: States vs. Features (Variables)
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➡ Constraint Satisfaction Problems (CSPs)

Constraint Satisfaction Problems: Definition

Definition:
A constraint satisfaction problem (CSP) consists of:

- a set of variables
- a domain for each variable
- a set of constraints

Definition:
A model of a CSP is an assignment of values to all of its variables that satisfies all of its constraints.

Constraint Satisfaction Problems: Variants

- We may want to solve the following problems with a CSP:
  - determine whether or not a model exists
  - find a model
  - find all of the models
  - count the number of models
  - find the best model, given some measure of model quality
    - this is now an optimization problem
  - determine whether some property of the variables holds in all models

Constraint Satisfaction Problems: Game Plan

- Even the simplest problem of determining whether or not a model exists in a general CSP with finite domains is NP-hard
  - There is no known algorithm with worst case polynomial runtime
  - We can't hope to find an algorithm that is efficient for all CSPs

- However, we can try to:
  - find consistency algorithms that reduce the size of the search space
  - identify special cases for which algorithms are efficient (polynomial)
  - work on approximation algorithms that can find good solutions quickly, even though they may offer no theoretical guarantees
  - find algorithms that are fast on typical cases

Constraint Satisfaction Problems (CSPs): Definition

Definition:

A constraint satisfaction problem (CSP) consists of:

- a set of variables $\mathcal{V}$
- a domain dom(V) for each variable $V \in \mathcal{V}$
- a set of constraints $\mathcal{C}$

Simple example:

- $\mathcal{V} = \{V_1\}$
  - $dom(V_1) = \{1,2,3,4\}$
- $\mathcal{C} = \{C_1, C_2\}$
  - $C_1: V_1 \neq 2$
  - $C_2: V_1 > 1$

Another example:

- $\mathcal{V} = \{V_1, V_2\}$
  - $dom(V_1) = \{1,2,3\}$
  - $dom(V_2) = \{1,2\}$
- $\mathcal{C} = \{C_1, C_2, C_3\}$
  - $C_1: V_2 \neq 2$
  - $C_2: V_1 + V_2 < 5$
  - $C_3: V_1 > V_2$

Constraint Satisfaction Problems (CSPs): Definition

Definition:
A constraint satisfaction problem (CSP) consists of:

- a set of variables $\mathcal{V}$
- a domain dom(V) for each variable V $\in \mathcal{V}$
- a set of constraints $\mathcal{C}$

Definition:
A model of a CSP is an assignment of values to all of its variables that satisfies all of its constraints.

Simple example:

- $\mathcal{V} = \{V_1\}$
  - $dom(V_1) = \{1,2,3,4\}$
- $\mathcal{C} = \{C_1, C_2\}$
  - $C_1: V_1 \neq 2$
  - $C_2: V_1 > 1$

All models for this CSP:

$\{V_1 = 3\}$

$\{V_1 = 4\}$

Constraint Satisfaction Problems (CSPs): Definition

Definition:
A constraint satisfaction problem (CSP) consists of:
- a set of variables $\mathcal{V}$
- a domain dom(V) for each variable V $\in \mathcal{V}$
- a set of constraints $\mathcal{C}$

Definition:
A model of a CSP is an assignment of values to all of its variables that satisfies all of its constraints.

Another example:
- $\mathcal{V} = \{V_1, V_2\}$
  - $dom(V_1) = \{1,2,3\}$
  - $dom(V_2) = \{1,2\}$
- $\mathcal{C} = \{C_1, C_2, C_3\}$
  - C_1: $V_2 \neq 2$
  - C_2: $V_1 + V_2 < 5$
  - C_3: $V_1 > V_2$

Which are models for this CSP?

$\{V_1=1, V_2=1\}$

$\{V_1=2, V_2=1\}$

$\{V_1=3, V_2=1\}$

$\{V_1=3, V_2=2\}$

Possible Worlds

Definition:
A possible world of a CSP is an assignment of values to all of its variables.

Definition:
A model of a CSP is an assignment of values to all of its variables that satisfies all of its constraints.

i.e. *a model is a possible world that satisfies all constraints*

Another example:

- $\mathcal{V} = \{V_1, V_2\}$
  - $dom(V_1) = \{1,2,3\}$
  - $dom(V_2) = \{1,2\}$
- $\mathcal{C} = \{C_1, C_2, C_3\}$
  - C_1: $V_2 \neq 2$
  - C_2: $V_1 + V_2 < 5$
  - C_3: $V_1 > V_2$

Possible worlds for this CSP:

$\{V_1=1, V_2=1\}$
$\{V_1=1, V_2=2\}$
$\{V_1=2, V_2=1\}$ (a model)
$\{V_1=2, V_2=2\}$
$\{V_1=3, V_2=1\}$ (a model)
$\{V_1=3, V_2=2\}$

Constraints

- Constraints are restrictions on the values that one or more variables can take
  - Unary constraint: restriction involving a single variable
    - E.g.: $V_2 \neq 2$
  - k-ary constraint: restriction involving k different variables
    - E.g. binary (k=2): $V_1 + V_2 < 5$
    - E.g. 3-ary: $V_1 + V_2 + V_4 < 5$
    - We will mostly deal with binary constraints
  - Constraints can be specified by
    1. listing all combinations of valid domain values for the variables participating in the constraint
       - E.g. for constraint $V_1 > V_2$ and $dom(V_1) = \{1,2,3\}$ and $dom(V_2) = \{1,2\}$:

V_1	V_2
2	1
3	1
3	2

    2. giving a function (predicate) that returns true if given values for each variable which satisfy the constraint else false: $V_1 > V_2$

Constraints

- A possible world satisfies a set of constraints
  - if the values for the variables involved in each constraint are consistent with that constraint
    1. They are elements of the list of valid domain values
    2. Function returns true for those values

V_1	V_2
2	1
3	1
3	2

  - Examples
    - $\{V_1=1, V_2=1\}$ (does not satisfy above constraint)
    - $\{V_1=3, V_2=1\}$ (satisfies above constraint)

Scope of a constraint

Definition:
The scope of a constraint is the set of variables that are involved in the constraint

- Examples:
  - $V_2 \neq 2$ has scope $\{V_2\}$
  - $V_1 > V_2$ has scope $\{V_1, V_2\}$
  - $V_1 + V_2 + V_4 < 5$ has scope $\{V_1, V_2, V_4\}$

- How many variables are in the scope of a k-ary constraint ?
  k variables

Finite Constraint Satisfaction Problem: Definition

Definition:
A finite constraint satisfaction problem (FCSP) is a CSP with a finite set of variables and a finite domain for each variable.

We will only study finite CSPs here but many of the techniques carry over to countably infinite and continuous domains. We use CSP here to refer to FCSP.

The scope of each constraint is automatically finite since it is a subset of the finite set of variables.

Learning Goals for CSP so far

- Define possible worlds in term of variables and their domains
- Compute number of possible worlds on real examples
- Specify constraints to represent real world problems differentiating between:
  - Unary and k-ary constraints
  - List vs. function format
- Verify whether a possible world satisfies a set of constraints (i.e., whether it is a model, a solution)

- Coming up: CSP as search
  - Read Sections 4.3-2
- Get busy with assignment 1 (Due: Friday, October 5)