

Reasoning Under Uncertainty: Variable elimination

Computer Science cpsc322, Lecture 30
(Textbook Chpt 6.4)

March, 29, 2010



Lecture Overview

- **Recap Intro Variable Elimination**
- Variable Elimination
 - Simplifications
 - Example
 - Independence
- Where are we?

Bnet Inference: General

- Suppose the variables of the belief network are $X_1, \dots, X_n$
- Z is the query variable
- $Y_1=v_1, \dots, Y_j=v_j$ are the observed variables (with their values)
- $Z_1, \dots, Z_k$ are the remaining variables

• What we **want to compute**: $P(Z \mid Y_1 = v_1, \dots, Y_j = v_j)$

• We **can actually compute**: $P(Z, Y_1 = v_1, \dots, Y_j = v_j)$

$$\underline{P(Z \mid Y_1 = v_1, \dots, Y_j = v_j)} = \frac{P(Z, Y_1 = v_1, \dots, Y_j = v_j)}{\underline{P(Y_1 = v_1, \dots, Y_j = v_j)}} = \sum_Z \frac{P(Z, Y_1 = v_1, \dots, Y_j = v_j)}{\underline{P(Y_1 = v_1, \dots, Y_j = v_j)}}$$

Inference with Factors

We can compute $P(Z, \boxed{Y_1=v_1, \dots, Y_j=v_j})$ by

- expressing the joint as a factor,

$$\underline{f(Z, Y_1, \dots, Y_j, Z_1, \dots, Z_k)}$$

→ • assigning $\underline{Y_1=v_1, \dots, Y_j=v_j}$

• and summing out the variables $Z_1, \dots, Z_k$

$$\boxed{P(Z, Y_1=v_1, \dots, Y_j=v_j)} = \sum_{Z_k} \cdots \sum_{Z_1} \boxed{f(Z, Y_1, \dots, Y_j, Z_1, \dots, Z_k)}_{Y_1=v_1, \dots, Y_j=v_j}$$

Variable Elimination Intro (1)

$$\underline{P(Z, Y_1 = v_1, \dots, Y_j = v_j)} = \sum_{Z_k} \cdots \sum_{Z_1} \boxed{f(Z, Y_1, \dots, Y_j, Z_1, \dots, Z_k)}_{Y_1=v_1, \dots, Y_j=v_j}$$

- Using the **chain rule** and the **definition of a Bnet**, we can write $\underline{P(X_1, \dots, X_n)}$ as $\prod_{i=1}^n P(X_i | pX_i)$

- We can express the joint factor as a product of factors

$$f(Z, \textcolor{green}{Y_1, \dots, Y_j}, \textcolor{blue}{Z_1, \dots, Z_j}) \quad \prod_{i=1}^n f(X_i, pX_i)$$

$$\underline{P(Z, Y_1 = v_1, \dots, Y_j = v_j)} = \sum_{Z_k} \cdots \sum_{Z_1} \boxed{\prod_{i=1}^n f(X_i, pX_i)}_{Y_1=v_1, \dots, Y_j=v_j}$$

Variable Elimination Intro (2)

Inference in belief networks thus reduces to computing “the sums of products....”

$$P(Z, Y_1 = v_1, \dots, Y_j = v_j) = \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^n f(X_i, pX_i)$$

Diagram annotations: A blue arrow points from the text “the sums of products....” to the equation. The equation is annotated with circled numbers: (1) above $f(X_i, pX_i)$, (2) below the product term, (3) above the product symbol $\prod$, (4) above the first summation symbol $\sum_{Z_k}$. The entire equation is enclosed in a blue box.

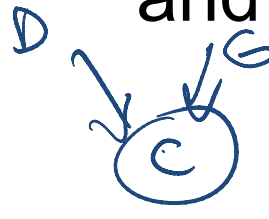
1. Construct a factor for each conditional probability.
2. In each factor assign the observed variables to their observed values.
3. Multiply the factors
4. For each of the other variables $Z_i \in \{Z_1, \dots, Z_k\}$, sum out Z_i

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How to simplify the Computation?

- Assume we have turned the CPTs into factors and performed the assignments

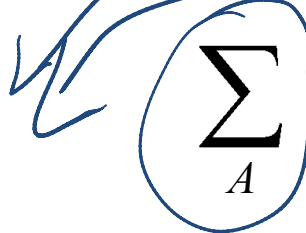


$$\boxed{f(X_i, pX_i)} \rightarrow \underline{(\text{vars } X_i)}$$

$$P(C|DG) \quad \underline{f(C, D, G)_{G=t}} \rightarrow f(C, D)$$

$$\sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^n f(X_i, pX_i)_{Y_1=v_1, \dots, Y_j=v_j} \rightarrow \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^n \underline{f(\text{vars } X_i)}$$

Let's focus on the basic case, for instance...



$$\sum_A \underline{f(C, D)} \times \underline{f(A, B, D)} \times \underline{f(E, A)} \times \underline{f(D)}$$

How to simplify: basic case

Let's focus on the basic case. $\sum_{Z_1} \prod_{i=1}^n f(\text{vars} X_i)$

$$\sum_A (C, D) \times (A, B, D) \times (E, A) \times (D)$$

• How can we compute efficiently?

Factor out those terms that don't involve Z_1 !

$$\left(\prod_{i | Z_1 \notin \text{vars} X_i} f(\text{vars} X_i) \right) \cdot \left(\sum_{Z_1} \prod_{i | Z_1 \in \text{vars} X_i} f(\text{vars} X_i) \right)$$

do not contain Z_1 do contain

$$t(C, D) \times t(D) \times \sum_A t(A, B, D) \times t(E, A)$$

General case: Summing out variables efficiently

$$\sum_{Z_k} \cdots \sum_{Z_1} \underbrace{f_1 \times \cdots \times f_h}_{\text{factors not containing } Z_2} = \sum_{Z_k} \cdots \sum_{Z_2} (f_1 \times \cdots \times f_i) \left(\underbrace{\sum_{Z_1} f_{i+1} \times \cdots \times f_h}_{\text{factors containing } Z_2} \right)$$

$$\sum_{Z_k} \cdots \sum_{Z_2} f_1 \times \cdots \times f_i \times \underbrace{f'}_{\text{factor containing } Z_2}$$

Now to sum out a variable Z_2 from a product $f_1 \times \cdots \times f_i \times f'$ of factors, again partition the factors into two sets

- $\hat{F}$: those that *contain* Z_2
- F : those that *do not*

$$\prod F \times \sum_{Z_2} \prod \hat{F}$$

Analogy with “Computing sums of products”

This simplification is similar to what you can do in basic algebra with *multiplication* and *addition*

- It takes 14 multiplications or additions to evaluate the expression

$$\underline{a b + a c + a d + a e h + a f h + a g h.}$$

- This expression be evaluated more efficiently....

$$a * (b + c + d + h * (e + f + g))$$

7 operations

Variable elimination ordering ↙

Is there only one way to simplify?

$$P(G, D=t) = \sum_{A,B,C} f(A, G) f(B, A) f(C, G) f(B, C)$$

CBA ↙ ↙ ↙

$$P(G, D=t) = \sum_A f(A, G) \sum_B f(B, A) \sum_C f(C, G) f(B, C)$$

BCA ↙ ↙ ↙

$$P(G, D=t) = \sum_A f(A, G) \sum_C f(C, G) \sum_B f(B, C) f(B, A)$$

Variable elimination algorithm: Summary

$$P(Z, \cancel{Y_1, \dots, Y_j}, \cancel{Z_1, \dots, Z_j})$$

To compute $P(Z / Y_1=v_1, \dots, Y_j=v_j)$:

1. Construct a factor for each conditional probability.
2. Set the observed variables to their observed values.
3. Given an elimination ordering, simplify/decompose sum of products
4. Perform products and sum out Z_i
5. Multiply the remaining factors (all in ? Z)
6. Normalize: divide the resulting factor $f(Z)$ by $\sum_Z f(Z)$.

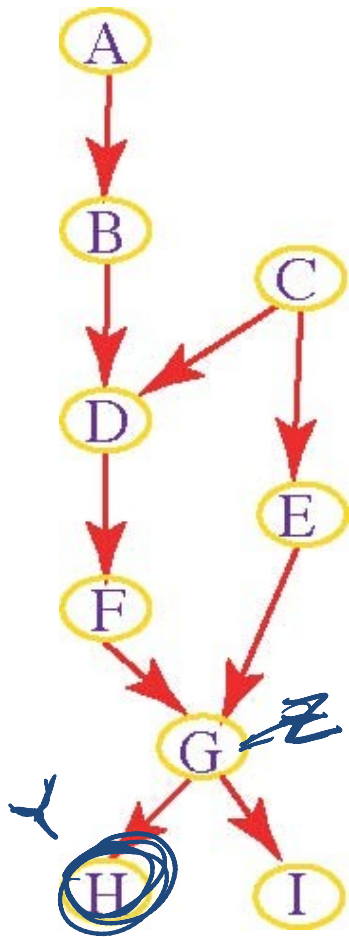
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Variable elimination example

Compute $P(G \mid H=h_1)$.

- $P(G, H) = \sum_{A, B, C, D, E, F, I} P(A, B, C, D, E, F, G, H, I)$



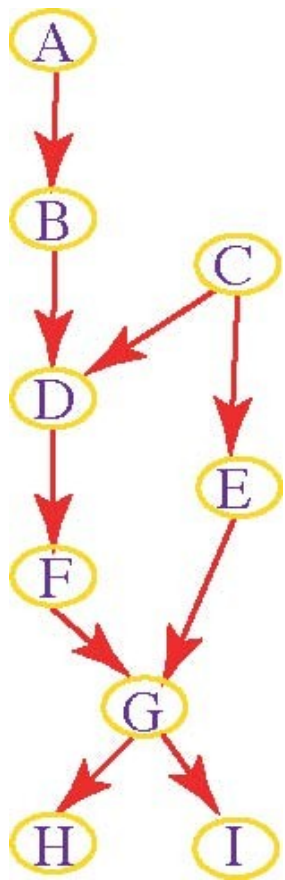
Variable elimination example

Compute $P(G \mid H=h_1)$.

- $P(G, H) = \sum_{A, B, C, D, E, F, I} \underline{P(A, B, C, D, E, F, G, H, I)}$

Chain Rule + Conditional Independence:

$$P(G, H) = \sum_{A, B, C, D, E, F, I} \underline{P(A)P(B|A)P(C)P(D|B, C)P(E|C)P(F|D)P(G|F, E)P(H|G)P(I|G)}$$



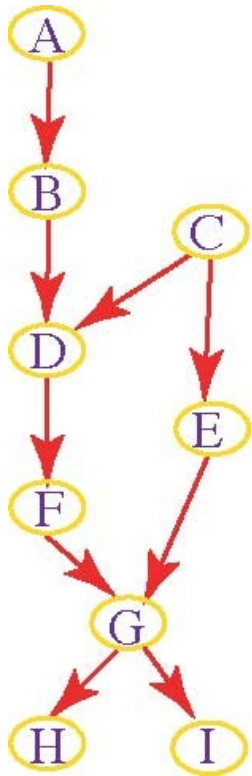
Variable elimination example (step1)

Compute $P(G \mid H=h_1)$.

• $P(G, H) = \sum_{A, B, C, D, E, F, I} P(A)P(B|A)P(C)P(D|B, C)P(E|C)P(F|D)P(G|F, E)P(H|G)P(I|G)$

Factorized Representation:

$$P(G, H) = \sum_{A, B, C, D, E, F, I} f_0(A) f_1(B, A) f_2(C) f_3(D, B, C) f_4(E, C) f_5(F, D) f_6(G, F, E) f_7(H, G) f_8(I, G)$$



- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$

Variable elimination example (step 2)

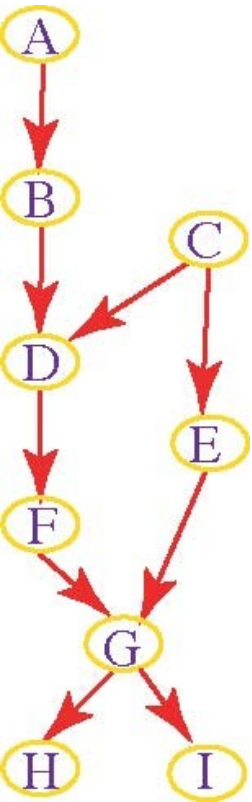
Compute $P(G \mid H=h_1)$.

Previous state:

$$P(G, H) = \sum_{A, B, C, D, E, F, I} f_0(A) f_1(B, A) f_2(C) f_3(D, B, C) f_4(E, C) f_5(F, D) f_6(G, F, E) \underline{f_7(H, G)} f_8(I, G)$$

Observe H:

$$P(G, H=h_1) = \sum_{A, B, C, D, E, F, I} f_0(A) f_1(B, A) f_2(C) f_3(D, B, C) f_4(E, C) f_5(F, D) f_6(G, F, E) \underline{f_9(G)} f_8(I, G)$$



- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$

• $f_9(G)$

Variable elimination example (steps 3-4)

Compute $P(G \mid H=h_1)$.

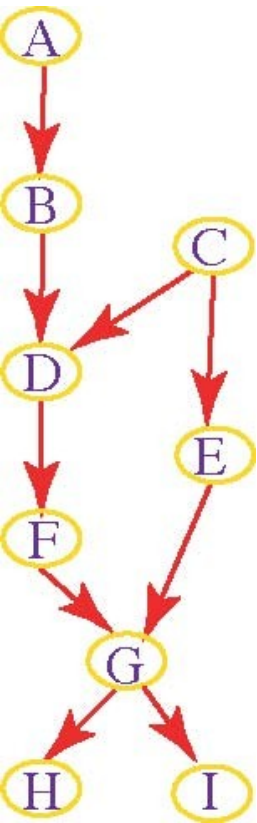
Previous state:

$$P(G, H) = \sum_{A, B, C, D, E, F, I} f_0(A) f_1(B, A) f_2(C) f_3(D, B, C) f_4(E, C) f_5(F, D) f_6(G, F, E) f_9(G) f_8(I, G)$$

Elimination ordering A, C, E, I, B, D, F :

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I, G) \underbrace{\sum_E f_6(G, F, E) \sum_C f_2(C) f_3(D, B, C) f_4(E, C)}_{\text{eliminated}} \underbrace{\sum_A f_0(A) f_1(B, A)}_{\text{eliminated}}$$

- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$
- $f_9(G)$



Variable elimination example(steps 3-4)

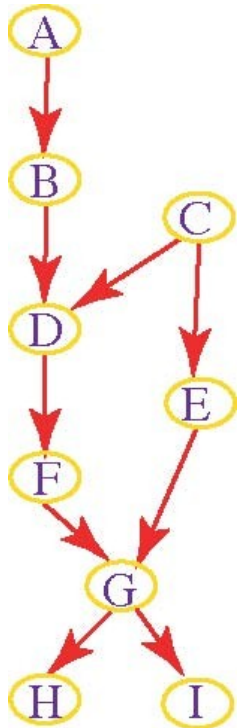
Compute $P(G / H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state:

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I, G) \sum_E f_6(G, F, E) \sum_C f_2(C) f_3(D, B, C) f_4(E, C) \sum_A \underline{f_0(A) f_1(B, A)}$$

Eliminate A:

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \underline{\sum_B f_{10}(B)} \sum_I f_8(I, G) \underline{\sum_E f_6(G, F, E)} \underline{\sum_C f_2(C)} f_3(D, B, C) f_4(E, C)$$



- $f_0(A)$

- $f_1(B, A)$

- $f_2(C)$

- $f_3(D, B, C)$

- $f_4(E, C)$

- $f_5(F, D)$

- $f_6(G, F, E)$

- $f_7(H, G)$

- $f_8(I, G)$

- $f_9(G)$

- $f_{10}(B)$

Variable elimination example(steps 3-4)

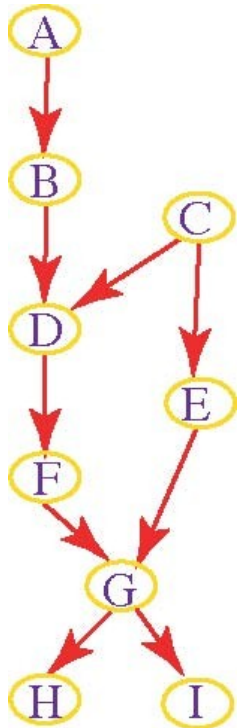
Compute $P(G / H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state:

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I, G) \sum_E f_6(G, F, E) \underbrace{\sum_C f_2(C) f_3(D, B, C) f_4(E, C)}$$

Eliminate C:

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I, G) \sum_E f_6(G, F, E) \underbrace{f_{12}(B, D, E)}$$



- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$
- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B, D, E)$

Variable elimination example(steps 3-4)

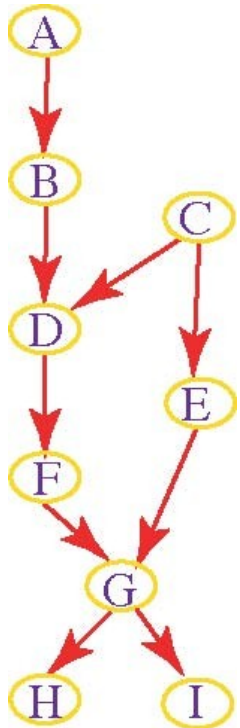
Compute $P(G \mid H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state:

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I, G) \sum_E f_6(G, F, E) f_{12}(B, D, E)$$

Eliminate E:

$$P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) f_{13}(B, D, F, G) \sum_I f_8(I, G)$$



- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$
- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B, D, E)$
- $f_{13}(B, D, F, G)$

Variable elimination example(steps 3-4)

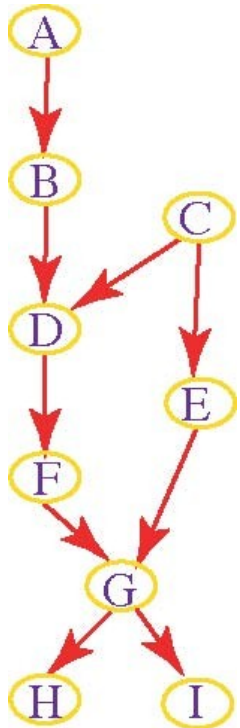
Compute $P(G \mid H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state: $P(G, H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) f_{13}(B, D, F, G) \underline{\sum_I f_8(I, G)}$

Eliminate I:

$$P(G, H=h_1) = f_9(G) \underline{f_{14}(G)} \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) f_{13}(B, D, F, G)$$

- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$
- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B, D, E)$
- $f_{13}(B, D, F, G)$
- $f_{14}(G)$



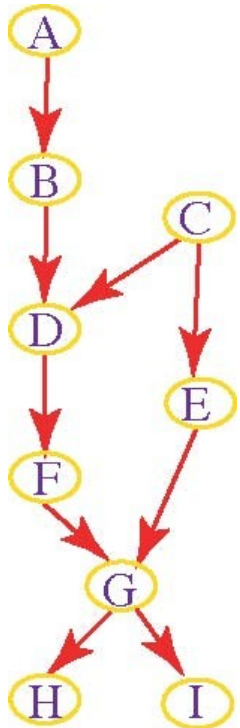
Variable elimination example(steps 3-4)

Compute $P(G \mid H=h_1)$. Elimination ordering $A, C, E, I, \textcolor{violet}{B}, D, F$.

Previous state: $P(G, H=h_1) = f_9(G) f_{14}(G) \sum_F \sum_D f_5(F, D) \sum_B \textcolor{violet}{f_{10}(B)} \textcolor{violet}{f_{13}(B, D, F, G)}$

Eliminate B:

$$P(G, H=h_1) = f_9(G) f_{14}(G) \sum_F \sum_D f_5(F, D) \textcolor{violet}{f_{15}(D, F, G)}$$



- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$
- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B, D, E)$
- $f_{13}(B, D, F, G)$
- $f_{14}(G)$
- $\textcolor{violet}{f_{15}(D, F, G)}$

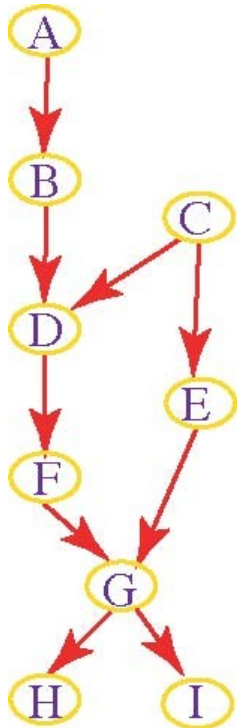
Variable elimination example(steps 3-4)

Compute $P(G \mid H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state: $P(G, H=h_1) = f_9(G) f_{14}(G) \sum_F \sum_D f_5(F, D) f_{15}(D, F, G)$

Eliminate D:

$$P(G, H=h_1) = f_9(G) f_{14}(G) \sum_F f_{16}(F, G)$$



- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$
- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B, D, E)$
- $f_{13}(B, D, F, G)$
- $f_{14}(G)$
- $f_{15}(D, F, G)$
- $f_{16}(F, G)$

Variable elimination example(steps 3-4)

Compute $P(G / H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state: $P(G, H=h_1) = f_9(G) f_{14}(G) \sum_F f_{16}(F, G)$

Eliminate F:

$$P(G, H=h_1) = f_9(G) f_{14}(G) f_{17}(G)$$

$$\bullet f_9(G)$$

$$\bullet f_0(A)$$

$$\bullet f_{10}(B)$$

$$\bullet f_1(B, A)$$

$$\bullet f_{12}(B, D, E)$$

$$\bullet f_2(C)$$

$$\bullet f_{13}(B, D, F, G)$$

$$\bullet f_3(D, B, C)$$

$$\bullet f_4(E, C)$$

$$\bullet f_{14}(G)$$

$$\bullet f_5(F, D)$$

$$\bullet f_{15}(D, F, G)$$

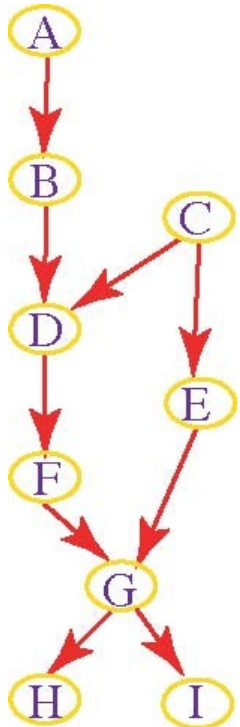
$$\bullet f_6(G, F, E)$$

$$\bullet f_{16}(F, G)$$

$$\bullet f_7(H, G)$$

$$\bullet f_{17}(G)$$

$$\bullet f_8(I, G)$$



Variable elimination example (step 5)

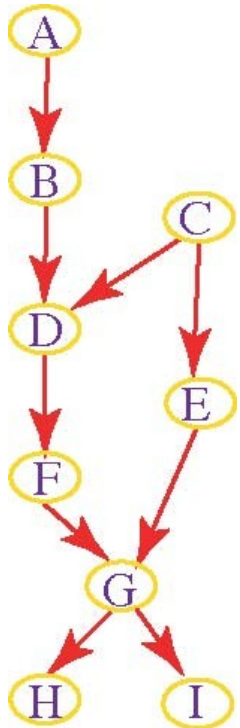
Compute $P(G / H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state: $P(G, H=h_1) = f_9(G) f_{14}(G) f_{17}(G)$

Multiply remaining factors:

$$P(G, H=h_1) = f_{18}(G)$$

- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B, D, E)$
- $f_{13}(B, D, F, G)$
- $f_{14}(G)$
- $f_{15}(D, F, G)$
- $f_{16}(F, G)$
- $f_{17}(G)$
- $f_{18}(G)$
- $f_0(A)$
- $f_1(B, A)$
- $f_2(C)$
- $f_3(D, B, C)$
- $f_4(E, C)$
- $f_5(F, D)$
- $f_6(G, F, E)$
- $f_7(H, G)$
- $f_8(I, G)$



Variable elimination example (step 6)

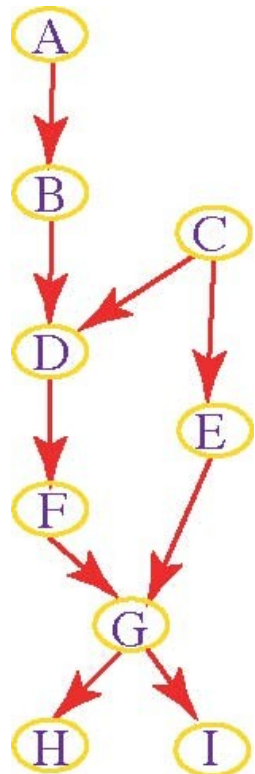
Compute $P(G \mid H=h_1)$. Elimination ordering A, C, E, I, B, D, F .

Previous state:

$$P(G, H=h_1) = f_{18}(G)$$

Normalize:

$$P(G \mid H=h_1) = f_{18}(G) / \sum_{g \in \text{dom}(G)} f_{18}(G)$$



• $f_0(A)$

• $f_1(B, A)$

• $f_2(C)$

• $f_3(D, B, C)$

• $f_4(E, C)$

• $f_5(F, D)$

• $f_6(G, F, E)$

• $f_7(H, G)$

• $f_8(I, G)$

• $f_9(G)$

• $f_{10}(B)$

• $f_{12}(B, D, E)$

• $f_{13}(B, D, F, G)$

• $f_{14}(G)$

• $f_{15}(D, F, G)$

• $f_{16}(F, G)$

• $f_{17}(G)$

• $f_{18}(G)$

Lecture Overview

- Recap Intro Variable Elimination
- Variable Elimination
 - Simplifications
 - Example
 - Independence
- Where are we?

Variable elimination and conditional independence

- Variable Elimination looks incredibly painful for large graphs?
- We used conditional independence.....

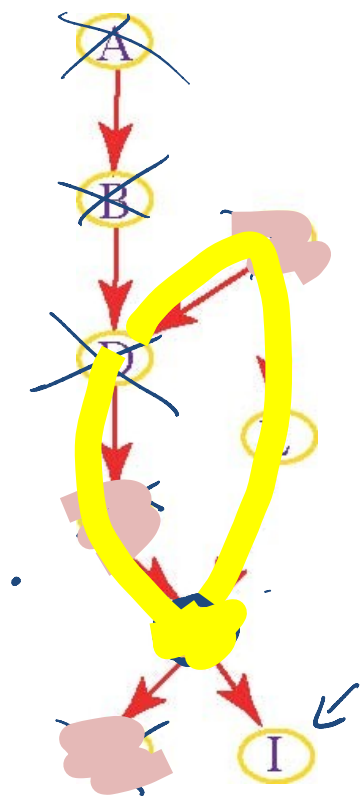
$$P(X_1, \dots, X_n) = \prod P(X_i | \text{parents}(X_i))$$

- Can we use it to make variable elimination simpler?

Yes, all the variables from which the query is conditional independent given the observations can be pruned from the Bnet

VE and conditional independence: Example

- All the variables from which the query is conditional independent given the observations can be pruned from the Bnet



e.g., $P(\text{I} / \text{A}, \text{B}, \text{C}, \text{D}, \text{E}, \text{F}, \text{G}, \text{H})$.

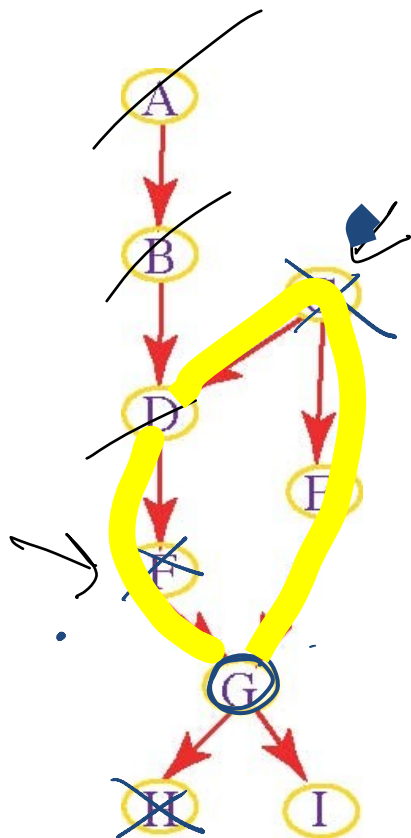


both paths
from G
to D are
blocked

G is conditionally
independent from $\overline{\text{ABD}}$ given
the observed vars
H, F, C

VE and conditional independence: Example

- All the variables from which the query is conditional independent given the observations can be pruned from the Bnet



e.g., $P(\underline{G} \mid \underline{H}=v_1, \underline{F}=v_2, \underline{C}=v_3)$.

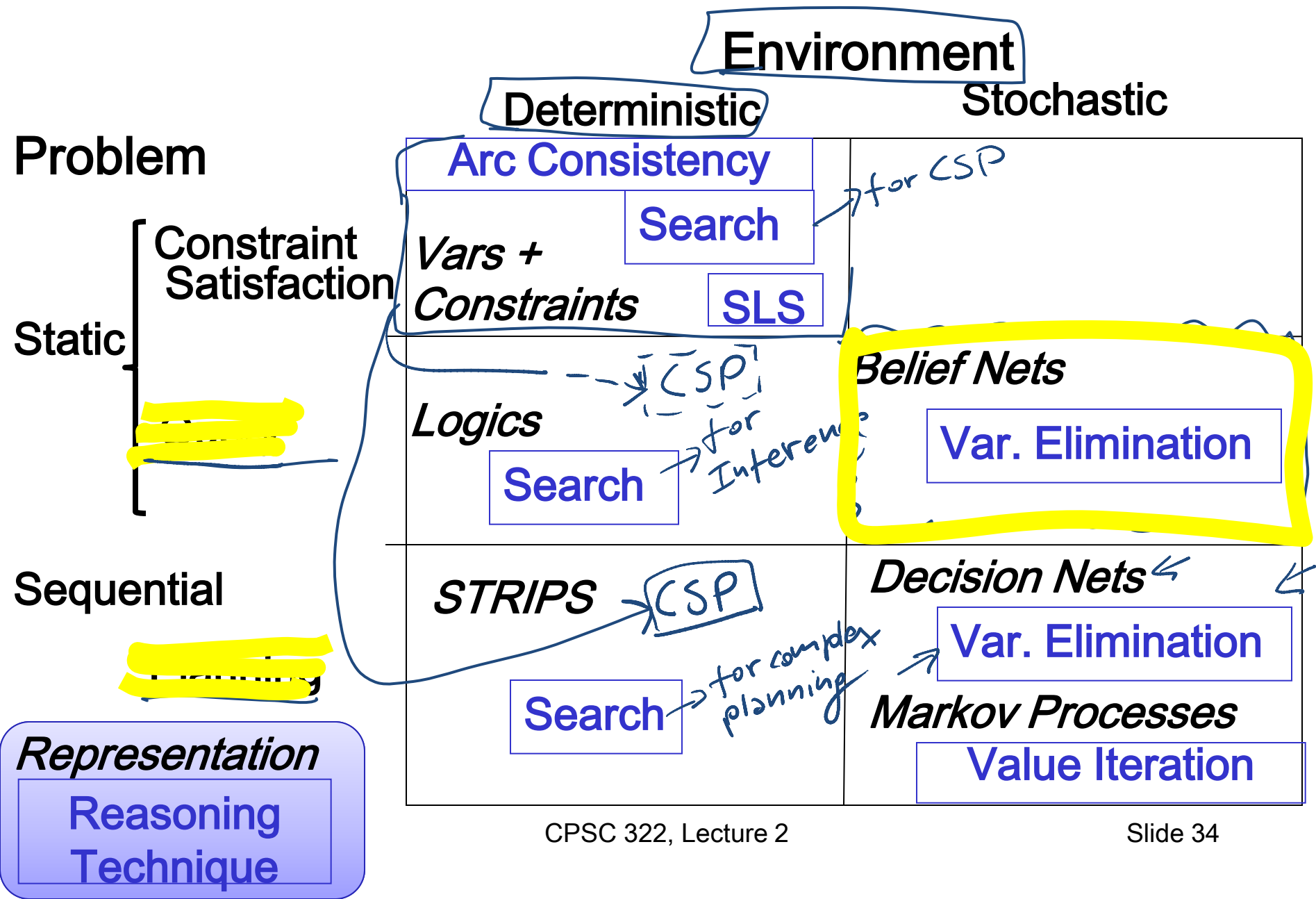
G is indep. $D B A$ given $C F$

Learning Goals for today's class

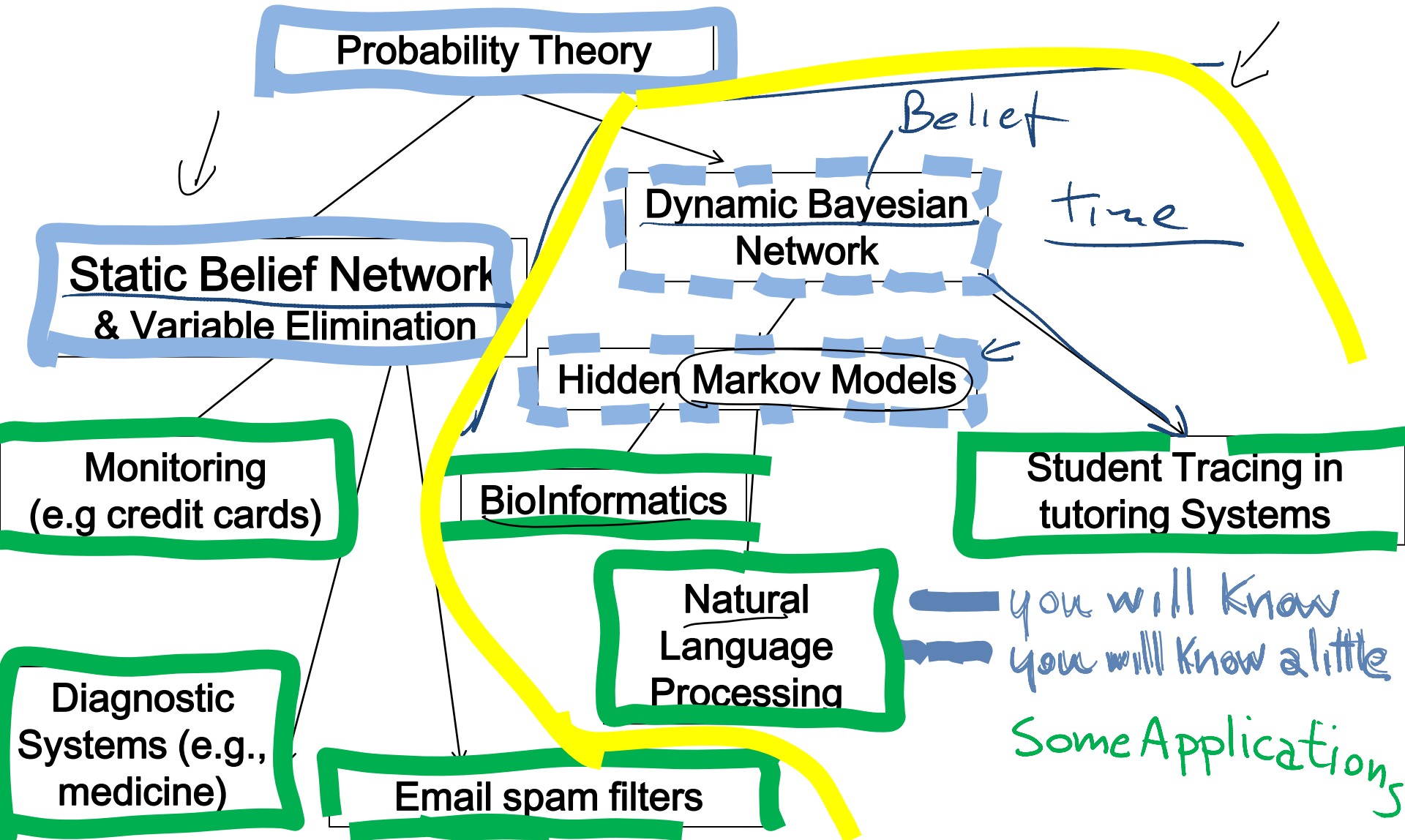
You can:

- Carry out **variable elimination** by using factor representation and using the factor operations.
- Use techniques to simplify variable elimination.

Big Picture: R&R systems



Answering Query under Uncertainty



Next Class

Probability and Time (*TextBook 6.5*)

Course Elements

- Practice Exercises available on Vista, two on Bnets. See course Bboard for instructions.
- Assignment 4 will be available on Wednesday and due on Apr the 14th (last class).