

# Bottom Up: Soundness and Completeness

Computer Science cpsc322, Lecture 21  
*(Textbook Chpt 5.2)*

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# Lecture Overview

- **Recap**
- Soundness of Bottom-up Proofs
- Completeness of Bottom-up Proofs

# (Propositional) Logic: Key ideas

Given a domain that can be represented with  $n$  propositions you have ..... interpretations (possible worlds)  
 $2^n$

If you do not know anything you can be in any of those

If you know that some logical formulas are true (your KB.....). You know that you can be only in .....  
in which the KB is true (i.e. the models of KB)

It would be nice to know what else is true in all those...  
models what is logically entailed

# PDCL syntax / semantics / proofs

Domain can be represented by three propositions:  $p, q, r$

Interpretations?

$$KB = \begin{cases} q \leftarrow \text{ } \\ r \leftarrow \text{ } \\ p \leftarrow q \wedge r. \end{cases}$$

$r$	$q$	$p$
T	T	T
<del>T</del>	<del>T</del>	<del>F</del>
<del>T</del>	<del>F</del>	<del>T</del>
<del>T</del>	<del>F</del>	<del>F</del>
<del>F</del>	<del>T</del>	<del>T</del>
<del>F</del>	<del>T</del>	<del>F</del>
<del>F</del>	<del>F</del>	<del>T</del>
<del>F</del>	<del>F</del>	<del>F</del>

Models?

What is logically entailed?

$r, q, p$

Prove

$$G = \boxed{q \wedge r}$$

$$C = \{q, r, p\} \quad G \subseteq C \quad KB \stackrel{\uparrow \downarrow}{\vdash_{BV}} G$$

# PDCL syntax / semantics / proofs

$$KB = \frac{\boxed{p \leftarrow q \wedge r.}}{q.}$$

## Interpretations

	$r$	$q$	$p$
→	T	T	T
	<del>T</del>	<del>T</del>	<del>F</del>
	<del>T</del>	<del>F</del>	<del>T</del>
	<del>T</del>	<del>F</del>	<del>F</del>
→	F	T	T
→	F	T	F
	<del>F</del>	<del>F</del>	<del>T</del>
	<del>F</del>	<del>F</del>	<del>F</del>

Models

What is logically entailed?

Prove  $G = q \wedge \text{'}$

$$C = \{q\}$$

$$G \notin C$$

$$KB \not\models_{BU} G$$

# Lecture Overview

- Recap
- **Soundness of Bottom-up Proofs**
- Completeness of Bottom-up Proofs

# Soundness of bottom-up proof procedure

Generic Soundness of proof procedure:

If G can be proved by the procedure ( $KB \vdash G$ )  
then G is logically entailed by the KB ( $KB \models G$ )

For Bottom-Up proof

if  $G \subseteq C$  at the end of procedure  
then G is logically entailed by the KB

So BU is sound, if all the atoms in... $C$ ..

are logically entailed by the KB

# Soundness of bottom-up proof procedure

Suppose this is not the case. 

1. Let  $h$  be the first atom added to  $C$  that is not entailed by  $KB$  (i.e., that's *not true* in every model of  $KB$ )
2. Suppose  $h$  isn't true in model  $M$  of  $KB$ .
3. Since  $h$  was added to  $C$ , there must be a clause in  $KB$  of form:  $h \leftarrow b_1 \wedge \dots \wedge b_m$
4. Each  $b_i$  is true in  $M$  (because of 1.).  $h$  is false in  $M$ .  
So..... the clause is false in  $M$
5. Therefore  $M$  is not a model
6. Contradiction! thus no such  $h$  exists.



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- Recap
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- **Completeness of Bottom-up Proofs**

# Completeness of Bottom Up

**Generic Completeness of proof procedure:**

If  $G$  is logically entailed by the KB ( $KB \models G$ )  
then  $G$  can be proved by the procedure ( $KB \vdash G$ )

$$G \subseteq C$$

**Sketch of our proof:**

1. Suppose  $KB \models G$ . Then  $G$  is true in all models of  $KB$ .
2. Thus  $G$  is true in any particular model of KB
3. We will define a model so that if  $G$  is true in that model,  $G$  is proved by the bottom up algorithm.  $G \subseteq C$
4. Thus  $KB \vdash_{BU} G$ .

## Let's work on step 3

3. We will define a model so that if  $G$  is true in that  
⇒ model,  $G$  is proved by the bottom up algorithm.

$$G \subseteq C$$

3.1 We will define an interpretation  $I$  so that if  $G$  is  
⇒ true in  $I$ ,  $G$  is proved by the bottom up algorithm.

$$G \subseteq C$$

3.2 We will then show that .....  $I$  ..... is a model

# Let's work on step 3.1

3.1 Define interpretation  $I$  so that if  $G$  is true in  $I$ ,  
Then  $G \subseteq C$ .

Let  $I$  be the interpretation in which every element  
of  $C$  is *true* and every other atom is *false*.

$a \leftarrow e \wedge g.$

$b \leftarrow f \wedge g.$

$c \leftarrow e.$

$f \leftarrow c \wedge e.$

$e.$

$d.$

$F \quad F \quad T \quad T \quad T \quad T \quad F$   
 $\{a, b, c, d, e, f, g\}$

$C$

- $\{\}$
- $\{e\}$
- $\{e, d\}$
- $\{e, d, c\}$
- $\{e, d, c, f\}$

## Let's work on step 3.2

**Claim:**  $I$  is a model of  $KB$  (we'll call it the minimal model).

**Proof:** Assume that  $I$  is not a model of  $KB$ .

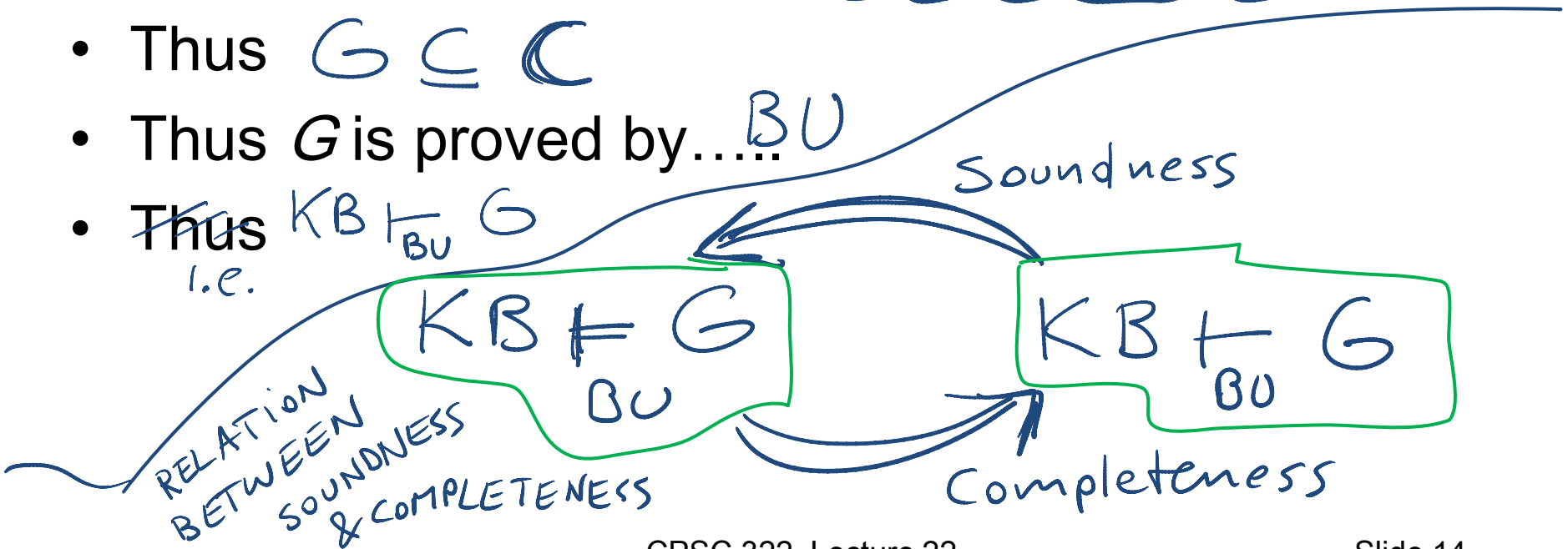
- Then there must exist some clause  $h \leftarrow b_1 \wedge \dots \wedge b_m$  in  $KB$  (having zero or more  $b_i$ 's) which is false in  $I$ .
- The only way this can occur is if  $b_1 \dots b_m$  are true in  $I$  (i.e., are in  $C$ ) and  $h$  is false in  $I$  (i.e., is not in  $C$ ).
- But if each  $b_i$  belonged to  $C$ , Bottom Up would have added  $h$  to  $C$  as well.
- So, there can be no clause in the  $KB$  that is false in interpretation  $I$  (which implies the claim :-)

# Completeness of Bottom Up

(proof summary)

If  $KB \models G$  then  $KB \vdash_{BU} G$

- Suppose  $KB \models G$ .
- Then  $G$  is true in all the models
- Thus  $G$  is true in the minimal model
- Thus  $G \in C$
- Thus  $G$  is proved by....  $BU$
- Thus  $KB \vdash_{BU} G$   
i.e.



# Learning Goals for today's class

**You can:**

- Prove that BU proof procedure is sound
- Prove that BU proof procedure is complete

# Next class

(still section 5.2)

- Using PDC Logic to model the electrical domain
- Reasoning in the electrical domain
- Top-down proof procedure (as Search!)