Stat 535 C - Statistical Computing & Monte Carlo Methods

Arnaud Doucet

Email: arnaud@cs.ubc.ca

1.1– Outline

- Introduction to Markov chain Monte Carlo
- The Gibbs Sampler
- Examples

2.1– Summary of Last Lecture

• Rejection Sampling and Importance Sampling are two general methods but limited to problems of moderate dimensions.

• "Problem": We try to sample all the components of a potentially high-dimensional parameter simultaneously.

• There are two ways to implement incremental strategies.

- Iteratively: Markov chain Monte Carlo.

- Sequentially: Sequential Monte Carlo.

2.2– Motivating Example: Nuclear Pump Data

• Multiple failures in a nuclear plant:

Pump	1	2	3	4	5	6	7	8	9	10
Failures	5	1	5	14	3	19	1	1	4	22
Times	94.32	15.72	62.88	125.76	5.24	31.44	1.05	1.05	2.10	10.48

• Model: Failures of the $i-$th pump follow a Poisson process with parameter λ_i $(1 \leq i \leq 10)$. For an observed time t_i, the number of failures p_i is thus a Poisson $\mathcal{P}(\lambda_i t_i)$ random variable.

• The unknowns consist of $\theta := (\lambda_1, \ldots, \lambda_{10}, \beta)$.

2.3– Bayesian Model for Nuclear Pump Data

- Hierarchical model

$$\lambda_i \overset{iid}{\sim} \mathcal{G}a(\alpha, \beta) \text{ and } \beta \sim \mathcal{G}a(\gamma, \delta)$$

with $\alpha = 1.8$ and $\gamma = 0.01$ and $\delta = 1$.

- The posterior distribution is proportional to

$$\prod_{i=1}^{10} \{(\lambda_i t_i)^{p_i} \exp(-\lambda_i t_i) \lambda_i^{\alpha-1} \exp(-\beta\lambda_i)\} \beta^{10\alpha} \beta^{\gamma-1} \exp(-\delta\beta)$$

$$\propto \prod_{i=1}^{10} \{\lambda_i^{p_i+\alpha-1} \exp(-(t_i+\beta)\lambda_i)\} \beta^{10\alpha+\gamma-1} \exp(-\delta\beta).$$

- This multidimensional distribution is rather complex. It is not obvious how the inverse cdf method, the rejection method or importance sampling could be used in this context.

2.4– Conditional Distributions

• The conditionals have a familiar form

$$\lambda_i|(\beta, t_i, p_i) \sim \mathcal{G}a(p_i + \alpha, t_i + \beta) \text{ for } 1 \leq i \leq 10,$$

$$\beta|(\lambda_1, \ldots, \lambda_{10}) \sim \mathcal{G}a(\gamma + 10\alpha, \delta + \sum_{i=1}^{10} \lambda_i).$$

• Instead of directly sampling the vector $\theta = (\lambda_1, \ldots, \lambda_{10}, \beta)$ at once, one could suggest sampling it iteratively, starting for example with the λ_i's for a given guess of β, followed by an update of β given the new samples $\lambda_1, \ldots, \lambda_{10}$.

2.4– Conditional Distributions

• Given a sample, at iteration t, $\theta^t := (\lambda_1^t, \ldots, \lambda_{10}^t, \beta^t)$ one could proceed as follows at iteration $t+1$,

1. $\lambda_i^{t+1} | (\beta^t, t_i, p_i) \sim \mathcal{G}a(p_i + \alpha, t_i + \beta^t)$ for $1 \leq i \leq 10$,

2. $\beta^{t+1} | (\lambda_1^{t+1}, \ldots, \lambda_{10}^{t+1}) \sim \mathcal{G}a(\gamma + 10\alpha, \delta + \sum_{i=1}^{10} \lambda_i^{t+1})$.

• Instead of directly sampling in a space with 11 dimensions, one samples in spaces of dimension 1

• Note that the deterministic version of such an algorithm would not generally converge towards the global maximum of the joint distribution.

2.4– Conditional Distributions

- The structure of the algorithm calls for many questions:
    - Are we sampling from the desired joint distribution?
    - If yes, how many times should the iteration above be repeated?

- The validity of the approach described here stems from the fact that the sequence $\{\theta^t\}$ defined above is a Markov chain and some Markov chains have very nice properties.

2.5– Introduction to Markov Chain Monte Carlo

• **Markov chain**: A sequence of random variables $\{X_n; n \in \mathbb{N}\}$ defined on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ which satisfies the property, for any $A \in \mathcal{B}(\mathbb{X})$

$$\mathbb{P}(X_n \in A | X_0, ..., X_{n-1}) = \mathbb{P}(X_n \in A | X_{n-1}).$$

and we will write

$$P(x, A) = \mathbb{P}(X_n \in A | X_{n-1}).$$

• **Markov chain Monte Carlo**: Given a target π, design a transition kernel P such that asymptotically as $n \to \infty$

$$\frac{1}{N} \sum_{n=1}^{N} \varphi(X_n) \to \int \varphi(x) \pi(x) dx \text{ and/or } X_n \sim \pi.$$

• It should be easy to simulate the Markov chain even if π is complex.

2.6– Example

- Consider the autoregression for $|\alpha| < 1$

$$X_n = \alpha X_{n-1} + V_n, \text{ where } V_n \sim \mathcal{N}\left(0, \sigma^2\right).$$

- The limiting distribution is

$$\pi\left(x\right) = \mathcal{N}\left(x; 0, \frac{\sigma^2}{1-\alpha^2}\right).$$

- To sample from π, we could just sample the Markov chain and asymptotically we would have $X_n \sim \pi$.

- Obviously, in this case this is useless because we can sample from π directly.

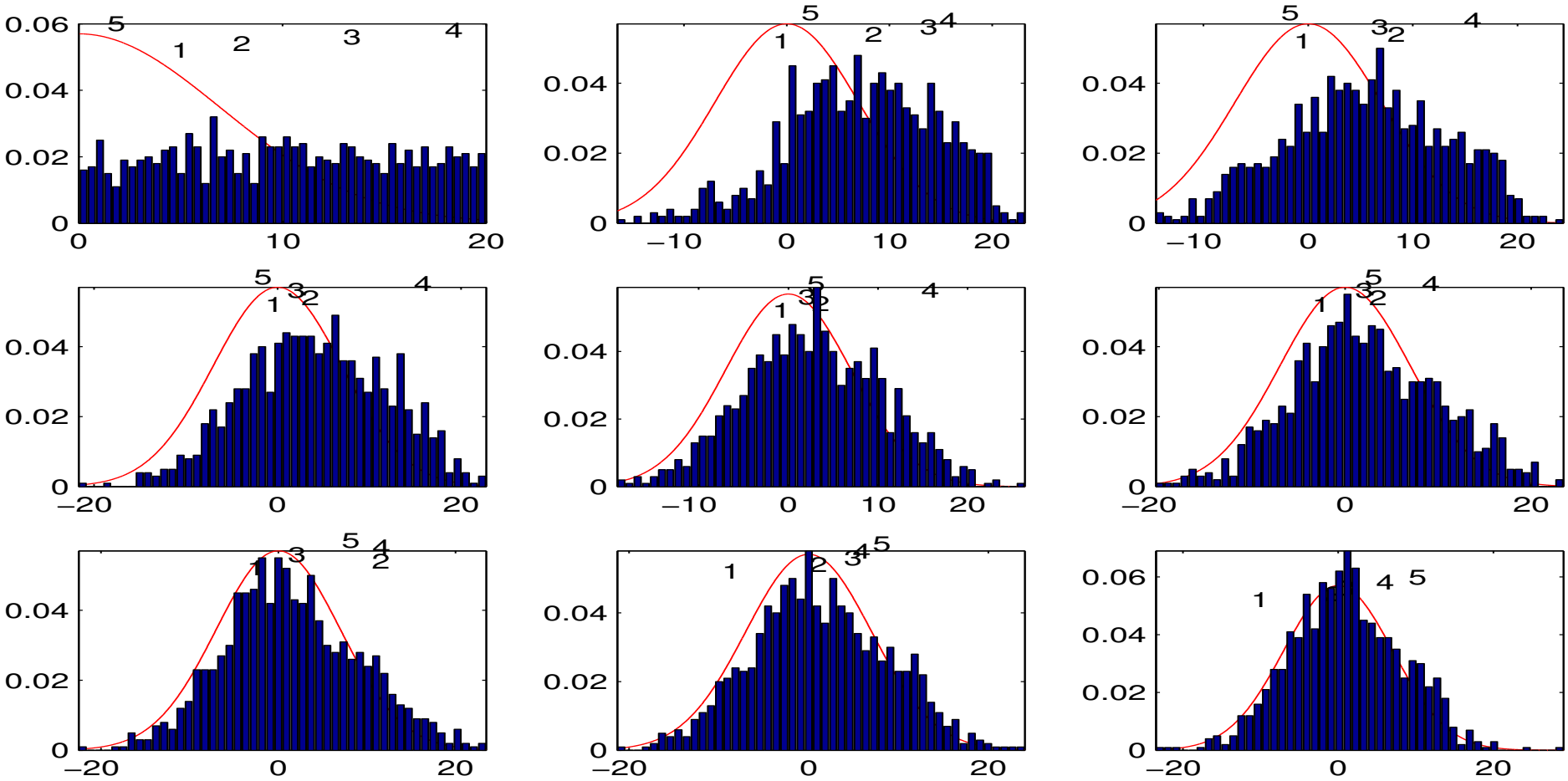
2.7– Example

• Graphically, consider 1000 independent Markov chains run in parallel.

• We assume that the initial distribution of these Markov chains is $\mathcal{U}_{[0,20]}$.
So initially, the Markov chains samples are not distributed according to π

2.7– Example

From top left to bottom right: histograms of 1000 independent Markov chains with a normal distribution as target distribution.



2.7– Example

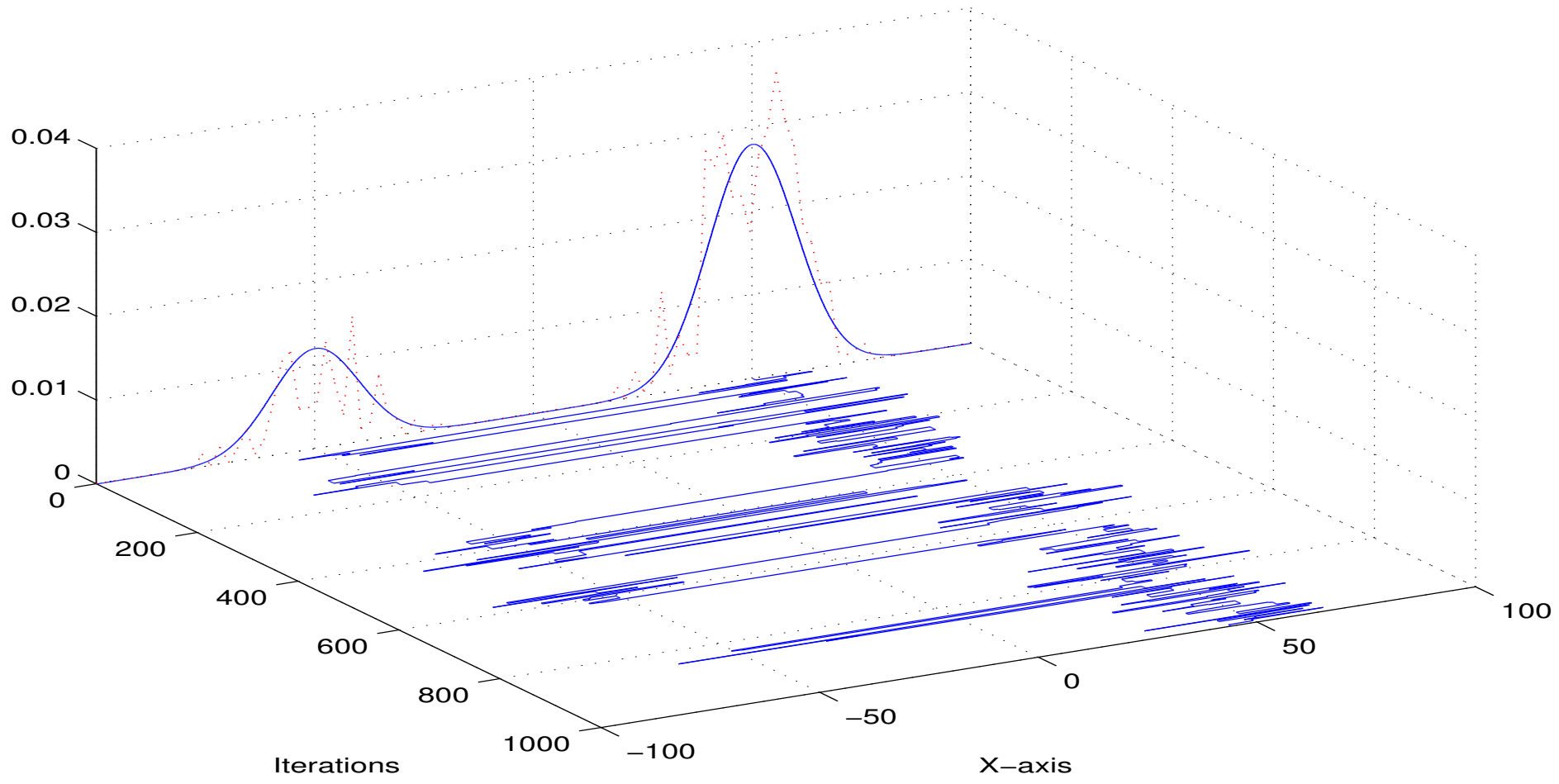
• The target normal distribution seems to "attract" the distribution of the samples and even to be a fixed point of the algorithm.

• This is is what we wanted to achieve, *i.e.* it seems that we have produced 1000 independent samples from the normal distribution.

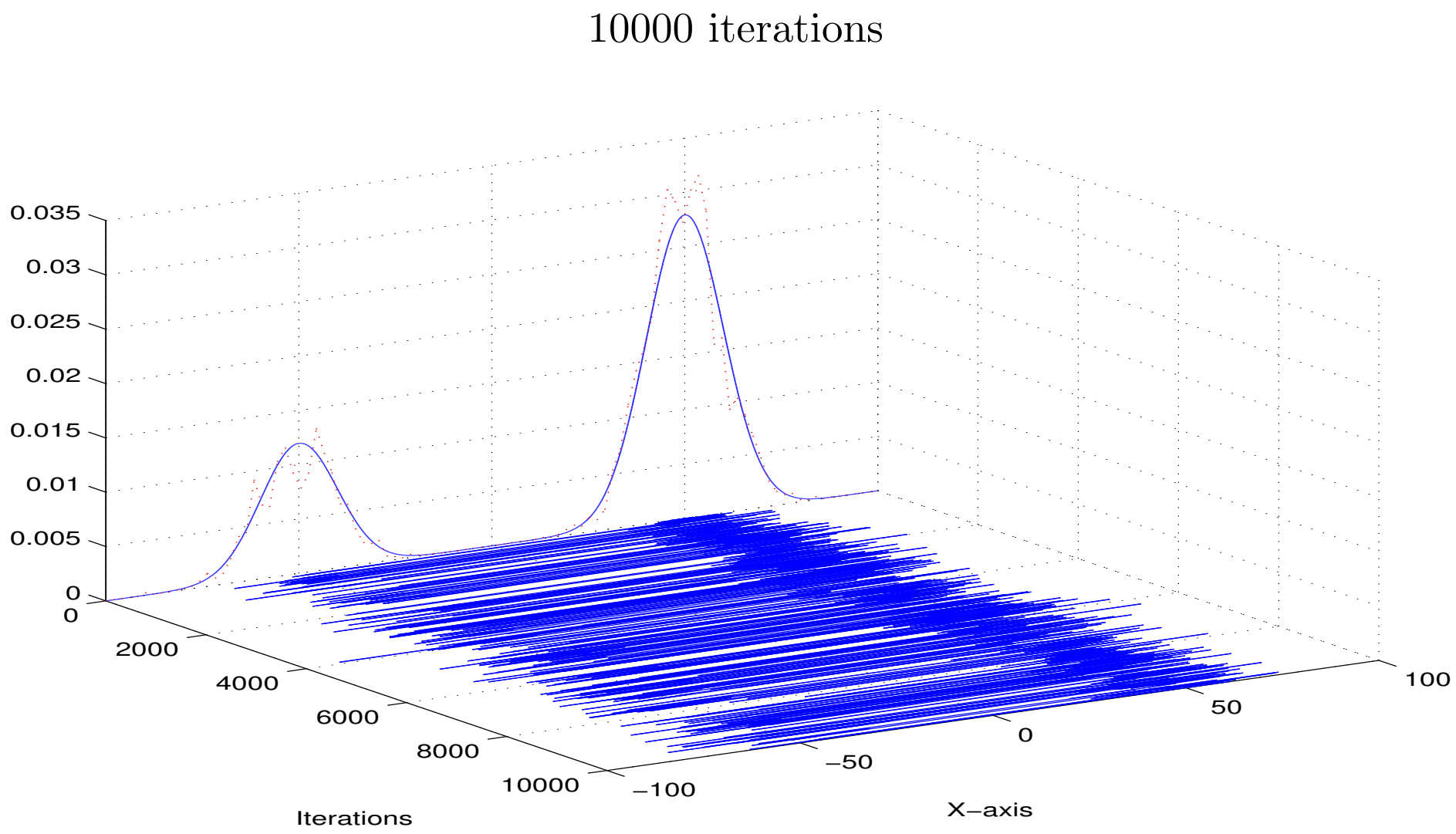
• In fact one can show that in many (all?) situations of interest it is not necessary to run N Markov chains in parallel in order to obtain 1000 samples, but that one can consider a unique Markov chain, and build the histogram from this single Markov chain by forming histograms from one trajectory.

2.8– Example: Mixture of Normals

1000 iterations



2.8– Example: Mixture of Normals



2.9– Markov chain Monte Carlo

• The estimate of the target distribution, through the series of histograms, improves with the number of iterations.

• Assume that we have stored $\{X_n, 1 \leq n \leq N\}$ for N large and wish to estimate $\int_{\mathsf{X}} \varphi(x)\pi(x)dx$.

• In the light of the numerical experiments, one can suggest the estimator

$$\frac{1}{N}\sum_{n=1}^{N} \varphi(X_n).$$

which is exactly the estimator that we would use if $\{X_n, 1 \leq n \leq N\}$ were independent.

• In fact, it can be proved, under relatively mild conditions, that such an estimator is consistent *despite the fact that the samples are NOT independent!* Under additional conditions, a CLT also holds with a rate of CV in $1/\sqrt{N}$.

2.9– Markov chain Monte Carlo

To summarize, we are interested in Markov chains with transition kernel P which have the following three important properties observed above:

• The desired distribution π is a "fixed point" of the algorithm or, in more appropriate terms, an *invariant distribution* of the Markov chain, *i.e.* $\int_{\mathsf{X}} \pi(x)P(x,y) = \pi(y)$.

• The successive distributions of the Markov chains are "attracted" by π, or converge towards π.

• The estimator $\frac{1}{N}\sum_{n=1}^{N} \varphi(X_n)$ converges towards $E_\pi(\varphi(X))$ and asymptotically $X_n \sim \pi$

2.9– Markov chain Monte Carlo

• Given $\pi(x)$, there is an infinite number of kernels $P(x, y)$ which admits $\pi(x)$ as their invariant distribution.

• The "art" of MCMC consists of coming up with good ones.

• Convergence is ensured under very weak assumptions; namely irreducibility and aperiodicity.

• It is usually very easy to establish that an MCMC sampler converges towards π but very difficult to obtain rates of convergence.

2.10– The Gibbs Sampler

• Consider the target distribution $\pi(\theta)$ such that $\theta = \left(\theta^1, \theta^2\right)$.
Then the 2 component Gibbs sampler proceeds as follows.

Initialization:

- Select deterministically or randomly $\theta_0 = \left(\theta_0^1, \theta_0^2\right)$.

Iteration i; $i \geq 1$:

- Sample $\theta_i^1 \sim \pi\left(\theta^1 \middle| \theta_{i-1}^2\right)$.

- Sample $\theta_i^2 \sim \pi\left(\theta^2 \middle| \theta_i^1\right)$.

• Sampling from these conditional is often feasible even
when sampling from the joint is impossible (e.g. nuclear pump data).

• Clearly $\left\{\left(\theta_i^1, \theta_i^2\right)\right\}$ is a Markov chain and its transition kernel is

$$P\left(\left(\theta^1, \theta^2\right), \left(\widetilde{\theta}^1, \widetilde{\theta}^2\right)\right) = \pi\left(\widetilde{\theta}^1 \middle| \theta^2\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right).$$

• Then $\int\int \pi\left(\theta^1, \theta^2\right) P\left(\left(\theta^1, \theta^2\right), \left(\widetilde{\theta}^1, \widetilde{\theta}^2\right)\right) d\theta^1 d\theta^2$ satisfies

$$\int\int \pi\left(\theta^1, \theta^2\right) \pi\left(\widetilde{\theta}^1 \middle| \theta^2\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right) d\theta^1 d\theta^2$$

$$= \int \pi\left(\theta^2\right) \pi\left(\widetilde{\theta}^1 \middle| \theta^2\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right) d\theta^2$$

$$= \int \pi\left(\widetilde{\theta}^1, \theta^2\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right) d\theta^2$$

$$= \pi\left(\widetilde{\theta}^1\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right) = \pi\left(\widetilde{\theta}^1, \widetilde{\theta}^2\right)$$

2.12– Irreducibility

• This does not ensure that the Gibbs sampler does converge towards the invariant distribution!

• Additionally it is required to ensure irreducibility: loosely speaking the Markov chain can move to any set A such that $\pi(A) > 0$ for (almost) any starting point.

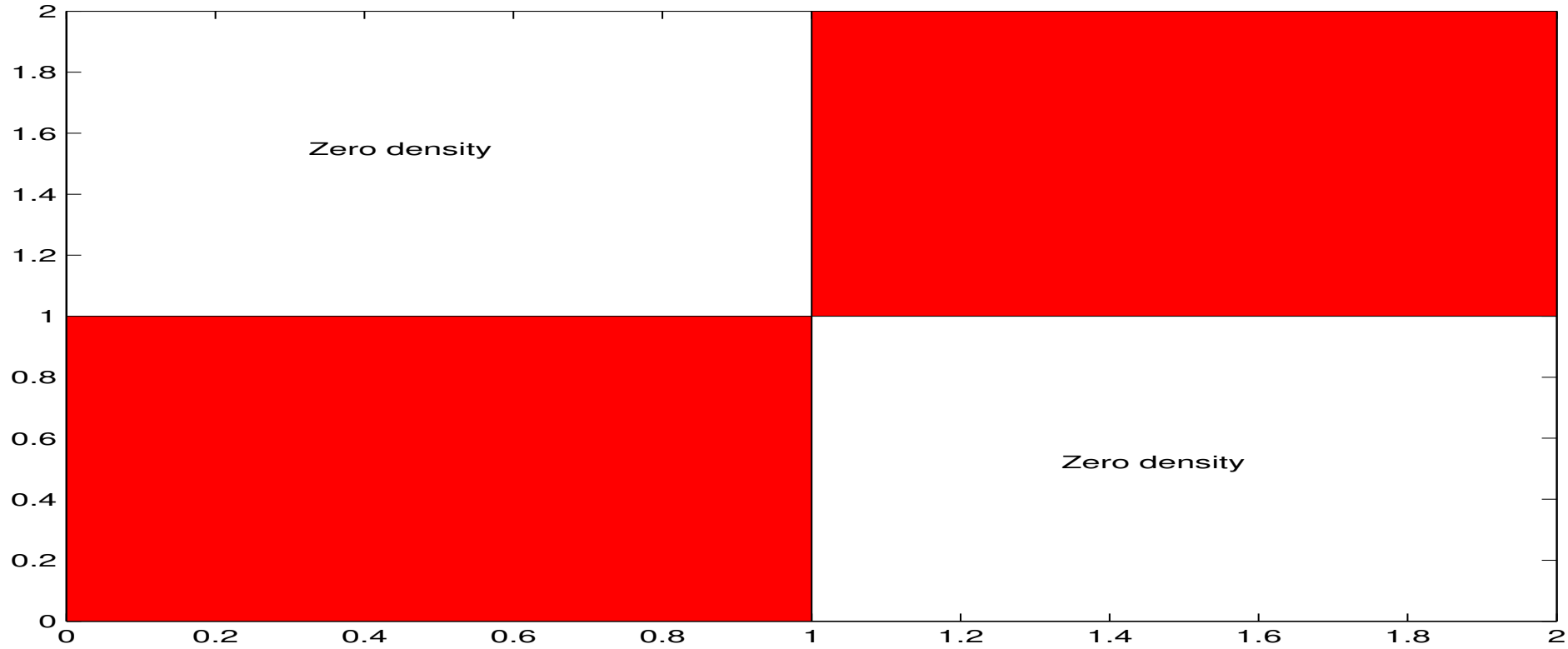
• This ensures that

$$\frac{1}{N}\sum_{n=1}^{N}\varphi\left(\theta_n^1,\theta_n^2\right)\rightarrow\int\varphi\left(\theta^1,\theta^2\right)\pi\left(\theta^1,\theta^2\right)d\theta^1 d\theta^2$$

but NOT that asymptotically $\left(\theta_n^1,\theta_n^2\right)\sim\pi$.

2.13– Irreducibility

A distribution that can lead to a reducible Gibbs sampler.



2.14– Aperiodicity

• Consider a simple example where $\mathbb{X} = \{1, 2\}$ and $P(1, 2) = P(2, 1) = 1$. Clearly the invariant distribution is given by $\pi(1) = \pi(2) = \frac{1}{2}$.

• However, we know that if the chain starts in $X_0 = 1$, then $X_{2n} = 1$ and $X_{2n+1} = 0$ for any n.

• We have

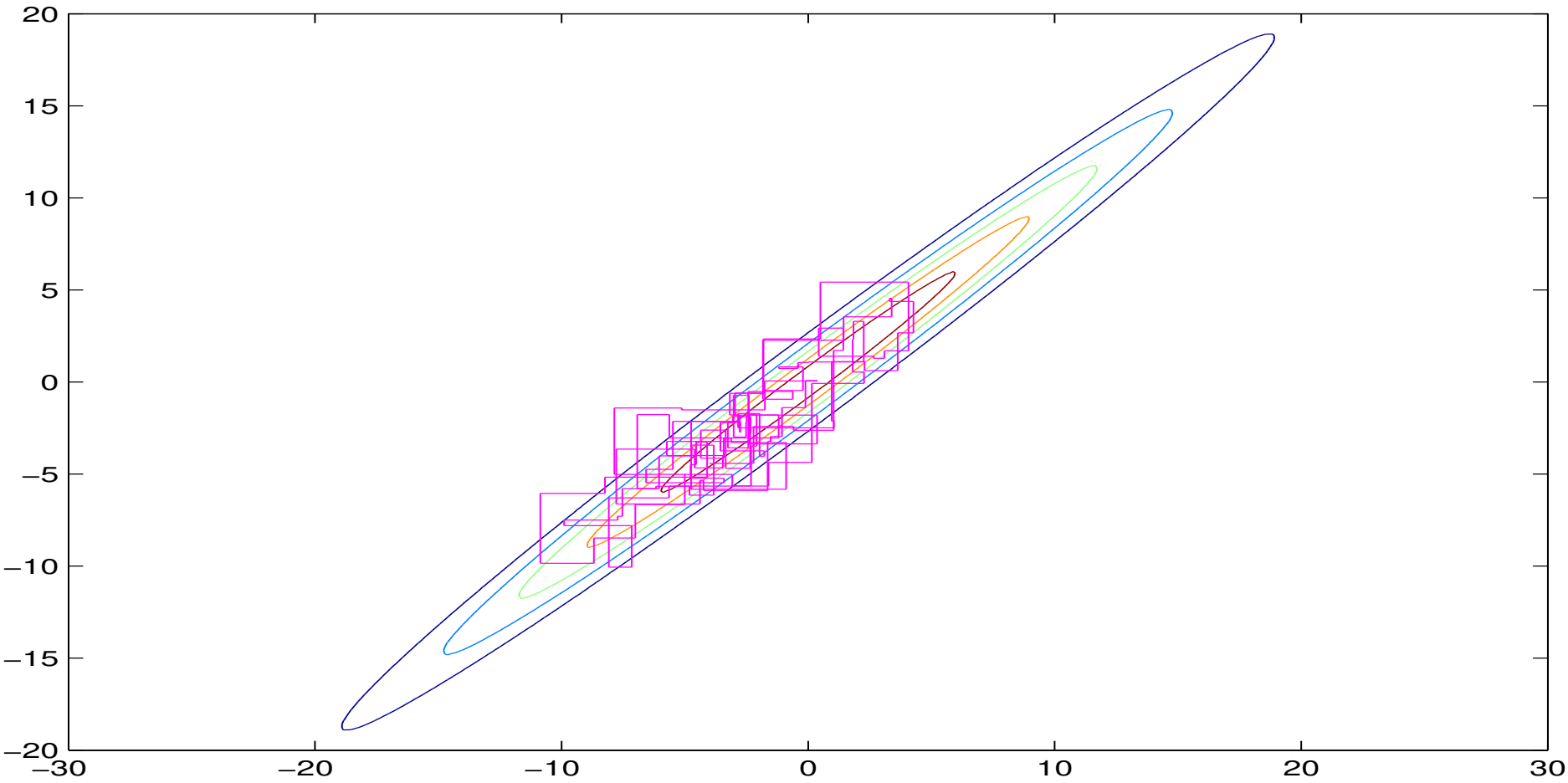
$$\frac{1}{N} \sum_{n=1}^{N} \varphi(X_n) \to \int \varphi(x) \pi(x) dx$$

but clearly X_n is NOT distributed according to π.

• You need to make sure that you do NOT explore the space in a periodic way to ensure that $X_n \sim \pi$ asymptotically.

2.15– About the Gibbs sampler

Even when irreducibility and aperiodicity are ensured,
the Gibbs sampler can still converge very slowly.



2.16– More about the Gibbs sampler

- If $\theta = (\theta_1, ..., \theta_p)$ where $p > 2$, the Gibbs sampling strategy still applies.

- Initialization:

  - Select deterministically or randomly $\theta_0 = \left(\theta_0^1, ..., \theta_0^p\right)$.

- Iteration i; $i \geq 1$:

  For $k = 1 : p$

  - Sample $\theta_i^k \sim \pi\left(\theta^k \middle| \theta_i^{-k}\right)$.

  where $\theta_i^{-k} = \left(\theta_i^1, ..., \theta_i^{k-1}, \theta_{i-1}^{k+1}, ..., \theta_{i-1}^p\right)$.

2.17– Random Scan Gibbs sampler

- Initialization:
  - Select deterministically or randomly $\theta_0 = \left(\theta_0^1, ..., \theta_0^p\right)$.
- Iteration i; $i \geq 1$:
  - Sample $K \sim U_{\{1,...,p\}}$.
  - Set $\theta_i^{-K} = \theta_{i-1}^{-K}$.
  - Sample $\theta_i^K \sim \pi\left(\theta^K \middle| \theta_i^{-K}\right)$.

  where $\theta_i^{-K} = \left(\theta_i^1, ..., \theta_i^{K-1}, \theta_i^{K+1}, ..., \theta_i^p\right)$.

2.18– Practical Recommendations

- Try to have as few "blocks" as possible.

- Put the most correlated variables in the same block.

- If necessary, reparametrize the model to achieve this.

- Integrate analytically as many variables as possible: pretty algorithms can be much more inefficient than ugly algorithms.

- There is no general result telling strategy A is better than strategy B in all cases: you need experience.

2.19– Bayesian Variable Selection Example

- We select the following model

$$Y = \sum_{i=1}^{p} \beta_i X_i + \sigma V \text{ where } V \sim \mathcal{N}(0,1)$$

where we assume $\mathcal{IG}\left(\sigma^2; \frac{\nu_0}{2}, \frac{\gamma_0}{2}\right)$ and for $\alpha^2 << 1$

$$\beta_i \sim \frac{1}{2}\mathcal{N}\left(0, \alpha^2\delta^2\sigma^2\right) + \frac{1}{2}\mathcal{N}\left(0, \delta^2\sigma^2\right)$$

- We introduce a latent variable $\gamma_i \in \{0,1\}$ such that

$$\Pr(\gamma_i = 0) = \Pr(\gamma_i = 1) = \tfrac{1}{2},$$

$$\beta_i | \gamma_i = 0 \sim \mathcal{N}\left(0, \alpha^2\delta^2\sigma^2\right), \quad \beta_i | \gamma_i = 1 \sim \mathcal{N}\left(0, \delta^2\sigma^2\right).$$

2.20– A Bad Gibbs Sampler

• We have parameters $\left(\beta_{1:p}, \gamma_{1:p}, \sigma^2\right)$ and observe n observations $D = \{x_i, y_i\}_{i=1}^n$.

• A potential Gibbs sampler consists of sampling iteratively
from $p\left(\beta_{1:p} \middle| D, \gamma_{1:p}, \sigma^2\right)$ (Gaussian), $p\left(\sigma^2 \middle| D, \gamma_{1:p}, \beta_{1:p}\right)$ (inverse-Gamma)
and $p\left(\gamma_{1:p} \middle| D, \beta_{1:p}, \sigma^2\right)$.

• In particular

$$p\left(\gamma_{1:p} \middle| D, \beta_{1:p}, \sigma^2\right) = \prod_{i=1}^{p} p\left(\gamma_i \middle| \beta_i, \sigma^2\right)$$

and

$$p\left(\gamma_i = 1 \middle| \beta_i, \sigma^2\right) = \frac{\frac{1}{\sqrt{2\pi}\delta\sigma} \exp\left(-\frac{\beta_i^2}{2\delta^2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi}\delta\sigma} \exp\left(-\frac{\beta_i^2}{2\delta^2\sigma^2}\right) + \frac{1}{\sqrt{2\pi}\alpha\delta\sigma} \exp\left(-\frac{\beta_i^2}{2\alpha^2\delta^2\sigma^2}\right)}.$$

• The Gibbs sampler becomes reducible as α goes to zero.

2.21– Bayesian Variable Selection Example

• This is the result of bad modelling and bad algorithm.
You would like to put $\alpha \underset{p}{\simeq} 0$ and write

$$Y = \sum_{i=1} \gamma_i \beta_i X_i + \sigma V \text{ where } V \sim \mathcal{N}(0,1)$$

where $\gamma_i = 1$ if X_i is included or $\gamma_i = 0$ otherwise. However this suggests that β_i is defined even when $\gamma_i = 0$.

• A neater way to write such models is to write

$$Y = \sum_{\{i:\gamma_i=1\}} \beta_i X_i + \sigma V = \beta_\gamma^{\mathrm{T}} X_\gamma + \sigma V$$

where, for a vector $\gamma = (\gamma_1, ..., \gamma_p)$, $\beta_\gamma = \{\beta_i : \gamma_i = 1\}$, $X_\gamma = \{X_i : \gamma_i = 1\}$ and $n_\gamma = \sum_{i=1}^p \gamma_i$.
• Prior distributions

$$\pi_\gamma\left(\beta_\gamma, \sigma^2\right) = \mathcal{N}\left(\beta_\gamma; 0, \delta^2\sigma^2 I_{n_\gamma}\right) \mathcal{IG}\left(\sigma^2; \frac{\nu_0}{2}, \frac{\gamma_0}{2}\right)$$

and $\pi(\gamma) = \prod_{i=1}^p \pi(\gamma_i) = 2^{-p}$.

2.22– A Better Gibbs Sampler

- We are interested in sampling from the trans-dimensional distribution $\pi\left(\gamma, \beta_\gamma, \sigma^2 \middle| D\right)$

- However, we know that

$$\pi\left(\gamma, \beta_\gamma, \sigma^2 \middle| D\right) = \pi\left(\gamma \middle| D\right) \pi\left(\beta_\gamma, \sigma^2 \middle| D, \gamma\right)$$

where

$$\pi\left(\gamma \middle| D\right) \propto \pi\left(D \middle| \gamma\right) \pi\left(\gamma\right)$$

and

$$\begin{aligned} \pi\left(D \middle| \gamma\right) &= \int \pi\left(D, \beta_\gamma, \sigma^2 \middle| \gamma\right) d\beta_\gamma d\sigma^2 \\ &\propto \Gamma\left(\frac{\nu_0 + n}{2} + 1\right) \delta^{-n_\gamma} \left|\Sigma_\gamma\right|^{1/2} \left(\frac{\gamma_0 + \sum_{i=1}^n y_i^2 - \mu_\gamma^{\mathrm{T}} \Sigma_\gamma^{-1} \mu_\gamma}{2}\right)^{-\left(\frac{\nu_0+n}{2}+1\right)}. \end{aligned}$$

2.23– Bayesian Variable Selection Example

- $\pi\left(\gamma\middle|\, D\right)$ is a discrete probability distribution with 2^p potential values.
- We can use the Gibbs sampler to sample from it.
- Initialization:
  - Select deterministically or randomly $\gamma_0 = \left(\gamma_0^1, ..., \gamma_0^p\right)$.
- Iteration i; $i \geq 1$:

  For $k = 1 : p$

  - Sample $\gamma_i^k \sim \pi\left(\gamma^k \middle|\, D, \gamma_i^{-k}\right)$.

  where $\gamma_i^{-k} = \left(\gamma_i^1, ..., \gamma_i^{k-1}, \gamma_{i-1}^{k+1}, ..., \gamma_{i-1}^p\right)$.

  - Optional step: Sample $\left(\beta_{\gamma,i}, \sigma_i^2\right) \sim \pi\left(\beta_\gamma, \sigma^2 \middle|\, D, \gamma_i\right)$.

2.23– Bayesian Variable Selection Example

• This very simple sampler is much more efficient than the previous one.

• However, it can also mix very slowly because the components are updated one at a time.

• Updating correlated components together would increase significantly the convergence speed of the algorithm at the cost of an increased complexity.

2.23– Bayesian Variable Selection Example

• The Gibbs sampler is a generic tool to sample approximately from high-dimensional distributions.

• Each time you face a problem, you need to think hard about it to design an efficient algorithm.

• Except the choice of the partitions of parameters, the Gibbs sampler is parameter free; this does not mean it is efficient.