Stat 461-561: Quiz 3 Solutions

Friday 16th March 2007

Exercise 1. Suppose that $x_{1:n} = (x_1, ..., x_n)$ is a a random sample from a Poisson distribution with unknown mean θ. Two models for the prior distribution of θ are contemplated:

$$\begin{aligned} H_0 &: \quad \pi_0(\theta) = \exp(-\theta)\, 1_{(0,\infty)}(\theta), \\ H_1 &: \quad \pi_1(\theta) = \theta \exp(-\theta)\, 1_{(0,\infty)}(\theta). \end{aligned}$$

(a) Calculate the posterior mean $\mathbb{E}[\theta | x_{1:n}]$ under both models.
We have

$$\begin{aligned} f(x_{1:n} | \theta) &= \prod_{i=1}^{n} \exp(-\theta) \frac{\theta^{x_i}}{x_i!} \\ &= \exp(-n\theta)\, \theta^{S_n} \end{aligned}$$

where $S_n = \sum_{i=1}^{n} x_i$. So

$$\begin{aligned} \pi_0(\theta | x_{1:n}) &\propto \exp(-\theta) \exp(-n\theta)\, \theta^{S_n} \\ &\propto \exp(-(n+1)\theta)\, \theta^{S_n}. \end{aligned}$$

So $\pi_0(\theta | x_{1:n}) = Gamma(\theta; S_n + 1, n + 1)$ and

$$\mathbb{E}[\theta | x_{1:n}] = \frac{S_n + 1}{n + 1}.$$

We also have

$$\begin{aligned} \pi_1(\theta | x_{1:n}) &\propto \theta \exp(-\theta) \exp(-n\theta)\, \theta^{S_n} \\ &\propto \exp(-(n+1)\theta)\, \theta^{S_n+1} \end{aligned}$$

so $\pi_1(\theta | x_{1:n}) = Gamma(\theta; S_n + 2, n + 1)$ and

$$\mathbb{E}[\theta | x_{1:n}] = \frac{S_n + 2}{n + 1}.$$

The first model puts more mass on smaller values of θ so the posterior distribution $\pi_0(\theta | x_{1:n})$ is shifted to the left of $\pi_1(\theta | x_{1:n})$.

(b) Calculate the Bayes factor

$$B_{01} = \frac{\pi(x_{1:n} | H_0)}{\pi(x_{1:n} | H_1)}.$$

The Bayes factor is given by

$$\begin{aligned} B_{01} &= \frac{\int \exp\left(-\left(n+1\right)\theta\right)\theta^{S_n}d\theta}{\int \exp\left(-\left(n+1\right)\theta\right)\theta^{S_n+1}d\theta} \\ &= \frac{\Gamma\left(S_n+1\right)/\left(\left(n+1\right)^{S_n+1}\right)}{\Gamma\left(S_n+2\right)/\left(\left(n+1\right)^{S_n+2}\right)} \\ &= \frac{n+1}{S_n+1}. \end{aligned}$$

(c) Assuming that $\pi(H_0) = \pi(H_1) = \frac{1}{2}$, compute the posterior probabilities $\pi(H_0|x_{1:n})$ and $\pi(H_1|x_{1:n})$.

We have

$$\begin{aligned} \pi\left(H_0|x_{1:n}\right) &= 1-\pi\left(H_1|x_{1:n}\right) \\ &= \frac{\pi\left(x_{1:n}|H_0\right)\pi\left(H_0\right)}{\pi\left(x_{1:n}|H_0\right)\pi\left(H_0\right)+\pi\left(x_{1:n}|H_1\right)\pi\left(H_1\right)} \\ &= \frac{\pi\left(x_{1:n}|H_0\right)}{\pi\left(x_{1:n}|H_0\right)+\pi\left(x_{1:n}|H_1\right)} \\ &= \frac{1}{1+B_{01}^{-1}} = \frac{1}{1+\frac{S_n+1}{n+1}} \\ &= \frac{n+1}{S_n+n+2}. \end{aligned}$$

Exercise 2.

Let θ be a real-valued parameter and let $f(x|\theta)$ be the probability density function of an observation x given θ. The prior distribution is of the form

$$\pi(\theta) = \beta\delta_{\theta_0}(\theta) + (1-\beta)\,g(\theta)$$

where $g(\theta)$ is a standard probability density function.

(a) Derive an expression for $\pi(\theta_0|x)$, the posterior probability of $H_0 : \theta = \theta_0$.

We have

$$\pi\left(\theta_0|x\right) = \frac{\beta f\left(x|\theta_0\right)}{\beta f\left(x|\theta_0\right)+\left(1-\beta\right)m\left(x\right)}$$

where

$$m(x) = \int f(x|\theta)\,m(\theta)\,d\theta.$$

(b) Derive the Bayes factor for the null hypothesis H_0 against the alternative $H_1 : \theta \sim g(\theta)$, that is

$$B(x) = \frac{\pi\left(x|H_0\right)}{\pi\left(x|H_1\right)}.$$

The Bayes factor is

$$B(x) = \frac{f\left(x|\theta_0\right)}{m\left(x\right)}.$$

(c) Express $\pi(\theta_0|x)$ in terms of $B(x)$.

We have

$$\pi(\theta_0|\,x) = \left(1 + \frac{1-\beta}{\beta B(x)}\right)^{-1}.$$

(d) Explain how you would use $B(x)$ to construct a most powerful test of size α for H_0, against the alternative H_1. [Hint: Apply Neyman-Pearson to test hypothesis on the distribution of X].

The simple hypothesis $\theta = \theta_0$ can be tested against the simple hypothesis that the density of x is $m(x)$; i.e. $H_0 : X \sim f(x|\,\theta_0)$ and $H_1 : X \sim m(x)$. The Neyman-Pearson lemma says that the most powerful test of size α rejects H_0 when $B(x) = \frac{f(x|\theta_0)}{m(x)} < C_\alpha$, where C_α is chosen such that

$$\int_{\{x:B(x)<C_\alpha\}} f(x|\,\theta_0)\,dx = \alpha.$$

Suppose that $X_1, ..., X_n$ are i.i.d. with $X_i \sim \mathcal{N}(\theta, v)$. Let $H_0 : \theta = 0$, let $\beta = \frac{1}{2}$ and let

$$g(\theta) = \mathcal{N}(\theta; 0, w).$$

Show that if the sample mean $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is observed to be $10\,(v/n)^{1/2}$, then

(f) the posterior probability of H_0 converges to 1, as $n \to \infty$.

The Bayes factor is given by

$$\frac{\frac{1}{\sqrt{\frac{v}{n}}}\exp\left(-\frac{10^2\frac{v}{n}}{2\frac{v}{n}}\right)}{\frac{1}{\sqrt{\frac{v}{n}+w}}\exp\left(-\frac{10^2\frac{v}{n}}{2\left(\frac{v}{n}+w\right)}\right)} \to \infty \text{ as } n \to \infty.$$

(e) the most powerful test of size $\alpha = 0.05$ will reject H_0 for any value of n [Hint: Apply Neyman-Pearson to test hypothesis on the distribution of $\overline{X}$]

We now have $x = (x_1, ..., x_n)$. The statistic $\overline{x}$ is sufficient. Under H_0, $\overline{x} \sim \mathcal{N}\left(0, \frac{v}{n}\right)$ and under H_1, $\overline{x} \sim \mathcal{N}\left(0, \frac{v}{n} + w\right)$. So Neyman-Pearson rejects when

$$|\overline{x}| > c$$

where c is selected as

$$\begin{aligned} & \int_{\{\overline{x}:|\overline{x}|>c\}} \mathcal{N}\left(\overline{x}; 0, \frac{v}{n}\right) d\overline{x} \\ = & \int_{\left\{u:|u|>c\sqrt{\frac{n}{v}}\right\}} \mathcal{N}(u; 0, 1)\,du. \\ = & \ \alpha \end{aligned}$$

so we have $c\sqrt{\frac{n}{v}} = z_{\alpha/2}$ and

$$|\overline{x}|\sqrt{\frac{n}{v}} > z_{\alpha/2}.$$

For $\overline{x} = 10\left(\upsilon/n\right)^{1/2}$, we have

$$\left|\overline{x}\right|\sqrt{\frac{n}{\upsilon}} = 10 > z_{0.05/2}$$

so H_0 is rejected.

Exercise 3. Consider $x = (x_1, x_2)$ with the following distribution

$$f\left(x_1, x_2\right|\theta) \propto \int_0^\infty t^{2n-1}\exp\left(-\frac{1}{2}\left(t^2 + n\left(x_1 t - \zeta\right)^2 + n\left(x_2 t - \xi\right)^2\right)\right)dt$$

with $\theta = (\zeta, \xi) \in \mathbb{R}\times\mathbb{R}$. The prior distribution is improper

$$\pi\left(\theta\right) \propto 1.$$

(a) Show that $\pi\left(\zeta\right|x) = \pi\left(\zeta\right|x_1)$ and that $f\left(x_1\right|\theta) = f\left(x_1\right|\zeta)$ but that $\pi\left(\zeta\right|x_1)$ is not proportional to $f\left(x_1\right|\zeta)$.

We have

$$\begin{aligned}\pi\left(\zeta\right|x) &\propto \int\int_0^\infty t^{2n-1}\exp\left(-\frac{1}{2}\left(t^2 + n\left(x_1 t - \zeta\right)^2 + n\left(x_2 t - \xi\right)^2\right)\right)dtd\xi \\ &\propto \int_0^\infty t^{2n-1}\exp\left(-\frac{1}{2}\left(t^2 + n\left(x_1 t - \zeta\right)^2\right)\right)dt\end{aligned}$$

so $\pi\left(\zeta\right|x)$ is only a function of x_1. We also have

$$\begin{aligned}f\left(x_1\right|\theta) &= \int f\left(x_1, x_2\right|\theta)\,dx_2 \\ &\propto \int\int_0^\infty t^{2n-1}\exp\left(-\frac{1}{2}\left(t^2 + n\left(x_1 t - \zeta\right)^2 + nt^2\left(x_2 - \frac{\xi}{t}\right)^2\right)\right)dtdx_2 \\ &\propto \int_0^\infty t^{2n-2}\exp\left(-\frac{1}{2}\left(t^2 + n\left(x_1 t - \zeta\right)^2\right)\right)dt\end{aligned}$$

so $f\left(x_1\right|\theta)$ is only a function of ζ. However, $\pi\left(\zeta\right|x)$ is not proportional to $f\left(x_1\right|\theta)$.

Hints.

• The Poisson distribution of parameter θ is a distribution on the set of integers $0, 1, 2, ...$ such that $f\left(x\right|\theta) = \exp\left(-\theta\right)\frac{\theta^x}{x!}$ and we have $\mathbb{E}\left[X\right] = \theta$.

• The Gamma distribution $Gamma\left(a, b\right)$ admits for density

$$\frac{b^a}{\Gamma\left(a\right)}x^{a-1}\exp\left(-bx\right)1_{(0,\infty)}\left(x\right)$$

and mean $\mathbb{E}\left[X\right] = \frac{a}{b}$.

• For integers $n \geq 1$, we have $\Gamma\left(n\right) = \left(n-1\right)!$