Stat 461-561: Quiz 3

Wednesday 19th March 2008

• **Exercise 1**. Let $X_1, X_2, \ldots, X_n$ be n independent observations from a normal of mean m and variance τ denoted $\mathcal{N}(m, \tau)$. We assume that m is *known*.

Question 1.1: [**2 points**] Establish the likelihood ratio test for $H_0 : \tau = 1$ versus $H_1 : \tau \neq 1$. Simplify as much as possible the expression of the test.

We have

$$\lambda(x_{1:n}) = \frac{f(x_{1:n}|\,\tau = 1)}{f(x_{1:n}|\,\widehat{\tau})}$$

where $\widehat{\tau}$ is the MLE given by

$$\widehat{\tau} = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2.$$

So

$$\begin{aligned}\lambda(x_{1:n}) &= \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i - m)^2\right)}{(\widehat{\tau})^{-n/2}\exp\left(-\frac{1}{2\widehat{\tau}}\sum_{i=1}^{n}(x_i - m)^2\right)} \\ &= (\widehat{\tau})^{n/2}\exp\left(-\frac{n\widehat{\tau}}{2} + \frac{n}{2}\right).\end{aligned}$$

Hence the test is of the form $\lambda(x_{1:n}) < c$ that is

$$\log(\widehat{\tau}) - \widehat{\tau} \leq c'.$$

Question 1.2: [**2 points**] Find the Wald test of (asymptotic) size α. Express the result in terms of a z_β where $\Pr(Z > z_\beta) = \beta$ for $Z \sim \mathcal{N}(0, 1)$.

The Wald test is given by

$$W_n = \sqrt{nI(1)}\,(\widehat{\tau} - 1).$$

We have

$$I(1) = -\mathbb{E}\left[\left.\frac{\partial^2 \log f(X|\,\tau)}{\partial \tau^2}\right|_{\tau=1}\right]$$

where

$$\begin{aligned}\log f(x|\,\tau) &= -\frac{1}{2}\log\tau - \frac{1}{2\tau}(x - m)^2, \\ \frac{\partial \log f(x|\,\tau)}{\partial \tau} &= -\frac{1}{2\tau} + \frac{1}{2\tau^2}(x - m)^2, \\ \frac{\partial^2 \log f(x|\,\tau)}{\partial \tau^2} &= \frac{1}{2\tau^2} - \frac{1}{\tau^3}(x - m)^2.\end{aligned}$$

so

$$I(1) = -\frac{1}{2\tau^2} + \frac{\tau^2}{\tau^3} = \frac{1}{2\tau^2} = \frac{1}{2}.$$

It follows that

$$\begin{aligned} W_n &= \sqrt{nI(1)}\,(\widehat{\tau}-1) \\ &= \sqrt{\frac{n}{2}}\,(\widehat{\tau}-1) \end{aligned}$$

and to obtain a test of asymptotic size α we select

$$|W_n| \geq z_{\alpha/2}.$$

Question 1.3: **[1 point]** Compute the power function of the Wald test (assuming n is large enough that the normal approximation holds). Express the result in terms of the cdf of the standard normal denoted $\Phi(\cdot)$; i.e. $\Phi(u) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} \exp\left(-z^2/2\right) dz$.

We have for the power function

$$\begin{aligned} \beta(\tau) &= P_\tau(|W_n| \geq c) \\ &= 1 - P_\tau(|W_n| \leq c) \\ &= 1 - (\Phi(c) - \Phi(-c)) \end{aligned}$$

If $\tau = 1$ then $W_n \sim \mathcal{N}(0,1)$ and by construction $\beta(\tau) = \alpha$. If $\tau \neq 1$ then

$$\begin{aligned} W_n &= \sqrt{nI(1)}\,(\widehat{\tau}-1) \\ &= \sqrt{nI(1)}\,(\widehat{\tau}-1) \\ &= \sqrt{nI(1)}\,(\widehat{\tau}-\tau) + \sqrt{nI(1)}\,(\tau-1) \\ &= \sqrt{\frac{I(1)}{I(\tau)}}\sqrt{nI(\tau)}\,(\widehat{\tau}-\tau) + \sqrt{nI(1)}\,(\tau-1) \end{aligned}$$

so asymptotically $W_n \sim \mathcal{N}(\underbrace{\sqrt{nI(1)}\,(\tau-1)}_{m}, \underbrace{\frac{I(1)}{I(\tau)}}_{\sigma^2})$. Thus

$$\beta(\tau) = P_\tau(|W_n| \geq c) = 1 - P_\tau(-c \leq m + \sigma Z \leq c)$$

with

$$P_\tau\left(\frac{-c-m}{\sigma} \leq Z \leq \frac{c-m}{\sigma}\right) = \Phi\left(\frac{c-m}{\sigma}\right) - \Phi\left(\frac{-c-m}{\sigma}\right).$$

So

$$\beta(\tau) = P_\tau(|W_n| \geq c) = 1 - \left(\Phi\left(\frac{c-m}{\sigma}\right) - \Phi\left(\frac{-c-m}{\sigma}\right)\right).$$

Question 1.4: **[1 point]** Establish the Rao test of (asymptotic) size α. Express the result in terms of a z_β where $\Pr(Z > z_\beta) = \beta$ for $Z \sim \mathcal{N}(0,1)$.

We have

$$R_n = \frac{\left.\frac{\partial \log f(X_{1:n}|\tau)}{\partial \tau}\right|_{\tau=1}}{\sqrt{nI(1)}} = \sqrt{\frac{2}{n}}\left(-\frac{n}{2}+\frac{1}{2}\sum_{i=1}^{n}(x_i-m)^2\right).$$

To obtain a test of asymptotic size α we select

$$|R_n| \geq z_{\alpha/2}.$$

• **Exercise 2**. Let $X_1, X_2, \ldots, X_n$ be n independent observations from a distribution $g(x)$ and consider

$$\Psi(x, \theta_1, \theta_2) = \begin{pmatrix} x-\theta_1 \\ (x-\theta_1)^2-\theta_2 \end{pmatrix}.$$

The solution is in the lecture notes.