Stat 561: Quiz 2

Friday 16th January 2007

Exercise 1. Assume we receive a single observation from the density

$$f\left(x|\,\theta\right)=\theta x^{\theta-1}1_{(0,1)}\left(x\right)$$

where $\theta>1$.

• *State the Neyman-Pearson lemma and choose an associated test statistic to test $H_0:\theta=\theta_0$ versus $H_1:\theta=\theta_1$ when $\theta_1>\theta_0$.*

Look at the lecture notes for the statement. We have

$$\frac{L\left(\theta_1|\,\mathbf{x}\right)}{L\left(\theta_0|\,\mathbf{x}\right)}=\frac{\theta_1}{\theta_0}x^{\theta_1-\theta_0}.$$

It follows that a test statistic is X and the rejection region of the test is of the form

$$R=\{X:X>c\}$$

as $\theta_1-\theta_0>0$.

• *Tune the test so that it has size α $(0<\alpha<1)$ when $\theta_1>\theta_0$.*

Given the rejection region of the test, we have

$$P_{\theta_0}\left(X>c\right)=\int_c^1\theta_0x^{\theta_0-1}dx=\left[x^{\theta_0}\right]_c^1=1-c^{\theta_0}$$

so $P_{\theta_0}\left(X>c\right)=\alpha$ yields $c=\left(1-\alpha\right)^{1/\theta_0}$.

• *Derive a formula for the power of your α size test at θ_1 when $\theta_1>\theta_0$.*

The power of the test is simply

$$P_{\theta_1}\left(X>c\right)=1-\left(1-\alpha\right)^{\theta_1/\theta_0}.$$

Exercise 2. Let $X_1,X_2,...,X_n$ be independent identically distributed from the uniform distribution on $(0,\theta)$ (where $\theta>0$); i.e.

$$f\left(x|\,\theta\right)=\theta^{-1}1_{(0,\theta)}\left(x\right).$$

Let $M_n=\max_{i=1,...,n}X_i$.

• *We want to test $H_0:\theta\geq\theta_0$ versus $H_1:\theta<\theta_0$. Consider a test with the following rejection region*

$$\{X_1,X_2,...,X_n:M_n<c\}.$$

Find the critical value c so that the test is a size α test.

We have

$$\begin{aligned} P_\theta(M_n < c) &= P_\theta(X_1 < c, ..., X_n < c) \\ &= \prod_{k=1}^{n} P_\theta(X_k < c) \\ &= P_\theta(X_1 < c)^n. \end{aligned}$$

Now $P_\theta(X < c)^n \leq \alpha$ for $\theta \geq \theta_0$ yields $c = \alpha^{1/n}\theta_0$.

• *How should we change the test to test $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Find the critical value c at level α for this new test.*

Similarly $P_\theta(M_n > c) = P_\theta(X_1 > c)^n = \alpha$ yields $c = \theta_0(1-\alpha)^{1/n}$.

Exercise 3.

Let $X_1, X_2, ..., X_n$ be independent identically distributed from the following exponential distribution $f(x|\theta)$ admitting the density

$$f(x|\theta) = \theta^{-1}\exp\left(-\theta^{-1}x\right)1_{(0,\infty)}(x)$$

where $\theta > 0$.

• *Establish the expression of the Neyman-Pearson test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Distinguish the cases $\theta_1 > \theta_0$ and $\theta_1 < \theta_0$.*

We have

$$\frac{L(\theta_1|\mathbf{x})}{L(\theta_0|\mathbf{x})} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left(\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i\right).$$

So the test is of the form $\sum_{i=1}^n x_i > c$ if $\theta_1 > \theta_0$ and $\sum_{i=1}^n x_i < c$ if $\theta_1 < \theta_0$.

• *Tune the test so that it has level α (express the critical value in terms of the quantile of a Gamma distribution).*

Under H_0, we have $X_i \sim \Gamma(n, \theta_0)$. So we can choose c to be the quantile of order $1-\alpha$ of a $\Gamma(n,\theta_0)$ distribution if $\theta_1 > \theta_0$ whereas c is the quantile of order α of a $\Gamma(n,\theta_0)$ if $\theta_1 < \theta_0$.

• *We now wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_1$. Recall the definition of the Likelihood Ratio Test statistic $\lambda(\mathbf{x})$ and show that*

$$\lambda(\mathbf{x}) = \left(\frac{\overline{x}}{\theta_0}\right)^n \exp\left(-\frac{n\overline{x}}{\theta_0} + n\right)$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$.

We have $\widehat{\theta}_{MLE} = \overline{x}$ so

$$\lambda(\mathbf{x}) = \frac{L(\theta_0|\mathbf{x})}{L(\overline{x}|\mathbf{x})} = \frac{\theta_0^{-n}\exp\left(-n\theta_0^{-1}\overline{x}\right)}{\overline{x}^{-n}\exp(-n)}.$$

• *Using the approximation* $\log(1-u) \approx -u - u^2/2$ *as $u \to 0$, show that as $n \to \infty$ we have under H_0*

$$-2\log\lambda(\mathbf{x}) \xrightarrow{\text{D}} \chi^2_{(1)}.$$

It follows that

$$
\begin{aligned}
-2\log\lambda(\mathbf{x}) &\approx n\left(1-\frac{\overline{x}}{\theta_0}\right)^2 \\
&= \left(\frac{\sqrt{n}}{\theta_0}(\overline{x}-\theta_0)\right)^2 \\
&\xrightarrow{\mathrm{D}} \chi^2_{(1)}
\end{aligned}
$$

• *Assuming that the model is misspecified; that is $X_i \sim g$ where $g(x)$ does not belong to the class of distributions of the form $f(x|\,\theta)$. Establish the asymptotic distribution of $-2\log\lambda(\mathbf{x})$ under H_0 and propose a test based on this result.*

It follows that

$$
-2\log\lambda(\mathbf{x}) \approx \left(\frac{\sqrt{n}}{\theta_0}(\overline{x}-\theta_0)\right)^2
$$

where

$$
\sqrt{n}(\overline{x}-\theta_0) \to \mathcal{N}\left(0, \frac{\int \left.\frac{\partial \log f(x|\theta)}{\partial\theta}\right|^2_{\theta=\theta_0} g(x)\,dx}{\left(\int \left.\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right|_{\theta=\theta_0} g(x)\,dx\right)^2}\right)
$$

so under H_0 we have

$$
\frac{\sqrt{n}}{\widehat{\sigma}_0}(\overline{x}-\theta_0) \xrightarrow{\mathrm{D}} \mathcal{N}(0,1)
$$

where $\widehat{\sigma}_0^2$ can be estimated consistently from the data.

Hints.

• The exponential distribution is a Gamma distribution $\Gamma(1,\theta)$. If $X_i \overset{\text{i.i.d.}}{\sim} \Gamma(1,\theta)$ then $\sum_{i=1}^n X_i \sim \Gamma(n,\theta)$.