Stat 461-561: Exercises 5

In Casella & Berger, Exercises 7.23, 7.24, 7.25 (Week 7) and 8.10, 8.11, 8.53 and 8.54 (Week 8)

Exercise 1 (Week 7) Let θ be a random variable in $(0,\infty)$ with density

$$\pi(\theta) \propto \theta^{\gamma-1} \exp(-\beta\theta)$$

where $\beta, \gamma \in (1,\infty)$.

• Calculate the mean and mode of θ.

• Suppose that $X_1, ..., X_n$ are random variables, which, conditional on θ, are independent and each have the Poisson distribution with parameter θ. Find the form of the posterior density of θ given $X_1 = x_1$,...,$X_n = x_n$. What is the posterior mean?

• Suppose that $T_1, \ldots, T_n$ are random variables, which, conditional on θ, are independent and each is exponentially distributed with parameter θ. What is the mode of the posterior distribution of θ, given $T_1 = t_1$,...,$T_n = t_n$?

Exercise 2 (Week 7). Let X_1,...,$X_n \overset{\text{i.i.d.}}{\sim}$ *Poisson* (λ).

• Let $\lambda \sim$ *Gamma* (α, β) be the prior. Show that the posterior is also a Gamma. Find the posterior mean.

• Find the Jeffreys's prior. Find the posterior.

Exercise 3 (Week 7). Suppose that, conditional on μ, X_1,...,X_n are i.i.d. $\mathcal{N}(\mu, \sigma_0^2)$ with σ_0^2 known. Suppose that $\mu \sim \mathcal{N}(\xi_0, \nu_0)$ where ξ_0, ν_0 are known. Let X_{n+1} be a single future observation from $\mathcal{N}(\mu, \sigma_0^2)$. Show that given $(X_1, ..., X_n)$, X_{n+1} is normally distributed with mean

$$\left(\frac{1}{\sigma_0^2/n} + \frac{1}{\nu_0}\right)^{-1} \left(\frac{\overline{X}}{\sigma_0^2/n} + \frac{\xi_0}{\nu_0}\right)$$

where $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ and variance

$$\sigma_0^2 + \left(\frac{1}{\sigma_0^2/n} + \frac{1}{\nu_0}\right)^{-1}.$$

Exercise 4 (Week 7). Suppose that X_1,...,X_n are i.i.d. $\mathcal{N}(\mu, \sigma^2)$ with both μ and σ^2 unknown. Let $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ and $s^2 = (n-1)^{-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$. Assume the improper prior

$$\pi(\mu, \sigma) \propto \sigma^{-1}.$$

Show that the marginal posterior distribution of $\sqrt{n}\left(\mu - \overline{X}\right)/s$ is the t distribution with $n-1$ degrees of freedom and find the marginal posterior distribution of σ.

Exercise 5 (Week 7). Let X_1,...,X_n be i.i.d. $\mathcal{N}(\mu, 1/\tau)$ and suppose that independent priors are placed on μ and τ. wit $\mu \sim \mathcal{N}\left(\xi, \kappa^{-1}\right)$ and $\tau \sim \mathcal{G}(\alpha, \beta)$. Show that the conditional posterior distributions $\pi(\mu|\, x_1, ..., x_n, \tau)$ and $\pi(\tau|\, x_1, ..., x_n, \mu)$ admit standard forms, namely normal and Gamma, and give their exact expressions.

Exercise 6 (Week 8). Let $X_1, ..., X_n$ be i.i.d. from $\mathcal{N}(\theta, \sigma^2)$ with σ^2 known. Consider $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$. Also suppose the prior for θ under H_1 is $\mathcal{N}(\mu, \tau^2)$. Show that the Bayes factor

$$\begin{aligned} B &= \frac{p(X_1, ..., X_n|\, H_0)}{p(X_1, ..., X_n|\, H_1)} \\ &= \left(1 + \frac{n\,\tau^2}{\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{n\overline{X}^2}{\sigma^2} - \frac{n}{n\tau^2 + \sigma^2}\left(\overline{X} - \mu\right)^2\right)\right). \end{aligned}$$

Exercise 7 (Week 8). Suppose $X \sim Bin(n, \theta)$ and consider testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_1$. Suppose under H_1, θ is uniformly distributed on $(0, 1)$. Show that the Bayes factor is given by

$$\begin{aligned} B &= \frac{p(X_1, ..., X_n|\, H_0)}{p(X_1, ..., X_n|\, H_1)} \\ &= \frac{(n+1)!}{x!\,(n-x)!}\theta_0^x (1 - \theta_0)^{n-x}. \end{aligned}$$

Using Stirling's approximation, i.e. $n! \approx \sqrt{2\pi} n^{n+1/2} \exp(-n)$, show that

$$B \approx \left(\frac{n}{2\pi\theta_0 (1 - \theta_0)}\right)^{1/2} \exp\left(-\frac{(x - n\theta_0)^2}{2n\theta_0 (1 - \theta_0)}\right).$$

Compare the Bayes factor to a standard 0.05 level test and show that for $n = 10000$ and $x = 5100$ the Bayes approach and the classical approach lead to opposite conclusions.

Exercise 8 (Week 8). Let $X_1, ..., X_n \overset{\text{i.i.d.}}{\sim} f(x|\,\theta)$ where

$$f(x|\,\theta) = \theta \exp(-\theta x)\, 1_{(0,\infty)}(x)$$

and $\theta \in (0, \infty)$ is an unknown parameter. We want to test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$. Define $S_n = X_1 + X_2 + \cdots + X_n$.

• Show that the test which rejects H_0 whenever $|S_n - n| > z_{\alpha/2}$ where $z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution, has size approximately α for large n.

• Suppose that the prior distribution on θ under H_1 is a Gamma distribution of parameter (a, b)

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1} \exp(-b\theta).$$

Show that the Bayes factor for H_0 against H_1, conditional on $S_n = s_n$, is

$$\frac{\Gamma(a)}{\Gamma(a+n)} \frac{(b + s_n)^{a+n} \exp(-s_n)}{b^a}.$$

• Suppose now that $a = b = 1$ and write $s_n = n + z_n\sqrt{n}$, so that, if $z_n = \pm z_{\alpha/2}$, the classical test will be just on the borderline of rejecting H_0 at the two-sided significance level α. Show that, as $n \to \infty$, provided $z_n \to \infty$ sufficiently slowly,

$$B \approx \sqrt{\frac{n}{2\pi}} \exp\left(1 - z_n^2/2\right).$$

• Hence show that there exists a sequence $\{s_n, n \geq 1\}$ such that, for the sequence of problems with $S_n = s_n$ for all n, H_0 is rejected at level α, for all sufficiently large n, *whatever* the value of $\alpha > 0$, but the Bayes factor $B \to \infty$ as $n \to \infty$.