Stat 461-561: Exercises 5

Remarks: Exercises 8.10 & 8.11 in C&B make implicit use of the incomplete Gamma function. No such question will be given at the exam.

Exercise C&B 8.53.

(a). We have

$$\begin{aligned}\pi(\theta) &= \pi(H_0)\,\pi(\theta|\,H_0) + \pi(H_1)\,\pi(\theta|\,H_1) \\ &= \frac{1}{2}\delta_0(\theta) + \frac{1}{2}\mathcal{N}\left(\theta; 0, \tau^2\right)\end{aligned}$$

so it is by construction a proper prior distribution.

(b) We have

$$\begin{aligned}\pi(H_0|\,x_{1:n}) &= \frac{\pi(x_{1:n}|\,H_0)\,\pi(H_0)}{\pi(x_{1:n}|\,H_0)\,\pi(H_0) + \pi(x_{1:n}|\,H_1)\,\pi(H_1)} \\ &= \frac{\pi(x_{1:n}|\,H_0)}{\pi(x_{1:n}|\,H_0) + \pi(x_{1:n}|\,H_1)}\end{aligned}$$

where

$$\pi(x_{1:n}|\,H_0) = \prod_{i=1}^{n} f(x_i|\,\theta = 0) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right)$$

and

$$\begin{aligned}\pi(x_{1:n}|\,H_1) &= \frac{1}{(2\pi\sigma^2)^{n/2}}\frac{1}{(2\pi)^{1/2}\tau} \\ &\quad\times\int \exp\left(-\frac{\sum_{i=1}^n (x_i-\theta)^2}{2\sigma^2}\right)\exp\left(-\frac{\theta^2}{2\tau^2}\right)d\theta.\end{aligned}$$

We have

$$\begin{aligned}&\frac{\sum_{i=1}^n (x_i-\theta)^2}{\sigma^2} + \frac{\theta^2}{\tau^2} \\ &= \theta^2\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right) - 2\left(\frac{\sum_{i=1}^n x_i}{\sigma^2}\right)\theta + \frac{\sum_{i=1}^n x_i^2}{\sigma^2} \\ &= \frac{(\theta-m)^2}{\upsilon^2} + \frac{\sum_{i=1}^n x_i^2}{\sigma^2} - \frac{m^2}{\upsilon^2}\end{aligned}$$

where $\upsilon^2 = \left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-1}$ and $m = \upsilon^2\left(\frac{\sum_{i=1}^n x_i}{\sigma^2}\right)$. So integrating out θ, we obtain

$$\pi(x_{1:n}|\,H_1) = \frac{\upsilon}{(2\pi\sigma^2)^{n/2}\tau}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2} + \frac{m^2}{2\upsilon^2}\right).$$

So

$$\begin{aligned}\pi\left(H_0 \mid x_{1:n}\right) &= \frac{\pi\left(x_{1:n} \mid H_0\right)}{\pi\left(x_{1:n} \mid H_0\right)+\pi\left(x_{1:n} \mid H_1\right)} \\ &= \frac{\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right)}{\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right)+\frac{\upsilon}{\tau}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}+\frac{m^2}{2\upsilon^2}\right)} \\ &= \frac{1}{1+\frac{\upsilon}{\tau}\exp\left(\frac{m^2}{\upsilon^2}\right)}\end{aligned}$$

(c) For the p-value, we take $P_{\theta_0}\left(\left|\overline{X}\right| \geq |\overline{x}|\right)$ where under H_0 we have $\overline{X} \sim \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)$. So

$$P_{\theta_0}\left(\left|\overline{X}\right| \geq |\overline{x}|\right) = 2P\left(Z \geq \frac{\sqrt{n}}{\sigma}\overline{x}\right).$$

where $Z \sim \mathcal{N}(0,1)$.

(d) (i) Graph omitted.

(ii). If $\overline{x} = \frac{\sigma z_{\alpha/2}}{\sqrt{n}}$ then

$$P_{\theta_0}\left(\left|\overline{X}\right| \geq |\overline{x}|\right) = 2P\left(Z \geq z_{\alpha/2}\right) = \alpha.$$

For the posterior probability, we have that

$$\pi\left(H_0 \mid x_{1:n}\right) = \frac{1}{1+\frac{\upsilon}{\tau}\exp\left(\frac{m^2}{\upsilon^2}\right)}$$

where

$$\begin{aligned}\frac{m^2}{\upsilon^2} &\sim \frac{\overline{x}^2}{\sigma^2/n} = \frac{n\overline{x}^2}{\sigma^2} \\ &= z_{\alpha/2}^2\end{aligned}$$

whereas $\frac{\upsilon}{\tau} = \frac{\sigma}{n\tau} \to 0$ so

$$\lim_{n\to\infty} \pi\left(H_0 \mid x_{1:n}\right) = 1.$$

Exercise C&B 8.54

(a) For $\pi(\theta) = \mathcal{N}\left(\theta; 0, \tau^2\right)$, we have shown (implicitly) in 8.53 (c) that

$$\pi\left(\theta \mid x_{1:n}\right) = \mathcal{N}\left(\theta; m, \upsilon^2\right)$$

where $\upsilon^2 = \left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-1}$ and $m = \upsilon^2\left(\frac{\sum_{i=1}^n x_i}{\sigma^2}\right)$. So

$$\begin{aligned}P\left(\theta \leq 0 \mid x_{1:n}\right) &= \int_{-\infty}^{0} \mathcal{N}\left(\theta; m, \upsilon^2\right) d\theta. \\ &= P\left(Z \leq -\frac{m}{\upsilon}\right)\end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$.

(b) For the p-value, we take $\sup_{\theta\leq 0} P_\theta\left(\overline{X}\geq\overline{x}\right)$. In this case, the sup is reached in $\theta=0$ where $\overline{X}\sim\mathcal{N}\left(0,\frac{\sigma^2}{n}\right)$. So the p-value is given by

$$P_{\theta=0}\left(\overline{X}\geq\overline{x}\right)=P\left(Z\geq\frac{\sqrt{n}}{\sigma}\overline{x}\right)=P\left(Z\leq-\frac{\sqrt{n}}{\sigma}\overline{x}\right).$$

where $Z\sim\mathcal{N}(0,1)$.

(c) For $\sigma^2=\tau^2=1$, we have $v^2=\frac{1}{n+1}$, $m=\frac{n\overline{x}}{n+1}$ and

$$P\left(\theta\leq 0|\, x_{1:n}\right)=P\left(Z\leq-\frac{m}{v}\right)=P\left(Z\leq-\frac{n\overline{x}}{(n+1)^{1/2}}\right)$$

whereas

$$P_{\theta=0}\left(\overline{X}\geq\overline{x}\right)=P\left(Z\leq-\sqrt{n}\overline{x}\right).$$

We have

$$\frac{n}{(n+1)^{1/2}}\leq\sqrt{n}$$

so

$$P\left(\theta\leq 0|\, x_{1:n}\right)\geq P_{\theta=0}\left(\overline{X}\geq\overline{x}\right).$$

(d) As $\tau^2\to\infty$, we have

$$v^2\to\frac{1}{n},\;\; m\to\overline{x}$$

and thus

$$\lim_{\tau^2\to\infty}P\left(\theta\leq 0|\, x_{1:n}\right)=\text{p-value}.$$

Exercise 6. Let $X_i\overset{\text{i.i.d.}}{\sim}\mathcal{N}\left(\theta,\sigma^2\right)$ with σ^2 known. We have

$$p\left(x_{1:n}|\, H_0\right)=\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right)$$

and

$$p\left(x_{1:n}|\, H_1\right)=\int\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{\sum_{i=1}^n(x_i-\theta)^2}{2\sigma^2}\right)\frac{1}{(2\pi\tau^2)^{1/2}}\exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right)d\theta.$$

We have for any k

$$\begin{aligned}&\sum_{i=1}^n(x_i-\theta)^2+k(\theta-\mu)^2\\=\;&(n+k)(\theta-m)^2+\frac{nk}{n+k}(\overline{x}-\mu)^2+\sum_{i=1}^n(x_i-\overline{x})^2\end{aligned}$$

where $m = (n\overline{x} + k\mu) / (n+k)$. Thus

$$\begin{aligned} & \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{\sigma^2} + \frac{(\theta-\mu)^2}{\tau^2} \\ = & \frac{\left(n\tau^2+\sigma^2\right)}{\sigma^2\tau^2}(\theta-m)^2 + \frac{n}{n\tau^2+\sigma^2}(\overline{x}-\mu)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\overline{x})^2 . \end{aligned}$$

Also we have

$$\int \exp\left(-\frac{\left(n\tau^2+\sigma^2\right)}{2\sigma^2\tau^2}(\theta-m)^2\right) d\theta = \left(\frac{2\pi\sigma^2\tau^2}{n\tau^2+\sigma^2}\right)^{1/2}.$$

Hence it follows that

$$\begin{aligned} p\left(x_{1:n}\middle| H_1\right) &= \frac{1}{\left(2\pi\sigma^2\right)^{n/2}}\left(\frac{\sigma^2}{n\tau^2+\sigma^2}\right)^{1/2} \\ &\quad \times \exp\left(-\frac{1}{2}\left(\frac{n}{n\tau^2+\sigma^2}(\overline{x}-\mu)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\overline{x})^2\right)\right) \end{aligned}$$

and so

$$\frac{p\left(x_{1:n}\middle| H_0\right)}{p\left(x_{1:n}\middle| H_1\right)} = \left(1+\frac{n\tau^2}{\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{n\overline{x}^2}{\sigma^2} - \frac{n}{n\tau^2+\sigma^2}(\overline{x}-\mu)^2\right)\right).$$

Exercise 7
We have

$$\begin{aligned} B_{01} &= \frac{\binom{n}{x}\theta_0^x(1-\theta_0)^{n-x}}{\int_0^1 \binom{n}{x}\theta_0^x(1-\theta_0)^{n-x}\,d\theta} \\ &= \frac{\theta_0^x(1-\theta_0)^{n-x}}{\int_0^1 \theta_0^{x+1-1}(1-\theta_0)^{n+1-x-1}\,d\theta} \\ &= \frac{\Gamma(n+2)\,\theta_0^x(1-\theta_0)^{n-x}}{\Gamma(x+1)\,\Gamma(n+1-x)} \\ &= \frac{(n+1)!}{x!\,(n-x)!}\theta_0^x(1-\theta_0)^{n-x}. \end{aligned}$$

Now we could use Stirling's approximation and further approximations to establish this approximation (see Exercise 8). Alternatively a much simpler solution consists of realising that B_{01} has the form of a Binomial distribution of parameter θ_0 up to a constant; i.e.

$$B_{01} = (n+1)\,Bin(\theta_0; x, n).$$

We know that for large n, we can approximate $Bin\left(\theta_0; x, n\right)$ by a Gaussian of mean $n\theta_0$ and variance $n\theta_0\left(1-\theta_0\right)$ so

$$\begin{aligned} B_{01} &\approx (n+1)\frac{1}{\sqrt{2\pi n\theta_0\left(1-\theta_0\right)}}\exp\left(-\frac{\left(x-n\theta_0\right)^2}{2n\theta_0\left(1-\theta_0\right)}\right) \\ &\approx \left(\frac{n}{2\pi\theta_0\left(1-\theta_0\right)}\right)^{1/2}\exp\left(-\frac{\left(x-n\theta_0\right)^2}{2n\theta_0\left(1-\theta_0\right)}\right). \end{aligned}$$

Exercise 8

(a) Under H_0, we have

$$\begin{aligned} \mathbb{E}\left[S_n\right] &= \mathbb{E}\left[X_1\right]=1, \\ \mathbb{V}\left[S_n\right] &= n\mathbb{V}\left[X_1\right]=n \end{aligned}$$

so, using the CLT, we know that asymptotically

$$S_n - n \sim \mathcal{N}\left(0, n\right)$$

and the result follows.

(b) We have

$$\pi\left(x_{1:n}\middle| H_0\right)=\prod_{i=1}^{n}\exp\left(-x_i\right)=\exp\left(-s_n\right),$$

and

$$\begin{aligned} \pi\left(x_{1:n}\middle| H_1\right) &= \int\theta^n\exp\left(-\theta\sum_{i=1}^{n}x_i\right)\frac{b^a\theta^{a-1}e^{-b\theta}}{\Gamma\left(a\right)}d\theta \\ &= \frac{b^a}{\Gamma\left(a\right)}\int\theta^{n+a-1}\exp\left(-\theta\left(\sum_{i=1}^{n}x_i+b\right)\right)d\theta \\ &= \frac{b^a}{\Gamma\left(a\right)}\frac{\Gamma\left(n+a\right)}{\left(s_n+b\right)^{n+a}}. \end{aligned}$$

So the result follows and

$$B_{01}=\frac{\pi\left(x_{1:n}\middle| H_0\right)}{\pi\left(x_{1:n}\middle| H_1\right)}=\frac{\Gamma\left(a\right)}{\Gamma\left(a+n\right)}\frac{\left(b+s_n\right)^{a+n}}{b^a}\exp\left(-s_n\right)$$

(c) If $a=b=1$ then

$$\begin{aligned} B_{01} &= \frac{\Gamma\left(1\right)}{\Gamma\left(1+n\right)}\left(1+s_n\right)^{1+n}\exp\left(-s_n\right) \\ &= \frac{\left(1+s_n\right)^{1+n}}{n!}\exp\left(-s_n\right). \end{aligned}$$

We have

$$\left(1+s_n\right)^{1+n}=\exp\left(\left(1+n\right)\log\left(1+s_n\right)\right)$$

where

$$\begin{aligned}\log(1+s_n) &= \log\left(1+n+\sqrt{n}z_n\right)\\ &= \log(1+n)+\log\left(1+\frac{\sqrt{n}z_n}{1+n}\right)\\ &\approx \log(1+n)+\frac{\sqrt{n}z_n}{1+n}-\frac{nz_n^2}{2(1+n)^2}\end{aligned}$$

It follows that

$$\begin{aligned}&(1+n)\log(1+s_n)-s_n\\ \approx\ &(1+n)\log(1+n)+\sqrt{n}z_n-\frac{nz_n^2}{2(1+n)}-n-\sqrt{n}z_n\\ \approx\ &(1+n)\log(1+n)-n-\frac{z_n^2}{2}.\end{aligned}$$

So

$$B_{01}\approx\frac{(n+1)^{n+1}\exp(-n)}{n!}\exp\left(-\frac{z_n^2}{2}\right).$$

Now using Stirling's approximation, we have

$$n!\approx n^n\exp(-n)\sqrt{2\pi n}$$

so

$$B_{01}\approx\frac{(n+1)^{n+1}}{n^n\sqrt{2\pi n}}\exp\left(-\frac{z_n^2}{2}\right).$$

Now

$$\begin{aligned}\frac{(n+1)^{n+1}}{n^n} &= (n+1)\left(1+\frac{1}{n}\right)^{n+1}\\ &\approx (n+1)\exp(1)\end{aligned}$$

so

$$B_{01}\approx\sqrt{\frac{n}{2\pi}}\exp\left(1-\frac{z_n^2}{2}\right).$$

(d) For s_n as defined above, H_0 will be rejected (it is just on the borderline) whereas we have

$$B_{01}\approx\sqrt{\frac{n}{2\pi}}\exp\left(1-\frac{z_\alpha^2}{2}\right)\to\infty.$$