Stat 461-561: Exercises 5

In Casella & Berger, Exercises 7.23, 7.24, 7.25 (Week 7) and 8.10, 8.11, 8.53 and 8.54 (Week 8)

Remark: There are several conventions available for parameterising Gamma and inverse Gamma distributions. I have adopted here the ones from C&B.

Exercise C&B 7.23.

The question is not very precise. We have $f\left(x|\,\sigma^2\right)=\mathcal{N}\left(x;m,\sigma^2\right)$.

If we assume that m is known, then

$$\begin{aligned}\pi\left(\sigma^2\middle|\,x_{1:n}\right) &\propto \frac{1}{\left(\sigma^2\right)^{\alpha+1}}\exp\left(-\frac{1}{\beta\sigma^2}\right)\\ &\quad\times\prod_{i=1}^{n}\frac{1}{\sigma}\exp\left(-\frac{\left(x_i-m\right)^2}{2\sigma^2}\right)\\ &\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}+\alpha+1}}\exp\left(-\frac{2/\beta+\sum_{i=1}^{n}\left(x_i-m\right)^2}{2\sigma^2}\right)\\ &\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}+\alpha+1}}\exp\left(-\frac{1/\beta+\left(n-1\right)S^2/2}{\sigma^2}\right)\end{aligned}$$

where $S^2:=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-m\right)^2$so $\pi\left(\sigma^2\middle|\,x_{1:n}\right)$ is an inverse Gamma distribution of parameters $\frac{n}{2}+\alpha$ and $\left(\frac{(n-1)S^2}{2}+\frac{1}{\beta}\right)^{-1}$.

If we assume that m is known *and* that $\pi\left(m\right)\propto 1$ then

$$\begin{aligned}\pi\left(\sigma^2,m\middle|\,x_{1:n}\right) &\propto \frac{1}{\left(\sigma^2\right)^{\alpha+1}}\exp\left(-\frac{1}{\beta\sigma^2}\right)\\ &\quad\times\prod_{i=1}^{n}\frac{1}{\sigma}\exp\left(-\frac{\left(x_i-m\right)^2}{2\sigma^2}\right)\end{aligned}$$

and

$$\begin{aligned}\left(x_i-m\right)^2 &= n^2\left(m-\overline{x}\right)^2+\sum_{i=1}^{n}x_i^2-n\overline{x}^2\\ &= n^2\left(m-\overline{x}\right)^2+\left(n-1\right)S^2\end{aligned}$$

where $\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $S^2:=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2$. So

$$\begin{aligned}\pi\left(\sigma^2,m\middle|\,x_{1:n}\right) &\propto \frac{1}{\left(\sigma^2\right)^{\alpha+1}}\exp\left(-\frac{1}{\beta\sigma^2}\right)\\ &\quad\times\frac{1}{\sigma^n}\exp\left(-\frac{n^2\left(m-\overline{x}\right)^2}{2\sigma^2}-\frac{1/\beta+\left(n-1\right)S^2/2}{\sigma^2}\right).\end{aligned}$$

By integrating out m, we obtain

$$\begin{aligned}\pi\left(\sigma^2, m \middle| x_{1:n}\right) &\propto \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left(-\frac{1}{\beta\sigma^2}\right) \\ &\quad \frac{1}{\sigma^{n-1}} \exp\left(-\frac{1/\beta + (n-1)\, S^2/2}{\sigma^2}\right).\end{aligned}$$

Hence, $\pi\left(\sigma^2 \middle| x_{1:n}\right)$ is an inverse Gamma distribution of parameters $\frac{n-1}{2} + \alpha$ and $\left(\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right)^{-1}$.

The mean of this distribution is given by

$$\begin{aligned}\mathbb{E}\left[\sigma^2 \middle| x_{1:n}\right] &= \frac{\frac{(n-1)S^2}{2} + \frac{1}{\beta}}{\frac{n-1}{2} + \alpha} \\ &= \frac{2/\beta + (n-1)\, S^2}{n - 1 + 2\alpha}.\end{aligned}$$

Exercise C&B 7.24.

- We have

$$\begin{aligned}\pi\left(\lambda | x_{1:n}\right) &\propto \lambda^{\alpha-1} \exp\left(-\lambda/\beta\right) \prod_{i=1}^{n} \exp\left(-\lambda\right) \frac{\lambda^{x_i}}{x_i!} \\ &\propto \lambda^{\alpha + \sum_{i=1}^{n} x_i - 1} \exp\left(-\lambda\left(1/\beta + n\right)\right).\end{aligned}$$

Hence we have $\pi\left(\lambda | x_{1:n}\right) = \mathit{Gamma}\left(\lambda; \alpha + \sum_{i=1}^{n} x_i, \left(1/\beta + n\right)^{-1}\right)$.

- Using C&B, page 624

$$\begin{aligned}\mathbb{E}\left[\lambda | x_{1:n}\right] &= \frac{\alpha + \sum_{i=1}^{n} x_i}{\beta^{-1} + n}, \\ \mathbb{V}\left[\lambda | x_{1:n}\right] &= \frac{\left(\alpha + \sum_{i=1}^{n} x_i\right)}{\left(1/\beta + n\right)^2}.\end{aligned}$$

Exercise C&B 7.25.

- We have

$$\begin{aligned}\pi\left(x_i\right) &= \int f\left(x_i | \theta_i\right) \pi\left(\theta_i\right) d\theta_i \\ &= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(x_i - \theta_i\right)^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi}\tau} \exp\left(-\frac{\left(\theta_i - \mu\right)^2}{2\tau^2}\right) d\theta_i\end{aligned}$$

where

$$\begin{aligned}&\frac{\left(x_i - \theta_i\right)^2}{\sigma^2} + \frac{\left(\theta_i - \mu\right)^2}{\tau^2} \\ &= \theta_i^2 \left(\frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) - 2\theta_i \left(\frac{x_i}{\sigma^2} + \frac{\mu}{\tau^2}\right) + \frac{x_i^2}{\sigma^2} + \frac{\mu^2}{\tau^2} \\ &= \left(\theta_i - m_i\right)^2 \left(\frac{1}{\widetilde{\sigma}^2}\right) - \frac{m_i^2}{\widetilde{\sigma}^2} + \frac{x_i^2}{\sigma^2} + \frac{\mu^2}{\tau^2}\end{aligned}$$

where

$$\begin{aligned} \frac{1}{\widetilde{\sigma}^2} &= \frac{1}{\sigma^2} + \frac{1}{\tau^2}, \\ m_i &= \widetilde{\sigma}^2 \left(\frac{x_i}{\sigma^2} + \frac{\mu}{\tau^2} \right). \end{aligned}$$

It follows that by integrating out θ_i, we have

$$\pi(x_i) = \frac{\widetilde{\sigma}}{\sqrt{2\pi}\sigma\tau} \exp\left(-\frac{1}{2} \left(\frac{x_i^2}{\sigma^2} + \frac{\mu^2}{\tau^2} - \frac{m_i^2}{\widetilde{\sigma}^2} \right) \right).$$

After rearranging the term in brackets, we obtain

$$\pi(x_i) = \mathcal{N}\left(x_i; \mu, \sigma^2 + \tau^2\right).$$

There is a much simpler way to do this calculations. We can rewrite

$$X_i = \theta_i + V_i$$

where $V_i \sim \mathcal{N}\left(0, \sigma^2\right)$ is independent of θ_i. We know that the sum of two Gaussian rvs is a Gaussian so we just need to compute its mean and variance to get its distribution

$$\begin{aligned} \mathbb{E}[X_i] &= \mathbb{E}[\theta_i + V_i] \\ &= \mathbb{E}[\theta_i] + \mathbb{E}[V_i] \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}[X_i] &= \mathbb{V}[\theta_i + V_i] \\ &= \mathbb{V}[\theta_i] + \mathbb{V}[V_i] \\ &= \tau^2 + \sigma^2. \end{aligned}$$

Moreover, the X_is are iid as

$$\begin{aligned} \pi(x_1, ..., x_n) &= \int \cdots \int \prod_{i=1}^n f(x_i | \theta_i) \pi(\theta_i) d\theta_{1:n} \\ &= \prod_{i=1}^n \int f(x_i | \theta_i) \pi(\theta_i) d\theta_i \\ &= \prod_{i=1}^n \pi(x_i) \end{aligned}$$

where $\pi(x_i)$ and $\pi(x_j)$ have the same functional form.

- Straightforward.

Exercise 1 (Week 7) Let θ be a random variable in $(0, \infty)$ with density

$$\pi(\theta) \propto \theta^{\gamma-1} \exp(-\beta\theta)$$

where $\beta, \gamma \in (1, \infty)$.

• Calculate the mean and mode of θ.

$\pi(\theta)$ is a Gamma distribution. We have

$$\mathbb{E}[\theta] = \frac{\gamma}{\beta}.$$

Moreover

$$\begin{aligned} \log \pi(\theta) &= cst + (\gamma - 1)\log\theta - \beta\theta, \\ \frac{d\log\pi(\theta)}{d\theta} &= \frac{(\gamma-1)}{\theta} - \beta \end{aligned}$$

thus the mode is

$$\theta = \frac{(\gamma-1)}{\beta}.$$

• Suppose that $X_1, ..., X_n$ are random variables, which, conditional on θ, are independent and each have the Poisson distribution with parameter θ. Find the form of the posterior density of θ given $X_1 = x_1$,...,$X_n = x_n$. What is the posterior mean?

We have

$$\pi(\theta | x_{1:n}) \propto \theta^{\gamma + \sum_{i=1}^n x_i - 1} \exp(-(\beta + n)\theta)$$

which with C&B convention is a Gamma distribution of parameters $\left(\gamma + \sum_{i=1}^n x_i, (\beta+n)^{-1}\right)$.

• Suppose that $T_1, \ldots, T_n$ are random variables, which, conditional on θ, are independent and each is exponentially distributed with parameter θ where we adopt the convention $f(x|\theta) = \theta\exp(-x\theta)\, 1_{(0,\infty)}(x)$. What is the mode of the posterior distribution of θ, given $T_1 = t_1$,...,$T_n = t_n$?

$$\begin{aligned} \pi(\theta | t_{1:n}) &\propto \theta^{\gamma-1}\exp(-\beta\theta)\prod_{i=1}^n \theta\exp(-\theta t_i) \\ &\propto \theta^{\gamma+n-1}\exp\left(-\theta\left(\beta + \sum_{i=1}^n t_i\right)\right) \end{aligned}$$

which is Gamma distribution of parameters $\left(\gamma + n, \left(\beta + \sum_{i=1}^n t_i\right)^{-1}\right)$.

We have

$$\log\pi(\theta | t_{1:n}) = cst + (\gamma + n - 1)\log\theta - \left(\beta + \sum_{i=1}^n t_i\right)\theta$$

thus the mode is located at

$$\theta = \left(\frac{\beta + \sum_{i=1}^n t_i}{\gamma + n - 1}\right)^{-1}.$$

Exercise 2 (Week 7). Let X_1,...,$X_n \overset{\text{i.i.d.}}{\sim} Poisson(\lambda)$.

• Let $\lambda \sim Gamma(\alpha, \beta)$ be the prior. Show that the posterior is also a Gamma. Find the posterior mean.

Trivial.

• Find the Jeffreys's prior. Find the posterior.

We have

$$\begin{aligned} \log f\left(x_{1:n}\middle|\,\lambda\right) &= \sum_{i=1}^{n} \log f\left(x_i\middle|\,\lambda\right) \\ &= \sum_{i=1}^{n} \left(-\lambda + x_i \log \lambda - x_i!\right) \end{aligned}$$

so

$$\begin{aligned} \frac{d \log f\left(x_{1:n}\middle|\,\lambda\right)}{d\lambda} &= -n + \sum_{i=1}^{n} \frac{x_i}{\lambda}, \\ \frac{d^2 \log f\left(x_{1:n}\middle|\,\lambda\right)}{d\lambda^2} &= -\sum_{i=1}^{n} \frac{x_i}{\lambda^2} \end{aligned}$$

so

$$I(\theta) = \mathbb{E}\left[\sum_{i=1}^{n} \frac{X_i}{\lambda^2}\right] = \frac{n}{\lambda}.$$

So Jeffreys' prior is

$$\pi(\lambda) \propto \frac{1}{\sqrt{\lambda}}.$$

Exercise 3 (Week 7). Suppose that, conditional on μ, X_1,...,X_n are i.i.d. $\mathcal{N}\left(\mu, \sigma_0^2\right)$ with σ_0^2 known. Suppose that $\mu \sim \mathcal{N}(\xi_0, \nu_0)$ where ξ_0, ν_0 are known. Let X_{n+1} be a single future observation from $\mathcal{N}\left(\mu, \sigma_0^2\right)$. Show that given $(X_1, ..., X_n)$, X_{n+1} is normally distributed with mean

$$\left(\frac{1}{\sigma_0^2/n} + \frac{1}{\nu_0}\right)^{-1} \left(\frac{\overline{X}}{\sigma_0^2/n} + \frac{\xi_0}{\nu_0}\right)$$

where $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ and variance

$$\sigma_0^2 + \left(\frac{1}{\sigma_0^2/n} + \frac{1}{\nu_0}\right)^{-1}.$$

• We have

$$\pi\left(\mu\middle|\, x_{1:n}\right) \propto \exp\left(-\frac{\left(\mu - \xi_0\right)^2}{2\nu_0^2}\right) \prod_{i=1}^{n} \exp\left(-\frac{\left(x_i - \mu\right)^2}{2\sigma_0^2}\right).$$

We have

$$\begin{aligned} & \frac{\left(\mu - \xi_0\right)^2}{\nu_0} + \frac{\sum_{i=1}^{n} \left(x_i - \mu\right)^2}{\sigma_0^2} \\ = \;& \mu^2 \left(\frac{1}{\nu_0} + \frac{n}{\sigma_0^2}\right) - 2\mu \left(\frac{\xi_0}{\nu_0} + \frac{\sum_{i=1}^{n} x_i}{\sigma_0^2}\right) \\ = \;& \left(\mu - \widetilde{m}\right)^2 / \widetilde{\sigma}^2 + cst \end{aligned}$$

where

$$\begin{aligned} \widetilde{\sigma}^2 &= \left(\frac{1}{\sigma_0^2/n}+\frac{1}{\nu_0}\right)^{-1}, \\ \widetilde{m} &= \widetilde{\sigma}^2\left(\frac{\overline{X}}{\sigma_0^2/n}+\frac{\xi_0}{\nu_0}\right). \end{aligned}$$

So $\pi\left(\mu|\, x_{1:n}\right)=\mathcal{N}\left(\mu;\widetilde{m},\widetilde{\sigma}^2\right)$. Now we want to compute

$$\pi\left(x|\, x_{1:n}\right)=\int \mathcal{N}\left(x;\mu,\sigma_0^2\right)\mathcal{N}\left(\mu;\widetilde{m},\widetilde{\sigma}^2\right)d\mu.$$

This is similar to the first part of Ex. 7.25 (corrected above) and the result follows.

Exercise 5 (Week 7). Let X_1,...,X_n be i.i.d. $\mathcal{N}(\mu,1/\tau)$ and suppose that independent priors are placed on μ and τ. wit $\mu\sim\mathcal{N}\left(\xi,\kappa^{-1}\right)$ and $\tau\sim\mathcal{G}(\alpha,\beta)$. Show that the conditional posterior distributions $\pi\left(\mu|\, x_1,...,x_n,\tau\right)$ and $\pi\left(\tau|\, x_1,...,x_n,\mu\right)$ admit standard forms, namely normal and Gamma, and give their exact expressions.

- We have

$$\begin{aligned} & \pi\left(\mu|\, x_{1:n},\tau\right) \\ \propto\ & \pi\left(\mu,\tau|\, x_{1:n}\right) \\ \propto\ & \exp\left(-\frac{\kappa\left(\mu-\xi\right)^2}{2}\right)\prod_{i=1}^{n}\exp\left(-\frac{\tau\left(x_i-\mu\right)^2}{2}\right) \\ \propto\ & \exp\left(-\frac{\left(\mu-\widetilde{m}\right)^2}{2\widetilde{\sigma}^2}\right) \end{aligned}$$

where

$$\begin{aligned} \widetilde{\sigma}^{-2} &= \kappa+\tau, \\ \widetilde{m} &= \widetilde{\sigma}^2\left(\kappa\xi+\tau\sum_{i=1}^{n}x_i\right). \end{aligned}$$

So we have $\pi\left(\mu|\, x_{1:n},\tau\right)=\mathcal{N}\left(x;\widetilde{m},\widetilde{\sigma}^2\right)$.

We have

$$\begin{aligned} & \pi\left(\tau|\, x_{1:n},\mu\right) \\ \propto\ & \pi\left(\mu,\tau|\, x_{1:n}\right) \\ \propto\ & \tau^{\alpha-1}\exp\left(-\frac{\tau}{\beta}\right)\prod_{i=1}^{n}\tau^{1/2}\exp\left(-\frac{\tau\left(x_i-\mu\right)^2}{2}\right) \\ \propto\ & \tau^{\alpha+\frac{n}{2}-1}\exp\left(-\left(\frac{1}{\beta}+\sum_{i=1}^{n}\frac{\left(x_i-\mu\right)^2}{2}\right)\tau\right) \end{aligned}$$

so $\pi\left(\tau|\, x_{1:n},\mu\right)=\mathcal{G}\text{amma}\left(\tau;\alpha+\frac{n}{2},\left(\frac{1}{\beta}+\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2}\right)^{-1}\right)$.