Stat 461-561: Solutions Exercises 3

Exercise 8.3

• Let $y = \sum_{i=1}^{m} y_i$. The likelihood is given by

$$L(\theta|\mathbf{y}) = \theta^y (1-\theta)^{m-y}$$

so the log-likelihood is

$$l(\theta|\mathbf{y}) = y\log(\theta) + (m-y)\log(1-\theta).$$

We want to compute

$$\lambda(\mathbf{y}) = \frac{\sup_{\theta\leq\theta_0} L(\theta|\mathbf{y})}{\sup_{\theta\in\Theta} L(\theta|\mathbf{y})}.$$

The unconstrained MLE is y/m, while the MLE under the null hypothesis is $\min\left(\sum_{i=1}^{m} y_i/m, \theta_0\right)$. Thus

$$\lambda(\mathbf{y}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{m} y_i/m \leq \theta_0 \\ \frac{\theta_0^y(1-\theta_0)^{m-y}}{(y/m)^y(1-y/m)^{m-y}} & \text{otherwise} \end{cases}$$

and we reject H_0 if $\frac{\theta_0^y(1-\theta_0)^{m-y}}{(y/m)^y(1-y/m)^{m-y}} < c$. To show that this is equivalent to rejecting if $y < b$, we could show that $\lambda(\mathbf{y})$ is decreasing in y so that $\lambda(\mathbf{y}) < c$ occurs for $y > b > m\theta_0$.

We have

$$\log\lambda(\mathbf{y}) = y\log(\theta_0) + (m-y)\log(m-\theta_0) - y\log(y/m) - (m-y)\log(1-y/m)$$

and

$$\frac{d\log\lambda(\mathbf{y})}{d\lambda} = \log\left(\frac{\theta_0\left(\frac{m-y}{m}\right)}{y/m\,(1-\theta_0)}\right).$$

For $y/m > \theta_0$, $1-y/m < 1-\theta_0$, we have $\frac{d\log\lambda(\mathbf{y})}{d\lambda} < 0$ and $\lambda(\mathbf{y}) < c$ if and only if $y > b$.

Exercise 8.7.a

• We have

$$L(\theta,\lambda|\mathbf{x}) = \frac{1}{\lambda^n}\exp\left(-\sum_{i=1}^{n} x_i + n\theta\right)\mathbb{I}_{[\theta,\infty)}\left(x_{(1)}\right)$$

which is increasing in θ if $x_{(1)} \geq \theta$ whatever being λ. So the MLE of θ is $\widehat{\theta} = x_{(1)}$ and we can easily check that $\widehat{\lambda} = \overline{x} - x_{(1)}$. Under the restriction $\theta \leq 0$, the MLE of θ regardless of λ is

$$\widehat{\theta}_0 = \begin{cases} 0 & \text{if } x_{(1)} > 0 \\ x_{(1)} & \text{otherwise} \end{cases}.$$

For $x_{(1)} > 0$, substituting $\widehat{\theta}_0 = 0$ and maximizing the likelihood with respect to λ, as above yields $\widehat{\lambda}_0 = \overline{x}$. Therefore

$$\lambda(\mathbf{x}) = \begin{cases} 1 & \text{if } x_{(1)} < 0 \\ \frac{L(0,\overline{x}|\mathbf{x})}{L(\widehat{\theta},\widehat{\lambda}|\mathbf{x})} & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} \frac{L(0,\overline{x}|\,\mathbf{x})}{L\left(\widehat{\theta},\widehat{\lambda}\middle|\,\mathbf{x}\right)} &= \frac{(1/\overline{x})^n \exp(-n\overline{x}/\overline{x})}{\left(1/\widehat{\lambda}\right)^n \exp\left(-n\left(\overline{x}-x_{(1)}\right)/\left(\overline{x}-x_{(1)}\right)\right)} \\ &= \left(\frac{\widehat{\lambda}}{\overline{x}}\right)^n = \left(1-\frac{x_{(1)}}{\overline{x}}\right)^n. \end{aligned}$$

So rejecting if $\lambda(\mathbf{x}) \le c$ is equivalent to rejecting if $\frac{x_{(1)}}{\overline{x}} \ge c^*$.

Exercise 8.13.a,b,c

• Let $Y = X_1 + X_2$ then

$$f_Y(y) = \begin{cases} y-2\theta & \text{if } 2\theta \le y < 2\theta+1 \\ 2\theta+2-y & \text{if } 2\theta+1 \le y < 2\theta+2 \\ 0 & \text{otherwise.} \end{cases}$$

• (a) The size of ϕ_1 is $\alpha_1 = \Pr(X_1 > .95|\,\theta = 0) = 0.05$. The size of ϕ_2 is $\alpha_2 = \Pr(X_1 + X_2 > C|\,\theta = 0)$. If $1 \le C \le 2$, then

$$\alpha_2 = \Pr(Y > C|\,\theta = 0) = \int_C^2 (2-y)\,dy = \frac{1}{2}(2-C)^2.$$

Setting this equal to α gives $C = 2-\sqrt{2\alpha}$, and for $\alpha = .05$, we get $C \approx 1.68$.

• (b) For the first test, we have

$$\beta_1(\theta) = \Pr(X_1 > 0.95|\,\theta) = \int_{0.95}^{\theta+1} 1dx = \theta + 0.05.$$

So we have

$$\beta_1(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 0 & \text{if } \theta \le -0.05 \\ \theta+0.05 & \text{if } -0.05 \le \theta \le 0.95 \\ 1 & \text{if } \theta > 0.95 \end{cases}.$$

For the second test, we have to consider $2\theta \le C < 2\theta+1$ and $2\theta+1 \le C < 2\theta+2$. For $2\theta \le C < 2\theta+1$, we have $\frac{C-1}{2} < \theta \le \frac{C}{2}$ and

$$\beta_2(\theta) = 1 - \int_{2\theta}^{C} (y-2\theta)\,dy = 1 - \frac{(C-2\theta)^2}{2}.$$

For $2\theta+1 \le C < 2\theta+2$, we have that $\frac{C}{2} - 1 < \theta \le \frac{C-1}{2}$ and

$$\beta_2(\theta) = \int_C^{2\theta+2} (2\theta+2-y)\,dy = \frac{(2\theta+2-C)^2}{2}.$$

Thus the power function for the second test is

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \text{if } \theta \le \frac{C}{2} - 1 \\ \frac{(2\theta+2-C)^2}{2} & \text{if } \frac{C}{2} - 1 < \theta \le \frac{C-1}{2} \\ 1 - \frac{(C-2\theta)^2}{2} & \text{if } \frac{C-1}{2} < \theta \le \frac{C}{2} \\ 1 & \text{if } \theta > \frac{C}{2} \end{cases}.$$

• I haven't drawn the figure but you can check that ϕ_1 is more powerful for θ near 0, but ϕ_2 is more powerful for larger θ's. Hence, neither test is uniformly more powerful than the other.

Exercise 8.15

• From the Neyman-Pearson lemma, the UMP test rejects H_0 if

$$\frac{f(x|\sigma_1)}{f(x|\sigma_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left(\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_i x_i^2\right)$$

for some $k \ge 0$. This is equivalent to rejecting if

$$\sum_i x_i^2 > \frac{2\log\left(k\left(\frac{\sigma_1}{\sigma_0}\right)^n\right)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c \text{ (remember that } \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0).$$

This is the UMP test of size α, where $\alpha = P_{\sigma_0}\left(\sum_i X_i^2 > c\right)$. Note that $Z = \sum_i X_i^2/\sigma_0^2 \sim \chi_n^2$. Thus

$$\alpha = P\left(Z > \frac{c}{\sigma_0^2}\right)$$

so we must have $c\sigma_0^2 = \chi_{n,\alpha}^2$ which means $c = \sigma_0^2\chi_{n,\alpha}^2$.

Exercise 8.17

• We have the following likelihood

$$L(\mu, \theta|\mathbf{x}, \mathbf{y}) = \mu^n \left(\prod_{i=1}^n x_i\right)^{\mu-1} \theta^m \left(\prod_{j=1}^m y_j\right)^{\theta-1}$$

and the log-likelihood is

$$l(\mu, \theta|\mathbf{x}, \mathbf{y}) = n\log(\mu) + (\mu - 1)\sum_{i=1}^n \log x_i + m\log\theta + (\theta - 1)\sum_{j=1}^m \log y_j.$$

Thus the unconstrained MLE is given by

$$\widehat{\mu} = \frac{-n}{\sum_{i=1}^n \log x_i} \text{ and } \widehat{\theta} = \frac{-m}{\sum_{j=1}^m \log y_j}.$$

Under H_0, the likelihood is $L(\theta|\mathbf{x}, \mathbf{y}) = \theta^{n+m}\left(\prod_{i=1}^n x_i \prod_{j=1}^m y_j\right)^{\theta-1}$ and maximizing wrt θ yields the restricted MLE

$$\widehat{\theta}_0 = -\frac{(n+m)}{\sum_{i=1}^n \log x_i + \sum_{j=1}^m \log y_j}.$$

Thus the LRT statistic is simply

$$\lambda(\mathbf{x},\mathbf{y}) = \frac{\widehat{\theta}_0^{n+m}}{\widehat{\mu}^n \widehat{\theta}^m} \left(\prod_{i=1}^n x_i\right)^{\widehat{\theta}_0-\widehat{\mu}} \theta^n \left(\prod_{j=1}^m y_j\right)^{\widehat{\theta}_0-\widehat{\mu}}.$$

• Substituting in the expressions for $\widehat{\mu}$, $\widehat{\theta}$ and $\widehat{\theta}_0$, we have $\left(\prod_{i=1}^n x_i\right)^{\widehat{\theta}_0-\widehat{\mu}} \left(\prod_{j=1}^m y_j\right)^{\widehat{\theta}_0-\widehat{\mu}} =$ 1 and

$$\lambda(\mathbf{x},\mathbf{y}) = \frac{\widehat{\theta}_0^{n+m}}{\widehat{\mu}^n \widehat{\theta}^m} \frac{\widehat{\theta}_0^{n+m}}{\widehat{\mu}^n \widehat{\theta}^m} = \left(\frac{m+n}{m}\right)^m \left(\frac{m+n}{m}\right)^m (1-T)^m T^n.$$

This is a unimodal function of T. So rejecting if $\lambda(\mathbf{x},\mathbf{y}) \leq c$ is equivalent to rejecting if $T \leq c_1$ or $T \geq c_2$ where c_1 and c_2 are appropriately chosen constants.

• It is easy to check that $-\log X_i \sim \exp(1/\mu)$ and $-\log Y_i \sim \exp(1/\theta)$. Therefore, $T = W/(W+V)$ where V and W are independent, $V \sim$Gamma$(n, 1/\mu)$ and $W \sim$Gamma$(m, 1/\theta)$. Under H_0, the scale parameters of V and W are equal. Then, a simple generalization of Exercise 4.19b yields $T \sim \mathit{Beta}(n,m)$. The constants c_1 and c_2 are determined by the two equations

$$P(T \leq c_1) + P(T \geq c_2) = \alpha$$

and we could select for example

$$(1-c_1)^m c_1^n = (1-c_2)^m c_2^n.$$