Stat 461-561 Exercises 1.

- *Exercise 5.12.* We have $X_i \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1)$ then we have

$$Z_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(0, \frac{1}{n}\right)$$

and $Y_1 = |Z_1|$ whereas

$$\mathbb{E}[Y_2] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[|X_i|].$$

For any variable $Z \sim \mathcal{N}(0,\sigma^2)$, we have

$$\begin{aligned} \mathbb{E}[|Z|] &= 2\int_0^\infty \frac{z}{\sqrt{2\pi}\sigma}\exp\left(-\frac{z^2}{2\sigma^2}\right)dz \\ &= \frac{2}{\sqrt{2\pi}\sigma}\left[-\sigma^2\exp\left(-\frac{z^2}{2\sigma^2}\right)\right]_0^\infty \\ &= \sqrt{\frac{2}{\pi}}\sigma. \end{aligned}$$

Thus

$$\mathbb{E}[Y_1] = \sqrt{\frac{2}{\pi n}}$$

and

$$\mathbb{E}[Y_2] = \sqrt{\frac{2}{\pi}}.$$

- *Exercise 5.13.* The sample variance is given by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

We know (e.g. Theorem 5.3.1 of C&B, p. 218) that

$$(n-1)S^2/\sigma^2 = Z \sim \chi^2_{p-1}.$$

Let us compute

$$\begin{aligned}
\mathbb{E}\left(\sqrt{S^2}\right) &= \frac{\sigma}{\sqrt{n-1}}\mathbb{E}\left(\sqrt{Z}\right) \\
&= \frac{\sigma}{\sqrt{n-1}}\frac{1}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}\int z^{1/2}z^{\frac{n-1}{2}-1}\exp\left(-z/2\right)dz \\
&= \frac{\sigma}{\sqrt{n-1}}\frac{1}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}\int z^{\frac{n}{2}-1}\exp\left(-z/2\right)dz \\
&= \frac{\sigma}{\sqrt{n-1}}\frac{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}} \\
&= \sigma\underbrace{\frac{\sqrt{2}\Gamma\left(\frac{n}{2}\right)}{\sqrt{n-1}\Gamma\left(\frac{n-1}{2}\right)}}_{c}
\end{aligned}$$

So if we define

$$W=\sqrt{\frac{S^2}{c}}$$

then

$$\mathbb{E}\left(W\right)=\sigma.$$

• *Exercise 5.30.* We can approximate the distribution of $\overline{X}_1-\overline{X}_2$ by a Gaussian of mean 0 and variance $\frac{2\sigma^2}{n}$. So we have

$$\begin{aligned}
&P\left(\left|\overline{X}_1-\overline{X}_2\right|<\sigma/5\right) \\
\approx\ &\int_{-\sigma/5}^{\sigma/5}\frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{2}\sigma}\exp\left(-\frac{nz^2}{4\sigma^2}\right)dz \\
=\ &\int_{-\frac{1}{5}\sqrt{\frac{n}{2}}}^{\frac{1}{5}\sqrt{\frac{n}{2}}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du
\end{aligned}$$

We know from tables that

$$\int_{-2.326}^{2.326}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du\approx 0.990$$

thus we can pick the smallest integer such that

$$\frac{1}{5}\sqrt{\frac{n}{2}}\geq 2.326\Leftrightarrow n\geq 271.$$

- *Exercise 7.1.* We have by direct inspection

x	0	1	2	3	4
$\widehat{\theta}$	1	1	2 & 3	3	3

- *Exercise 7.2.* We have

$$f\left(x \middle| \alpha, \beta\right) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp\left(-x/\beta\right)$$

thus the log-likelihood of n data is

$$L\left(\theta \middle| \mathbf{x}\right) = -n \log \Gamma(\alpha) - n\alpha \log(\beta) + (\alpha - 1)\left(\sum_{i=1}^{n} \log x_i\right) - \beta^{-1}\left(\sum_{i=1}^{n} x_i\right).$$

We have

$$\frac{\partial L\left(\theta \middle| \mathbf{x}\right)}{\partial \beta} = -n\frac{\alpha}{\beta} + \frac{1}{\beta^2}\left(\sum_{i=1}^{n} x_i\right).$$

So if α is known then

$$\widehat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha}.$$

If α is unknown, we can plug $\widehat{\beta}$ in $L\left(\theta \middle| \mathbf{x}\right)$ and maximizing numerically the resulting function

$$L\left(\alpha, \widehat{\beta} \middle| \mathbf{x}\right) = -n \log \Gamma(\alpha) - n\alpha \log\left(\frac{\sum_{i=1}^{n} x_i}{n\alpha}\right) + (\alpha - 1)\left(\sum_{i=1}^{n} \log x_i\right) - n\alpha.$$

We can alternatively find the root of this equation

$$\frac{\partial L\left(\alpha, \widehat{\beta} \middle| \mathbf{x}\right)}{\partial \alpha} = -n\psi(\alpha) - n\left(\log\left(\frac{\sum_{i=1}^{n} x_i}{n\alpha}\right) + 1\right) + \left(\sum_{i=1}^{n} \log x_i\right)$$

where $\psi(\alpha) = \frac{\partial \Gamma(\alpha)/\partial\alpha}{\Gamma(\alpha)}$ is the so-called digamma function which is tabulated in a few packages.

- *Exercise 7.9.* We have already shown that

$$\widehat{\theta}_{ML} = x_{(n)}$$

whereas

$$\mathbb{E}_{\theta}[X] = \frac{\theta}{2}$$

so the method of moments estimates is given by

$$\widehat{\theta}_{MM} = \frac{2}{n}\sum_{i=1}^{n} x_i.$$

We have

$$\mathbb{E}_\theta\left[\widehat{\theta}_{MM}\right] = \theta$$

and

$$\begin{aligned} var_\theta\left[\widehat{\theta}_{MM}\right] &= \frac{4}{n^2}\sum_{i=1}^{n} var_\theta\left[X_i\right] \\ &= \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}. \end{aligned}$$

For the ML, we use the fact that (e.g. Theorem 5.4.4. and example 5.4.5 in C&B, p. 229-230)

$$\frac{X_{(n)}}{\theta} \sim Beta\left(n, 1\right)$$

so

$$\begin{aligned} \mathbb{E}_\theta\left[\widehat{\theta}_{ML}\right] &= \frac{n\theta}{n+1}, \\ var_\theta\left[\widehat{\theta}_{ML}\right] &= \frac{n^2\theta^2}{\left(n+1\right)^2\left(n+2\right)}. \end{aligned}$$

$\widehat{\theta}_{ML}$ might be biased but it can be check that it has a lower MSE

$$\begin{aligned} MSE\left(\widehat{\theta}_{MM}\right) &= \frac{\theta^2}{3n}, \\ MSE\left(\widehat{\theta}_{ML}\right) &= \frac{\theta^2}{\left(n+1\right)^2} + \frac{n^2\theta^2}{\left(n+1\right)^2\left(n+2\right)}. \end{aligned}$$

- *Exercise 7.11.*

(a). We have

$$L\left(\theta \mid \mathbf{x}\right) = n\log\theta + \left(\theta - 1\right)\left(\sum_{i=1}^{n}\log\left(x_i\right)\right)$$

thus

$$\frac{\partial L\left(\theta \mid \mathbf{x}\right)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(x_i\right)$$

and

$$\widehat{\theta} = \frac{-n}{\sum_{i=1}^{n} \log(x_i)}.$$

Thus by the law of large numbers, we have

$$\widehat{\theta} \to -\frac{1}{\mathbb{E}_\theta[\log(X)]} \text{ a.s.}$$

so $var\left(\widehat{\theta}\right) \to 0$.

(b) We have

$$\begin{aligned} \mathbb{E}_\theta[X] &= \theta \int_0^1 x^\theta dx \\ &= \theta \left[\frac{x^{\theta+1}}{\theta+1}\right]_0^1 = \frac{\theta}{\theta+1} \end{aligned}$$

so

$$\widehat{\theta}_{MM} = \frac{\overline{X}}{1-\overline{X}}.$$

- *Exercise 7.19.*

(a) We have

$$\begin{aligned} L(\theta|\,\mathbf{x},\mathbf{y}) &= -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{\sum_{i=1}^n (y_i-\beta x_i)^2}{2\sigma^2} \\ &= -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{\sum_{i=1}^n y_i^2 + \beta^2\sum_{i=1}^n x_i^2 - 2\beta\sum_{i=1}^n x_i y_i}{2\sigma^2}. \end{aligned}$$

The sufficients statistics are $\left(\sum_{i=1}^n y_i^2, \sum_{i=1}^n x_i y_i\right)$ [and $\sum_{i=1}^n x_i^2$ but it is assumed fixed so it is not truly a sufficient statistics.]

(b) We have

$$\begin{aligned} \widehat{\beta} &= \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}, \\ \widehat{\sigma}^2 &= \frac{\sum_{i=1}^n \left(y_i - \widehat{\beta}x_i\right)^2}{n} \\ &= \frac{\sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n x_i y_i\right)^2/\sum_{i=1}^n x_i^2}{n} \end{aligned}$$

(c) $\widehat{\beta}$ is a linear combination of Gaussian variables and is thus Gaussian. We have

$$\begin{aligned} \mathbb{E}\left(\widehat{\beta}\right) &= \frac{\sum_{i=1}^n x_i \mathbb{E}(y_i)}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n \beta x_i^2}{\sum_{i=1}^n x_i^2} = \beta, \\ var\left(\widehat{\beta}\right) &= \frac{\sum_{i=1}^n x_i^2 var(y_i)}{\left(\sum_{i=1}^n x_i^2\right)^2} = \frac{\sigma^2\sum_{i=1}^n x_i^2}{\left(\sum_{i=1}^n x_i^2\right)^2}. \end{aligned}$$

- *Exercise 10.1.* We have

$$\begin{aligned}\mathbb{E}_\theta[X] &= \frac{1}{2}\int_{-1}^{1} x(1+\theta x)\,dx \\ &= \theta\int_0^1 x^2 dx \\ &= \frac{\theta}{2}\left[\frac{x^3}{2}\right]_0^1 = \frac{\theta}{4}.\end{aligned}$$

So we propose the estimate

$$\widehat{\theta} = 4\overline{X}.$$

We have

$$\begin{aligned}\mathbb{E}_\theta\left[\widehat{\theta}\right] &= \theta, \\ var_\theta\left[\widehat{\theta}\right] &= \frac{16}{n} var_\theta[X] = \frac{C}{n}\end{aligned}$$

for a finite C. So this estimate is consistent as $\mathbb{E}_\theta\left[\widehat{\theta}\right]^2 + var_\theta\left[\widehat{\theta}\right] \to 0$ as $n \to \infty$.

- *Exercise 10.3.*

(a) We have

$$\begin{aligned}L(\theta|\,\mathbf{x}) &= -\frac{n}{2}\log(2\pi\theta) - \frac{\sum_{i=1}^n (x_i-\theta)^2}{2\theta} \\ &= -\frac{n}{2}\log(2\pi\theta) - \frac{\sum_{i=1}^n x_i^2}{2\theta} + \sum_{i=1}^n x_i - \frac{n}{2}\theta.\end{aligned}$$

so

$$\frac{2}{n}\frac{\partial L(\theta|\,\mathbf{x})}{\partial\theta} = -\frac{1}{\theta} + \frac{1/n\sum_{i=1}^n x_i^2}{\theta^2} - 1.$$

We thus have to solve

$$\theta^2 + \theta - W = 0$$

and $W = \frac{1}{n}\sum_{i=1}^n x_i^2$. We have two roots

$$\begin{aligned}\theta_1 &= \frac{-1-\sqrt{1+4W}}{2}, \\ \theta_2 &= \frac{-1+\sqrt{1+4W}}{2}\end{aligned}$$

Now we know that $\theta > 0$ as it is a variance so $\widehat{\theta}_{ML} = \theta_2$.

(b) We can simply approximate the variance by

$$var\left[\widehat{\theta}_{ML}\right] \approx \frac{1}{-\left.\frac{\partial^2 L(\theta|\mathbf{x})}{\partial\theta^2}\right|_{\widehat{\theta}_{ML}}}$$

where

$$\frac{\partial^2 L\left(\theta\right|\mathbf{x})}{\partial\theta^2} = \frac{n}{2\theta^2}\left(1 - \frac{2W}{\theta}\right).$$

- *Exercise 10.8.* We did it during the course.