STAT461-561: Delta Method

AD

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Delta Method

- Assume first that $\theta \in \mathbb{R}$ and that you have

$$\theta_n \xrightarrow{\mathrm{P}} \theta$$

and

$$\sqrt{n}\left(\theta_n - \theta\right) \Rightarrow \mathcal{N}\left(0, \sigma^2\left(\theta\right)\right)$$

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- Then we have

$$\sqrt{n}\left(g\left(\theta_n\right) - g\left(\theta\right)\right) \Rightarrow \mathcal{N}\left(0, g'\left(\theta\right)^2 \sigma^2\left(\theta\right)\right)$$

Proof

- We use a Taylor's expansion

$$g\left(\theta_n\right) = g\left(\theta\right) + g'\left(\overline{\theta}_n\right)\left(\theta_n - \theta\right)$$

where $\overline{\theta}_n$ is on the line between θ and θ_n.

Proof

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$$g(\theta_n) = g(\theta) + g'\left(\overline{\theta}_n\right)(\theta_n - \theta)$$

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- Slutzky implies

$$g'\left(\overline{\theta}_n\right) \to g'(\theta)$$

  and

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- So Slutzky again yields

$$\sqrt{n}\,(g(\theta_n) - g(\theta)) \Rightarrow \mathcal{N}\left(0, g'(\theta)^2\,\sigma^2\right)$$

Multivariate delta method

- Assume that $\theta \in \mathbb{R}^d$ and that you have

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Multivariate delta method

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$$\theta_n \xrightarrow{\text{P}} \theta$$

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$$\sqrt{n}\left(\theta_n - \theta\right) \Rightarrow \mathcal{N}\left(0, \Sigma\left(\theta\right)\right)$$

- Now if we have $g : \mathbb{R}^d \to \mathbb{R}^m$ then

$$\sqrt{n}\left(g\left(\theta_n\right) - g\left(\theta\right)\right) \Rightarrow \mathcal{N}\left(0, \nabla g\left(\theta\right) \Sigma\left(\theta\right) \left(\nabla g\left(\overline{\theta}_n\right)\right)^{\mathsf{T}}\right)$$

where $\nabla g := \left(\frac{\partial g}{\partial \theta_1}, \ldots, \frac{\partial g}{\partial \theta_d}\right)^{\mathsf{T}}$ is a $d \times m$ matrix.

Proof

- We use a multivariate Taylor's expansion

$$g\left(\theta_n\right) = g\left(\theta\right) + \nabla g\left(\overline{\theta}_n\right)\left(\theta_n - \theta\right)$$

where $\overline{\theta}_n$ is on the line between θ and θ_n.

Proof

- We use a multivariate Taylor's expansion

$$g(\theta_n) = g(\theta) + \nabla g\left(\overline{\theta}_n\right)(\theta_n - \theta)$$

where $\overline{\theta}_n$ is on the line between θ and θ_n.

- Now we have $\nabla g\left(\overline{\theta}_n\right) \to \nabla g(\theta)$ and

$$\begin{aligned} cov\,(g(\theta_n) - g(\theta)) &= cov\,(\nabla g(\theta)(\theta_n - \theta)) \\ &= \nabla g(\theta)\, cov\,(\theta_n - \theta)\,(\nabla g(\theta))^{\mathsf{T}} \end{aligned}$$

and the result follows.