Lecture Stat 302
Introduction to Probability - Slides 2

AD

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Recapitulation

- **Principle of counting**: If experiment 1 has n_1 possible outcomes, experiment 2 has n_2 possible outcomes,...., experiment r n_r possible outcomes, then there is a total of $n_1 \cdot n_2 \cdot \cdots \cdot n_r$ possible outcomes of the r experiments.
- **Permutations**: For n objects, the number of permutations, i.e. ordered arrangements, of these n objects is given by $n!$
- **Permutations with objects alike**: For n objects, of which n_1 are alike, n_2 are alike,..., n_r are alike, there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

permutations.
- **Combinations**: The number of different groups of r objects that can be formed from n objects is

$$\begin{array}{ll} \frac{n!}{(n-r)!} & \text{if the order matters,} \\ \frac{n!}{(n-r)!r!} & \text{if the order does not matter.} \end{array}$$

Examples

- **Example** *(Pb. 13)*: Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?
- **Answer**: One handshake corresponds to one pair of people so we have to consider all possible pairs of people. There are $\binom{20}{2} = 190$ handshakes.
- **Example** *(Pb. 15)*: A dance class consists of 22 students, 10 W and 12 M. If 5 M and 5 W are to be chosen and then paired off, how many results are possible?
- **Answer**: There are $\binom{10}{5}$ ways to select 5 W and $\binom{12}{5}$ ways to select 5 M. Then, given 5 W and 5 M, there are 5! possible permutations to combine them so the answer is

$$\binom{10}{5}\binom{12}{5}5! = \frac{10!}{5!5!}\frac{12!}{7!5!}5! = 23,950,080.$$

Examples

- **Example** *(Pb. 20)*: A person has 8 friends, of whom 5 will be invited to a party.
  - (a) How many choices are there if 2 of the friends are feuding and will not attend together?
  - (b) How many choices if 2 of the friends will only attend together?
- **Answer**: (a) There are $\begin{pmatrix} 8 \\ 5 \end{pmatrix} = 56$ possible groups of 5 friends if there were no constraint. Among those 56 groups, we have to exclude the ones where 2 of the friends are feuding. There are $\begin{pmatrix} 2 \\ 2 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20$ such groups so the answer is $56 - 20 = 36$.
  (b) There are $\begin{pmatrix} 2 \\ 2 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20$ groups where the two friends attend together and $\begin{pmatrix} 6 \\ 5 \end{pmatrix} = 6$ groups where none of them attend the party. So there are $20 + 6 = 26$ possible groups.

The Binomial Theorem

Theorem

We have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

- **Example***:* Compute $(x+y)^3$.
- **Answer**: We have

$$\begin{aligned}(x+y)^3 &= \binom{3}{0} x^0 y^3 + \binom{3}{1} xy^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3 y^0 \\ &= y^3 + 3xy^2 + 3x^2 y + x^3.\end{aligned}$$

Proof of the binomial theorem by induction

This is true at rank $n = 1$. Assume this is true at rank $n-1$ and let us prove it at rank n. We have

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1} = (x+y)\left(\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}\right)\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}.\end{aligned}$$

For the 1st term on the lhs, perform a change of variables $l \leftarrow k+1$ then we have

$$(x+y)^n = x^n + \sum_{l=1}^{n-1}\left\{\binom{n-1}{l-1} + \binom{n-1}{l}\right\}x^l y^{n-l} + y^n$$

where

$$\binom{n-1}{l-1} + \binom{n-1}{l} = \frac{(n-1)!}{(l-1)!\,(n-1-l)!}\left(\frac{1}{n-l}+\frac{1}{l}\right) = \binom{n}{l}.$$

So the result is proven. You might know the result as *Pascal's triangle*.

Pascal's triangle

- This is a simple way to compute the binomial coefficients.

$$
\begin{array}{ccccccccc}
 & 1 & & & & & & \\
 & 1 & 1 & & & & & \\
 & 1 & 2 & 1 & & & & \\
n & 1 & 3 & 3 & 1 & & & \\
 & 1 & 4 & 6 & 4 & 1 & & \\
 & 1 & 5 & 10 & 10 & 5 & 1 & \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 & & & & & l & &
\end{array}
$$

- $\left(\begin{array}{c} n \\ l \end{array}\right)$ (where $n \geq 0, l \geq 0$) corresponds to entry (n, l) of the array and satisfies $\left(\begin{array}{c} n \\ l \end{array}\right) = \left(\begin{array}{c} n-1 \\ l-1 \end{array}\right) + \left(\begin{array}{c} n-1 \\ l \end{array}\right)$ for $l = 1, ..., n-1$.

Combinatorial proof of the identity

- We have established analytically $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
- Consider n objects and select one of them arbitrarily.
- The nb. of different groups of r objects that can be formed from n objects is $\binom{n}{r}$, it is also equal to the sum of the nb. of groups of r objects from n objects which include the arbitrarily selected object, given by $\binom{n-1}{r-1}$, and the nb. of groups of r objects from n objects which exclude the arbitrarily selected object, given by $\binom{n-1}{r}$.

Combinatorial proof of the binomial theorem

- Introduce artificial indexes on x, y and consider the product

$$\left(x_1 + y_1\right)\left(x_2 + y_2\right)\cdots\left(x_n + y_n\right)$$

- Expansion of this product leads to 2^n coefficients; e.g. for $n = 2$

$$\left(x_1 + y_1\right)\left(x_2 + y_2\right) = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2$$

and for $n = 3$

$$\begin{aligned}\left(x_1 + y_1\right)\left(x_2 + y_2\right)\left(x_3 + y_3\right) &= x_1 x_2 x_3 + x_1 y_2 x_3 + y_1 x_2 x_3 + y_1 y_2 x_3 \\ &\quad + x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 + y_1 y_2 y_3.\end{aligned}$$

- Among the 2^n terms, we will have terms with k x_i's and $(n-k)$ y_i's. Each such term corresponds to the selection of a group of k x_i's among n possible terms, there are $\left(\begin{array}{c} n \\ k \end{array}\right)$ such terms. Hence the result follows as $x_i = x$, $y_i = y$.

Examples

- **Example:** How many *non-empty* subsets are there of a set of n elements?
- **Answer**: The total number of non-empty sets is the number of sets with k elements where $k = 1, ..., n$

$$\sum_{k=1}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \underbrace{\binom{n}{0}}_{=1} = (1+1)^n - 1 = 2^n - 1.$$

- **Example** (Th. Ex. 13). Show that $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$.
- **Answer**. We use the binomial theorem for $x = -1$ and $y = 1$

$$(-1+1)^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k .$$

Example

- **Example** (Th. Ex. 12, question a). Establish $\sum_{k=1}^{n} k \left(\begin{array}{c} n \\ k \end{array} \right) = n.2^{n-1}$ by considering a set of n persons and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.
- **Answer**: For a committee of size k, there are $\left(\begin{array}{c} n \\ k \end{array} \right)$ possible choices for selecting the persons in the committee and k choices for the chairperson so $k \left(\begin{array}{c} n \\ k \end{array} \right)$ possible choices $\Rightarrow$.total is $\sum_{k=1}^{n} k \left(\begin{array}{c} n \\ k \end{array} \right)$. Alternatively, there are n possible choices for the chairperson and we have $\left(\begin{array}{c} n-1 \\ k-1 \end{array} \right)$ other possible persons to put in the committee if it is of size $k = 1, ..., n$. By summing over k

$$\sum_{k=1}^{n} \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right) = \sum_{k=0}^{n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) = (1+1)^{n-1} = 2^{n-1} \Rightarrow \text{total is } n.2^{n-1}$$

Multinomial Coefficients

- We now want to divide a set of n items into r distinct groups of respective sizes $n_1, n_2, ..., n_r$ where $n_1 + n_2 + \cdots + n_r = n$. The number of possibilities is

$$\frac{n!}{n_1!n_2!\cdots n_r!} := \begin{pmatrix} n \\ n_1, n_2, \ldots, n_r \end{pmatrix} \quad \text{Multinomial coefficient}$$

  This is the same as the number of permutations of n items with n_1 alike, n_2 alike etc.
- **Example** *(Pb. 25)*: The game of bridge is played by 4 players, each of who is dealt 13 cards. How many bridge deals are possible?
- **Answer**: There are $4 \times 13 = 52$ cards so 52! possible permutations. However, like all card games, any permutation of the cards received by a given player are irrelevant (order does not matter). So there are

$$\frac{52!}{13!13!13!13!}$$

  different possible deals.

Combinatiorial Proof

- First divide our set into 2 groups of resp. size n_1 and $n - n_1$, there are $\left(\begin{array}{c} n \\ n_1 \end{array} \right)$ possible choices for the 1st group. For each of the 1st group, we have $\left(\begin{array}{c} n - n_1 \\ n_2 \end{array} \right)$ possibilities for the 2nd group, then $\left(\begin{array}{c} n - n_1 - n_2 \\ n_3 \end{array} \right)$ for the 3rd group, etc. So we have

$$\left(\begin{array}{c} n \\ n_1 \end{array} \right) \left(\begin{array}{c} n - n_1 \\ n_2 \end{array} \right) \left(\begin{array}{c} n - n_1 - n_2 \\ n_3 \end{array} \right) \cdots \left(\begin{array}{c} n - n_1 - n_2 - \cdots - n_{r-1} \\ n_r \end{array} \right)$$
$$= \left(\begin{array}{c} n \\ n_1, n_2, \ldots, n_r \end{array} \right)$$

Examples

- **Example**: A small company has 9 employees. Everyday the company has 4 persons working during the day, 3 other persons working at night and 2 others not working. How many different divisions of the 9 employees in these 3 groups is possible?
- **Answer**: The answer is

$$\binom{9}{4,3,2} = 1260.$$

- **Example** (Pb. 28): 8 teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?
- **Answer**. Each teacher has 4 possible choices (outcome) so the answer to the first question is simply $4^8 = 65536$.If each school must receive 2 teachers then we have

$$\binom{8}{2,2,2,2} = 2520.$$

Examples

- **Example**: In order to organize a basketball tournament, 20 children at a playground divide themselves in four teams of 5 players. How many different divisions are possible?
- **Answer**: The answer is NOT

$$\binom{20}{5,5,5,5}$$

because the order of the four teams is irrelevant. It would be exact if being in the team A would be considered different from being in the team D. Here we are only interested in the possible divisions, so as there are 4! permutations between team "labels" then the answer is

$$\binom{20}{5,5,5,5}/4! = \binom{20}{5,5,5,5,4}.$$

The Multinomial Theorem

Theorem

We have

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r): \\ n_1 + n_2 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_k}$$

- **Example**: Compute $(x_1 + x_2 + x_3)^3$.
- **Answer**: We have

$$(x_1 + x_2 + x_3)^3 = \binom{2}{2, 0, 0} x_1^2 x_2^0 x_3^0 + \binom{2}{0, 2, 0} x_1 x_2^2 x_3^0 + \binom{2}{0, 0, 2} x_1^0 x_2^0 x_3^2 + \binom{2}{1, 1, 0} x_1^1 x_2^1 x_3^0 + \binom{2}{1, 0, 1} x_1^1 x_2^0 x_3^1 + \binom{2}{0, 1, 1} x_1^0 x_2^1 x_3^1$$
$$= x_1^2 + x_2^2 + x_3^2 + 2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3.$$

Combinatorial proof of the multinomial theorem

- Introduce artificial indexes on x_i and consider the product

$$\begin{aligned} & \left(x_{1,1} + x_{2,1} + \cdots + x_{r,1}\right)\left(x_{1,2} + x_{2,2} + \cdots + x_{r,2}\right) \times \cdots \\ & \cdots \times \left(x_{1,n} + x_{2,n} + \cdots + x_{r,n}\right) \end{aligned}$$

- Expansion of this product leads to r^n coefficients.
- Among the r^n terms, we will have terms with n_1 $x_{1,i}$'s, n_2 terms $x_{2,i}$'s,...., n_r terms $x_{r,i}$'s. This corresponds to the selection of groups of n_1 terms, n_2 terms,... n_r terms such that $n_1 + n_2 + \cdots + n_r = n$. Hence there $\begin{pmatrix} n \\ n_1, n_2, ..., n_r \end{pmatrix}$ such terms. The result follows as $x_{k,i} = x_k$ for $i = 1, ..., n$.

Example

- **Example**: In the 1st round of a knockout tournament involving $n = 2^m$ players, the n players are divided in $n/2$ pairs. Each pair plays a game. The losers of the games are eliminated while the winners go on to the next round and the process is repeated until only one single player remains. Assume $n = 8$. How many possible outcomes are there for the initial round?
- **Answer**: If there was an ordering of the pairs then the number of possible pairs for the initial round is given by

$$\binom{8}{2,2,2,2} = \frac{8!}{2^4}.$$

As there is no ordering of the pairs, then there are $\frac{8!}{2^4}/4!$ pairs. For each of these pairs then there are 2 possible outcomes in the game and there are 4 games, so there are

$$\frac{8!}{2^4 4!} \times 2^4 = \frac{8!}{4!}.$$

Example continued...

- Alternative way to establish the result: pick 4 winners among the 8 players, $\binom{8}{4}$ possibilities, and match them with the 4 losers, there are 4! ways to do this so $\binom{8}{4} 4! = \frac{8!}{4!}$.
- **Question**: How many outcomes of the tournaments are possible, where an outcome gives complete information for all rounds?
- **Answer**: Similarly, we have $\frac{4!}{2!}$ possible outcomes for the second round and $\frac{2!}{1!}$ for the third (and final) round. So by the principle of counting, there are

$$\frac{8!}{4!} \times \frac{4!}{2!} \times \frac{2!}{1!} = 8!$$

In the general case of $n = 2^m$, there are m rounds and $n!$ possible outcomes.