# CPSC 535
# Gibbs Sampling

AD

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- **Problem**: We try to sample all the components of a potentially high-dimensional parameter simultaneously/sequentially and we can never correct for components already sampled.
- A powerful class of methods is available to deal with such methods: Markov chain Monte Carlo.

# Bayesian Model

- Multiple failures in a nuclear plant

| Pump $i$ | 1 | 2 | 3 | 4 | 5 |
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| # Failures $p_i$ | 5 | 1 | 5 | 14 | 3 |
| Times $t_i$ | 94.32 | 15.72 | 62.88 | 125.76 | 5.24 |

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- Model: Failures of the $i-$th pump follow a Poisson process with parameter $\lambda_i$ $(1 \leq i \leq 10)$. For an observed time $t_i$, the number of failures $p_i$ is thus a Poisson $\mathcal{P}(\lambda_i t_i)$ random variable.

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- The unknown parameters consist of $\theta = (\lambda_1, \ldots, \lambda_{10}, \beta)$.

- Hierarchical model

$$\lambda_i | (\alpha, \beta) \overset{\text{iid}}{\sim} \mathcal{G}a(\alpha, \beta) \text{ and } \beta \sim \mathcal{G}a(\gamma, \delta)$$

with $\alpha = 1.8$ and $\gamma = 0.01$ and $\delta = 1$.

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- The posterior distribution is proportional to

$$\begin{aligned} & p\left(\lambda_{1:10}, \beta | \, p_{1:10}, t_{1:10}\right) \\ \propto & \prod_{i=1}^{10} \{(\lambda_i t_i)^{p_i} \exp(-\lambda_i t_i) \lambda_i^{\alpha-1} \exp(-\beta\lambda_i)\} \beta^{10\alpha} \beta^{\gamma-1} \exp(-\delta\beta) \\ \propto & \prod_{i=1}^{10} \{\lambda_i^{p_i+\alpha-1} \exp(-(t_i+\beta)\lambda_i)\} \beta^{10\alpha+\gamma-1} \exp(-\delta\beta). \end{aligned}$$

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- This multidimensional distribution is rather complex. It is not obvious how the rejection method or importance sampling could be used in this context.

- The conditionals have a familiar form

$$p\left(\lambda_{1:10}\middle|\, p_{1:10}, t_{1:10}, \beta\right) = \prod_{i=1}^{10} p\left(\lambda_i\middle|\, p_i, t_i, \beta\right)$$

where

$$\lambda_i|(\beta, t_i, p_i) \sim \mathcal{G}a(p_i + \alpha, t_i + \beta) \text{ for } 1 \leq i \leq 10,$$

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- Instead of directly sampling the vector $\theta = (\lambda_1, \ldots, \lambda_{10}, \beta)$ at once, one could suggest sampling it iteratively, starting for example with the $\lambda_i$'s for a given guess of $\beta$, followed by an update of $\beta$ given the new samples $\lambda_1, \ldots, \lambda_{10}$.

# My first Gibbs sampler

- Given a sample, at iteration $t$, $\theta^t := (\lambda_1^t, \ldots, \lambda_{10}^t, \beta^t)$ one could proceed as follows at iteration $t+1$,

1. $\lambda_i^{t+1} | (\beta^t, t_i, p_i) \sim \mathcal{G}a(p_i + \alpha, t_i + \beta^t)$ for $1 \leq i \leq 10$,
2. $\beta^{t+1} | (\lambda_1^{t+1}, \ldots, \lambda_{10}^{t+1}) \sim \mathcal{G}a(\gamma + 10\alpha, \delta + \sum_{i=1}^{10} \lambda_i^{t+1})$.

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- Instead of directly sampling in a space with 11 dimensions, one samples in spaces of dimension 1.
- Note that the deterministic version of such an algorithm where sampling is replaced by maximization would not generally converge towards the global maximum of the joint distribution.

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  - Are we sampling from the desired joint distribution?
  - If yes, how many times should the iteration above be repeated?
- The validity of the approach described here stems from the fact that the sequence $\{\theta^t\}$ defined above is a Markov chain and some Markov chains have very nice properties.

# Elements of Markov chains

- **Markov chain**: A sequence of random variables $\{X_n; n \in \mathbb{N}\}$ defined on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ which satisfies the property, for any $A \in \mathcal{B}(\mathbb{X})$

$$\mathbb{P}(X_n \in A | X_0, ..., X_{n-1}) = \mathbb{P}(X_n \in A | X_{n-1}).$$

and we will write

$$\mathbb{P}(X_n \in A | X_{n-1}) = P(x, A) = \int_A P(x, dy).$$

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- **Markov chain Monte Carlo**: Given a target $\pi$, design a transition kernel $P$ such that asymptotically as $n \longrightarrow \infty$

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- It should be easy to simulate the Markov chain even if $\pi$ is complex.

- Consider the autoregression for $|\alpha| < 1$

$$X_n = \alpha X_{n-1} + V_n, \text{ where } V_n \sim \mathcal{N}\left(0, \sigma^2\right)$$

then

$$P\left(x, dy\right) = P\left(x, y\right) dy = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(y - \alpha x\right)^2}{2\sigma^2}\right) dx.$$

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$$\pi\left(x\right) = \mathcal{N}\left(x; 0, \frac{\sigma^2}{1 - \alpha^2}\right)$$

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- To sample from $\pi$, we could just sample the Markov chain and asymptotically we would have $X_n \sim \pi$. [Obviously, in this case this is useless because we can sample from $\pi$ directly.]

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- We assume that the initial distribution of these Markov chains is $\mathcal{U}_{[0,20]}$. So initially, the Markov chains samples are not distributed according to $\pi$

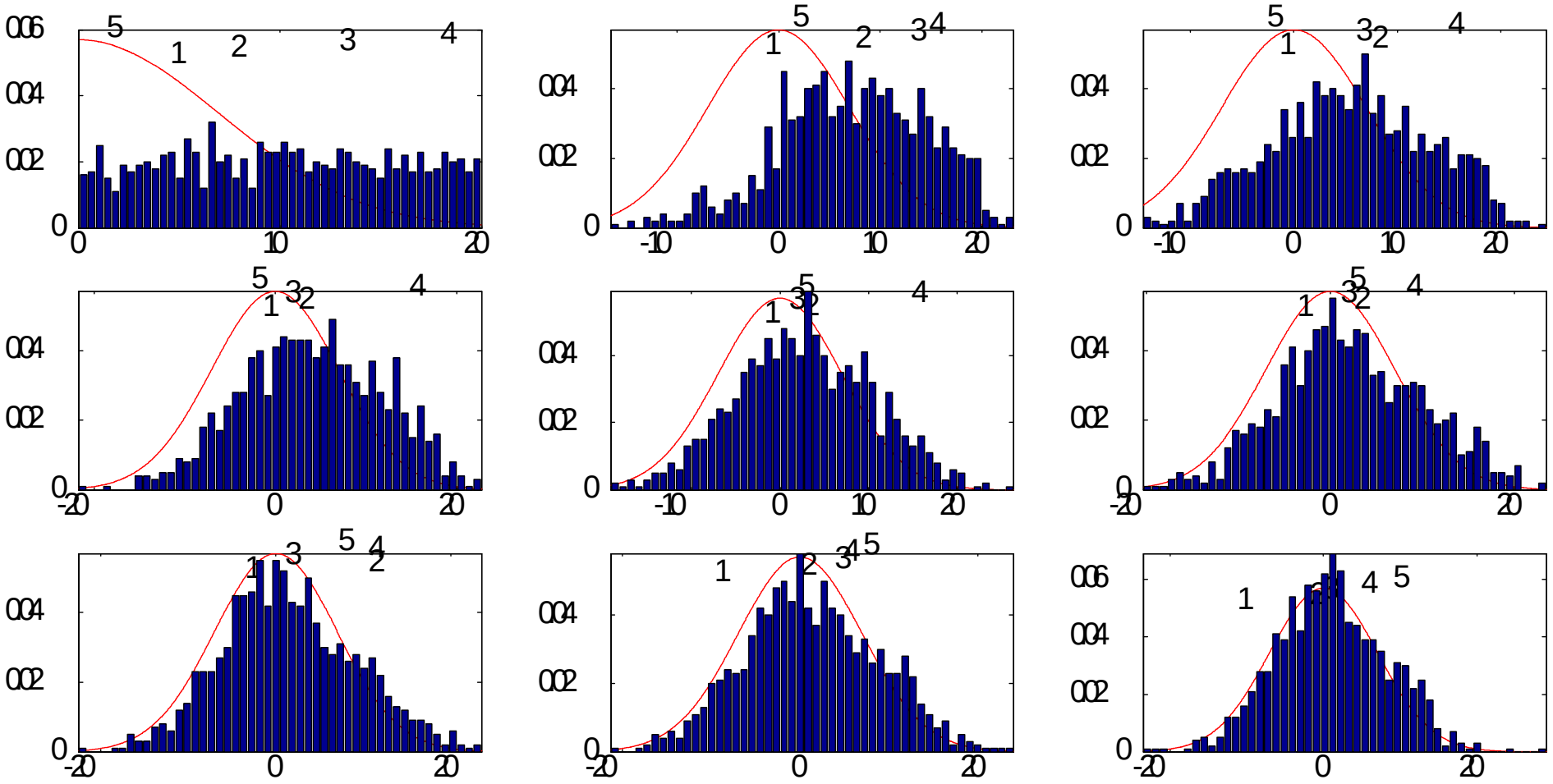

Figure: From top left to bottom right: histograms of 1000 independent Markov chains with a normal distribution as target distribution as $n$ increases.

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- This is is what we wanted to achieve, *i.e.* it seems that we have produced 1000 independent samples from the normal distribution.
- In fact one can show that in many (all?) situations of interest it is not necessary to run $N$ Markov chains in parallel in order to obtain 1000 samples, but that one can consider a unique Markov chain, and build the histogram from this single Markov chain by forming histograms from one trajectory.

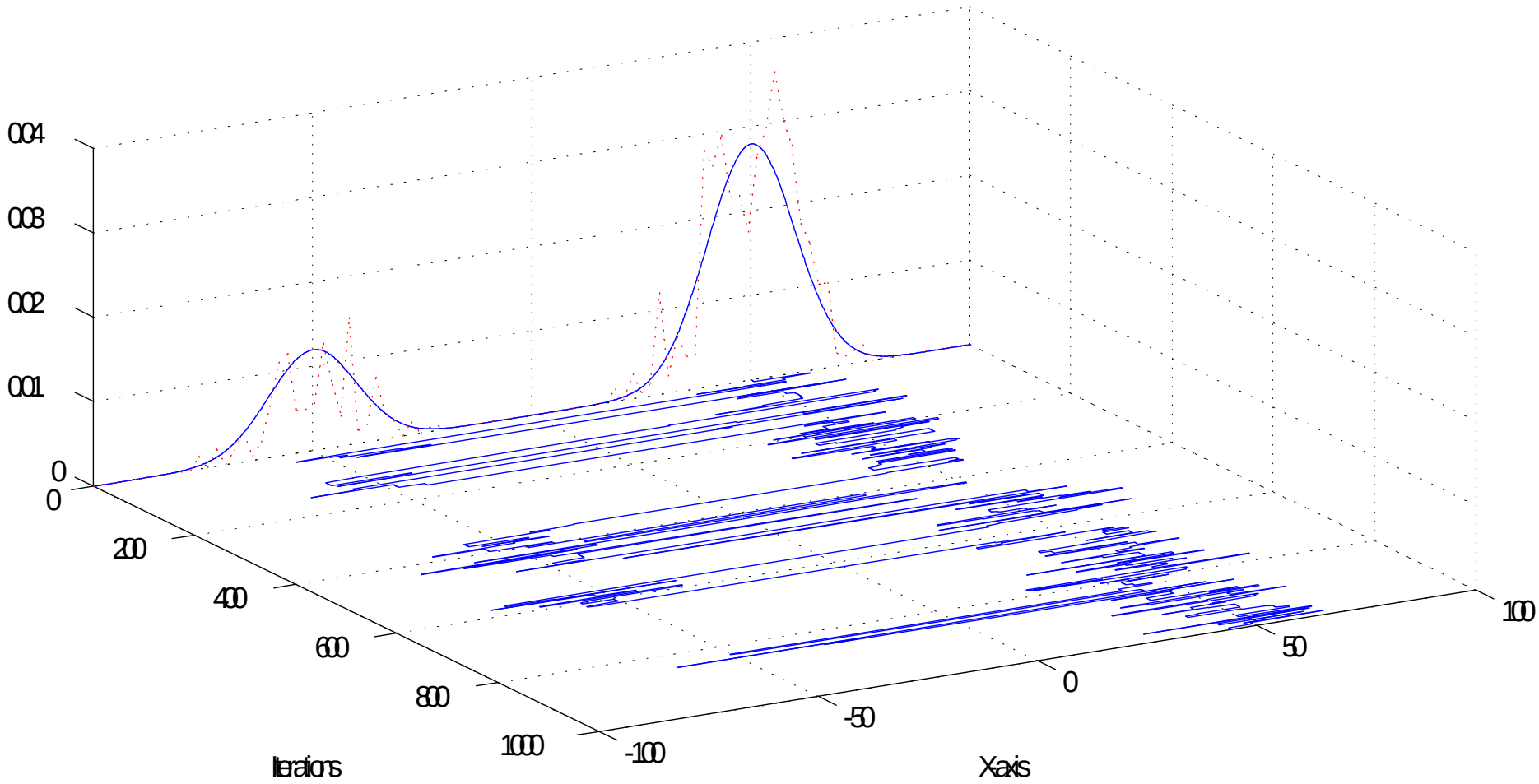
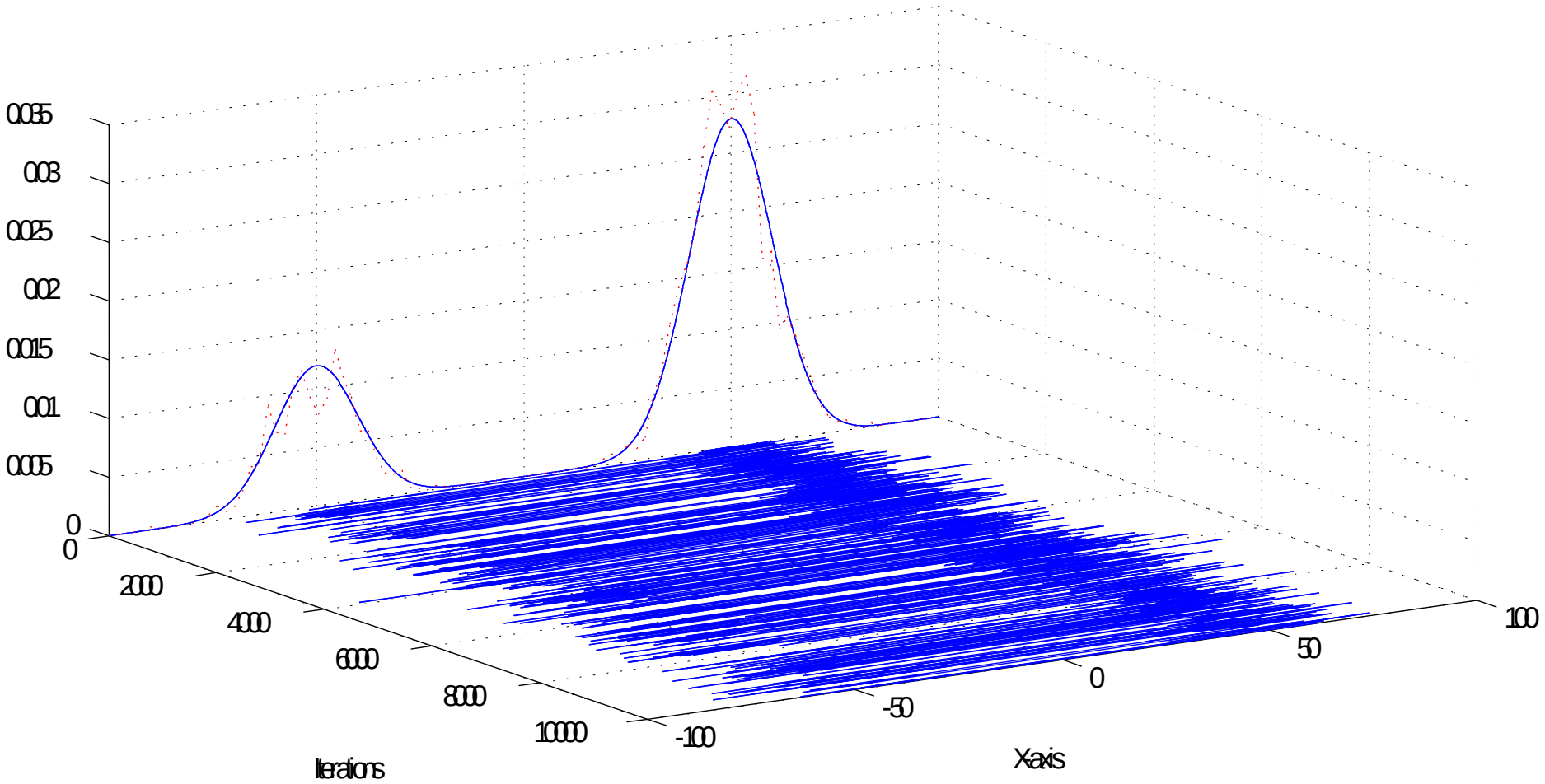



Figure: Bimodal target distributions and simulated Markov chain

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- In the light of the numerical experiments, one can suggest the estimator

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  which is exactly the estimator that we would use if $\{X_n, 1 \leq n \leq N\}$ were independent.
- In fact, it can be proved, under relatively mild conditions, that such an estimator is consistent *despite the fact that the samples are NOT independent!* Under additional conditions, a CLT also holds with a rate of CV in $1/\sqrt{N}$.

# Markov chains for Monte Carlo

To summarize, we are interested in Markov chains with transition kernel $P$ which have the following three important properties observed above:

- The desired distribution $\pi$ is a "fixed point" of the algorithm or, in more appropriate terms, an *invariant distribution* of the Markov chain, *i.e.* $\int_{\mathbb{X}} \pi(x)P(x,y)dx = \pi(y)$ .

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- The successive distributions of the Markov chains converge towards $\pi$.
- The estimator $\frac{1}{N}\sum_{n=1}^{N} \varphi(X_n)$ converges towards $\mathbb{E}_\pi(\varphi(X))$ and asymptotically $X_n \sim \pi$

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- The "art" of MCMC consists of coming up with good ones.
- Convergence is ensured under very weak assumptions; namely irreducibility and aperiodicity.
- It is usually very easy to establish that an MCMC sampler converges towards $\pi$ but very difficult to obtain rates of convergence.

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  - Sample $\theta_i^2 \sim \pi\left(\theta^2 \middle| \theta_i^1\right)$.
- Sampling from these conditional is often feasible even when sampling from the joint is impossible (e.g. nuclear pump data).

- Clearly $\left\{\left(\theta_i^1, \theta_i^2\right)\right\}$ is a Markov chain and its transition kernel is

$$P\left(\left(\theta^1, \theta^2\right), \left(\widetilde{\theta}^1, \widetilde{\theta}^2\right)\right) = \pi\left(\widetilde{\theta}^1 \middle| \theta^2\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right).$$

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- Then $\int\int \pi\left(\theta^1, \theta^2\right) P\left(\left(\theta^1, \theta^2\right), \left(\widetilde{\theta}^1, \widetilde{\theta}^2\right)\right) d\theta^1 d\theta^2$ satisfies

$$\begin{aligned}
& \int\int \pi\left(\theta^1, \theta^2\right) \pi\left(\widetilde{\theta}^1 \middle| \theta^2\right) \pi\left(\widetilde{\theta}^2 \middle| \widetilde{\theta}^1\right) d\theta^1 d\theta^2 \\
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\end{aligned}$$

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- This does not ensure that the Gibbs sampler does converge towards the invariant distribution!
- Additionally it is required to ensure *irreducibility*: loosely speaking the Markov chain can move to any set $A$ such that $\pi(A) > 0$ for (almost) any starting point.
- This ensures that

$$\frac{1}{N}\sum_{n=1}^{N}\varphi\left(\theta_n^1,\theta_n^2\right)\rightarrow\int\varphi\left(\theta^1,\theta^2\right)\pi\left(\theta^1,\theta^2\right)d\theta^1 d\theta^2$$

but NOT that asymptotically $\left(\theta_n^1,\theta_n^2\right)\sim\pi$.

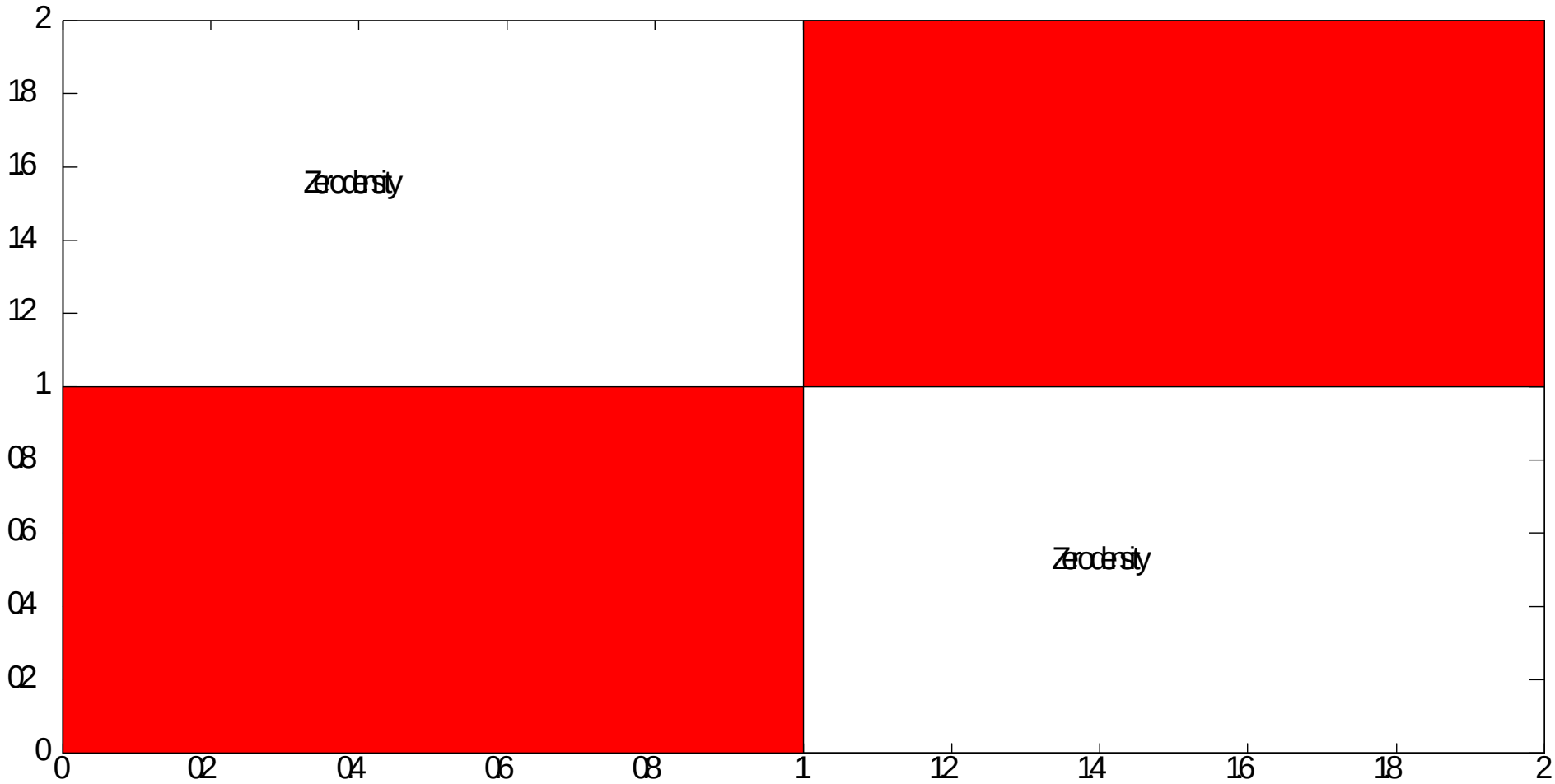


Figure: A distribution that can lead to a reducible Gibbs sampler.

# Aperiodicity

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but clearly $X_n$ is NOT distributed according to $\pi$.

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- We have

$$\frac{1}{N} \sum_{n=1}^{N} \varphi(X_n) \to \int \varphi(x) \pi(x) dx$$

  but clearly $X_n$ is NOT distributed according to $\pi$.
- You need to make sure that you do NOT explore the space in a periodic way to ensure that $X_n \sim \pi$ asymptotically.

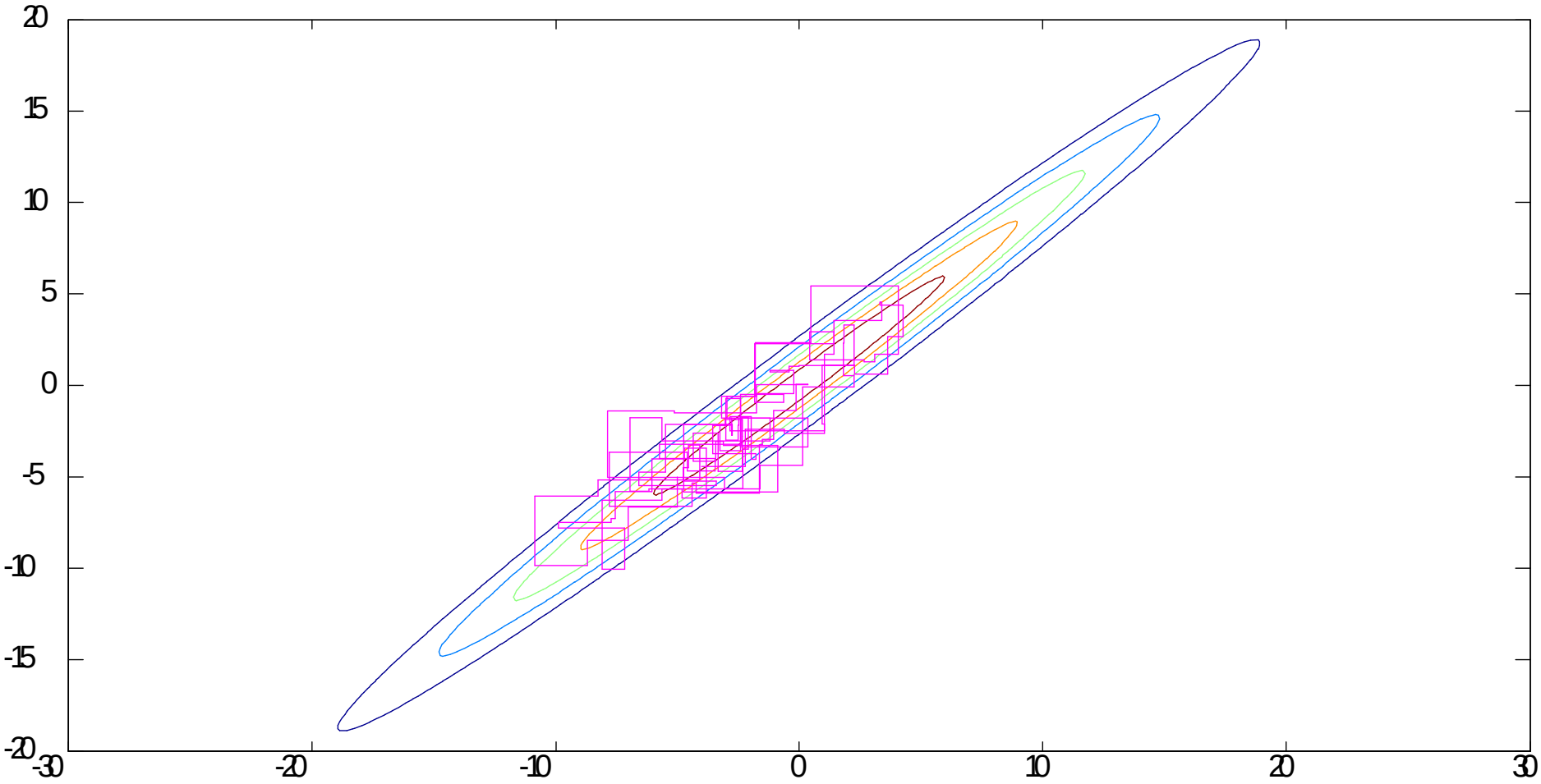


Figure: Even when irreducibility and aperiodicity are ensured, the Gibbs sampler can still converge very slowly.

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- If necessary, reparametrize the model to achieve this.
- Integrate analytically as many variables as possible: pretty algorithms can be much more inefficient than ugly algorithms.
- There is no general result telling strategy A is better than strategy B in all cases: you need experience.

# Application to Simulation of Fractal Images

- Consider a 2D black and white 'target' image. We define an distribution $\nu$ which assigns $1/P$ mass on each black point and zero on white points where $P$ is the number of black points.

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$$P(x, y) = \sum_{i=1}^{k} w_i \delta_{A_i x + b_i}(y)$$

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- To find $\{w_i, A_i, b_i\}$, we write

$$\begin{aligned} \int \pi(x) P(x, y) f(y) dxdy &= \sum_{i=1}^{k} w_i \int f(A_i x + b_i) \pi(x) dx \\ &= \int f(x) \pi(x) dx \approx \int f(x) \nu(x) dx \end{aligned}$$

and solve approximately the equations for some functions $f$ (linear or low order polynoms).

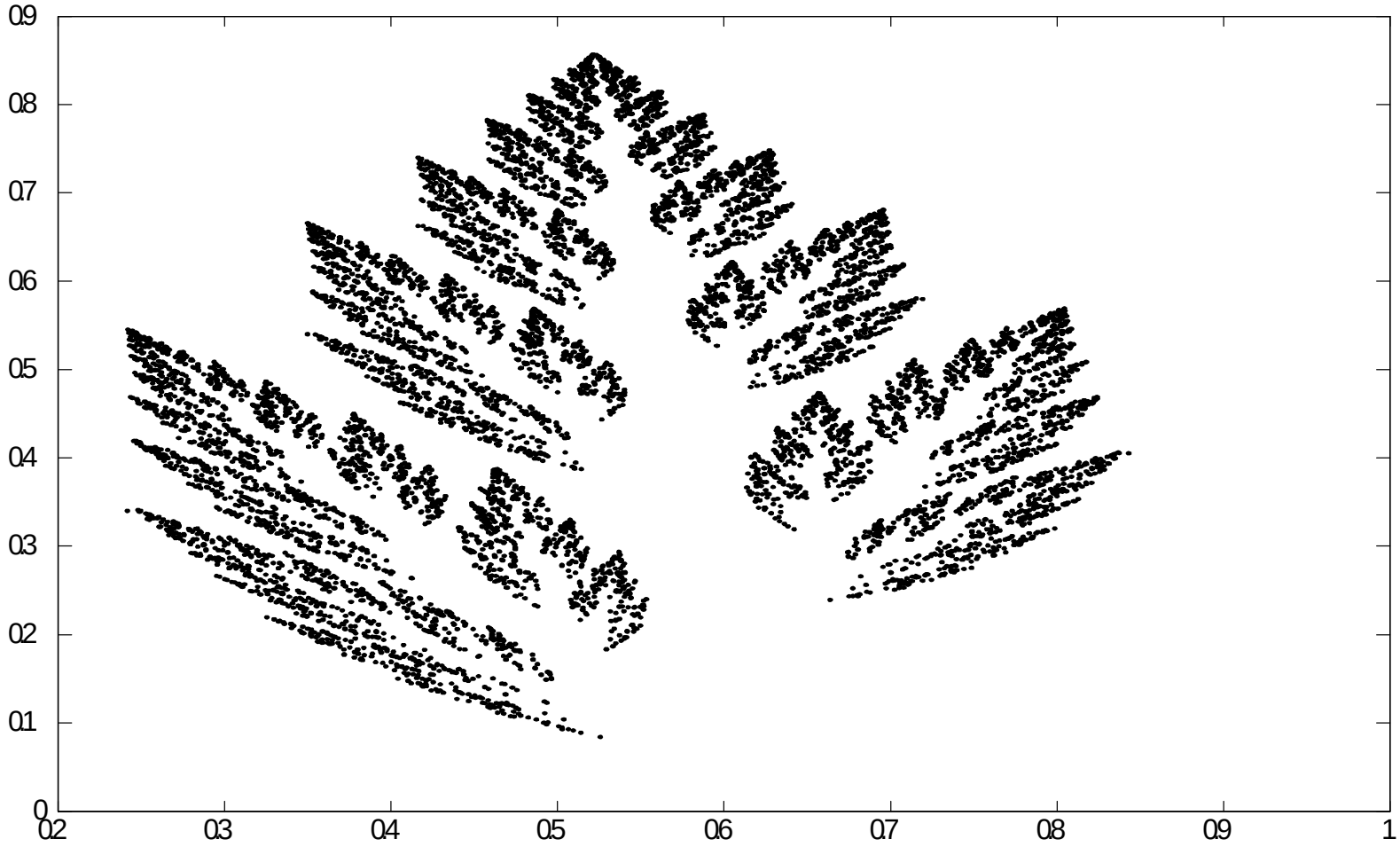


Figure: Fractal image generated using Iterated random functions with $k = 2$ and $N = 10000$ samples

# Gibbs Sampler for Bayesian Variable Selection

- We select the following model

$$Y = \sum_{i=1}^{p} \beta_i X_i + \sigma V \text{ where } V \sim \mathcal{N}(0,1)$$

where we assume $\mathcal{IG}\left(\sigma^2; \frac{\nu_0}{2}, \frac{\gamma_0}{2}\right)$ and for $\alpha^2 << 1$

$$\beta_i \sim \frac{1}{2}\mathcal{N}\left(0, \alpha^2\delta^2\sigma^2\right) + \frac{1}{2}\mathcal{N}\left(0, \delta^2\sigma^2\right)$$

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- We introduce a latent variable $\gamma_i \in \{0,1\}$ such that

$$\begin{array}{l} \Pr(\gamma_i = 0) = \Pr(\gamma_i = 1) = \frac{1}{2}, \\ \beta_i | \gamma_i = 0 \sim \mathcal{N}\left(0, \alpha^2\delta^2\sigma^2\right), \quad \beta_i | \gamma_i = 1 \sim \mathcal{N}\left(0, \delta^2\sigma^2\right). \end{array}$$

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- A potential Gibbs sampler consists of sampling iteratively from $p\left(\beta_{1:p} \middle| D, \gamma_{1:p}, \sigma^2\right)$ (Gaussian), $p\left(\sigma^2 \middle| D, \gamma_{1:p}, \beta_{1:p}\right)$ (inverse-Gamma) and $p\left(\gamma_{1:p} \middle| D, \beta_{1:p}, \sigma^2\right)$.

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- In particular

$$p\left(\gamma_{1:p} \middle| D, \beta_{1:p}, \sigma^2\right) = \prod_{i=1}^{p} p\left(\gamma_i \middle| \beta_i, \sigma^2\right)$$

and

$$p\left(\gamma_i = 1 \middle| \beta_i, \sigma^2\right) = \frac{\frac{1}{\sqrt{2\pi}\delta\sigma} \exp\left(-\frac{\beta_i^2}{2\delta^2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi}\delta\sigma} \exp\left(-\frac{\beta_i^2}{2\delta^2\sigma^2}\right) + \frac{1}{\sqrt{2\pi}\alpha\delta\sigma} \exp\left(-\frac{\beta_i^2}{2\alpha^2\delta^2\sigma^2}\right)}.$$

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- The Gibbs sampler becomes reducible as $\alpha$ goes to zero.

- This is the result of bad modelling and bad algorithm. You would like to put $\alpha \simeq 0$ and write

$$Y = \sum_{i=1}^{p} \gamma_i \beta_i X_i + \sigma V \text{ where } V \sim \mathcal{N}(0,1)$$

where $\gamma_i = 1$ if $X_i$ is included or $\gamma_i = 0$ otherwise. However this suggests that $\beta_i$ is defined even when $\gamma_i = 0$.

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- A neater way to write such models is to write

$$Y = \sum_{\{i:\gamma_i=1\}} \beta_i X_i + \sigma V = \beta_\gamma^T X_\gamma + \sigma V$$

where, for a vector $\gamma = (\gamma_1, ..., \gamma_p)$, $\beta_\gamma = \{\beta_i : \gamma_i = 1\}$, $X_\gamma = \{X_i : \gamma_i = 1\}$ and $n_\gamma = \sum_{i=1}^{p} \gamma_i$.

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- Prior distributions

$$\pi_\gamma\left(\beta_\gamma, \sigma^2\right) = \mathcal{N}\left(\beta_\gamma; 0, \delta^2\sigma^2 I_{n_\gamma}\right) \mathcal{IG}\left(\sigma^2; \frac{\nu_0}{2}, \frac{\gamma_0}{2}\right)$$

and $\pi(\gamma) = \prod_{i=1}^{p} \pi(\gamma_i) = 2^{-p}$.

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- However, we know that

$$\pi\left(\gamma, \beta_\gamma, \sigma^2 \middle| D\right) = \pi\left(\gamma \middle| D\right) \pi\left(\beta_\gamma, \sigma^2 \middle| D, \gamma\right)$$

where

$$\pi\left(\gamma \middle| D\right) \propto \pi\left(D \middle| \gamma\right) \pi\left(\gamma\right)$$

and

$$\pi\left(D \middle| \gamma\right) = \int \pi\left(D, \beta_\gamma, \sigma^2 \middle| \gamma\right) d\beta_\gamma d\sigma^2.$$

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  - Optional step: Sample $\left(\beta_{\gamma,i}, \sigma_i^2\right) \sim \pi\left(\beta_\gamma, \sigma^2 \middle| D, \gamma\right)$.

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- Updating correlated components together would increase significantly the convergence speed of the algorithm at the cost of an increased complexity.

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- Each time you face a problem, you need to think hard about it to design an efficient algorithm.
- Except the choice of the partitions of parameters, the Gibbs sampler is parameter free; this does not mean it is efficient.