

# Collusion in Unrepeated, First-Price Auctions with an Uncertain Number of Participants

Kevin Leyton-Brown\*    Moshe Tennenholtz†    Navin A.R. Bhat‡    Yoav Shoham§

## Abstract

We consider the question of whether collusion among bidders (a “bidding ring”) can be supported in equilibrium of an unrepeated first-price auctions. Unlike previous work on the topic such as that by McAfee and McMillan [1992] and Marshall and Marx [2007], we do not assume that non-colluding agents have perfect knowledge about the number of colluding agents whose bids are suppressed by the bidding ring, and indeed even allow for the existence of multiple cartels. Furthermore, while we treat the association of bidders with bidding rings as exogenous, we allow bidders to make strategic decisions about whether to join bidding rings when invited. We identify a bidding ring protocol that results in an efficient allocation in Bayes–Nash equilibrium, under which non-colluding agents bid straightforwardly, and colluding agents join bidding rings when invited and truthfully declare their valuations to the ring center. We show that bidding rings benefit ring centers and all agents, both members and non-members of bidding rings, at the auctioneer’s expense. The techniques we introduce in this paper may also be useful for reasoning about other problems in which agents have asymmetric information about a setting.

## 1 Introduction

We consider ways that agents can gain by coordinating their bidding in non-repeated single-good auctions, even when all agents still act selfishly. In practice, reduction of revenue due to bidder collusion is a significant threat to auctioneers. Understanding the topic theoretically can help auctioneers to choose an auction type and to modify the rules of their auctions in order to make collusion more difficult. Collusion has been observed to occur in both repeated and single-auction settings; the latter is the focus of our work.

Much work has studied the question of how collusion can be supported in second-price auctions. One of the first formal papers on this topic was by Graham and Marshall [1987], who described a *knockout procedure*: agents announce their bids in a knockout auction; only the highest bidder goes to the auction, but this bidder must pay a ring center the amount of his gain relative to the case where there was no collusion. The ring center pays each agent in advance; the amount of this payment is calculated so that on expectation the ring center will budget-balance *ex ante*, before knowing the agents’ valuations. This work has been extended to deal with variations in the knockout procedure, differential payments, and relations to the Shapley value [Graham *et al.*, 1990]. The case where only some of the agents are part of the cartel is discussed by Mailath and Zemsky [1991], who also derived a mechanism in which ring centers achieve *ex post* budget balance. Work by von Ungern-Sternberg [1988] considers collusion in second-price auctions where the designated winner of a cartel is not the agent with the highest valuation. Although we are not aware of any work that presents this result, Graham and Marshall’s protocol also extends directly to environments containing uncertain numbers of bidders, multiple rings, and non-ring bidders. Less formal discussion of collusion in auctions can be found in a wider variety of papers. For example, Hendricks and Porter [1989] surveyed mechanisms that are likely to facilitate collusion in auctions, as well as methods for the detection of such schemes. Robinson [1985] discussed and compared the stability of rings associated with classical auctions, concentrating on the case where the valuations of agents in the cartel are honestly reported. Collusion is also discussed in other settings, *e.g.*, in the context of general Bertrand or Cournot competition [Cramton & Palfrey, 1990].

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\*kevinlb@cs.ubc.ca; Computer Science Department, University of British Columbia

†moshet@ie.technion.ac.il; Faculty of Industrial Engineering and Management, Technion

‡Navin.Bhat@bmo.com; Bank of Montreal

§shoham@cs.stanford.edu; Computer Science Department, Stanford University

## 1.1 Collusion in First-Price Auctions

Conventional wisdom holds that it is harder to sustain collusion in first-price auctions. Intuitively, unlike in second-price auctions, bidders must lower their bids to profit from reduced competition, making them susceptible to deviations from fellow cartel members. An influential paper by McAfee and McMillan [1992] presented the first theoretical results on collusion in this setting. Their work assumes that a fixed number of agents participate in the auction and that all agents are part of a single cartel that coordinates its behavior in the auction. The authors show optimal collusion protocols for “weak” cartels (in which transfers between agents are not permitted: all bidders bid the reserve price, using the auctioneer’s tie-breaking rule to randomly select a winner) and for “strong” cartels (the cartel holds a knockout auction, the winner of which bids the reserve price in the main auction while all other bidders sit out; the winner distributes some of his gains to other cartel members through side payments). Relatedly, McAfee [1994] studied collusion in (effectively) a first-price setting, with agents facing endogenous choices about whether to join the cartel, and with various asymmetries in the setting. This work considered only a single cartel to which all agents were invited, did not consider the problem of cartel enforcement, and performed a full information analysis.

Though it was not the focus of their work, McAfee and McMillan [1992] also considered the case where in addition to a single cartel there are also non-colluding agents. However, they give results only for two cases: (1) where non-cartel members bid without taking the existence of a cartel into account (*e.g.*, they are irrational, or they hold the false belief that no cartels exist) and (2) where each agent  $i$  has valuation  $v_i \in \{0, 1\}$ . McAfee and McMillan explain that they do not attempt to deal with general strategic behavior in the case where the cartel consists of only a subset of the agents; furthermore, they do not consider the case where multiple cartels can operate in the same auction.

The only other work of which we are aware that proposes a self-enforcing collusion protocol for single (i.e., unrepeated) first-price auctions is Marshall and Marx [2007]. This paper considers both first- and second-price auctions, addresses both the repeated and unrepeated cases, and proposes both “bid submission mechanisms” (ring protocols in which the ring center is able to submit bids on the bidders’ behalf) and “bid coordination mechanisms” (ring protocols in which the ring center is only able to suggest bid amounts to agents). They assume that there is only one bidding ring involving some subset of the agents, and that the existence and formation or non-formation of the cartel is common knowledge among all agents participating in the auction.

Finally, a number of other papers have studied collusion in repeated first-price auctions or have presented results that bear directly on that setting [Aoyagi, 2003; Hörner & Jamison, 2007; Skrzypacz & Hopenhayn, 2004; Blume & Heidhues, 2008; Fudenberg *et al.*, 1994; Feinstein *et al.*, 1985]. Overall, these mechanisms tend to work by using folk-theorem-like constructions, incentivizing some bidders to rotate their participation in the auction through the threat of future punishment or the promise of future opportunities to collude.

## 1.2 Novelty of Our Work

Our paper<sup>1</sup> differs from related work on collusion in unrepeated first-price auctions by relaxing several assumptions. Comparing to McAfee and McMillan [1992], we allow for the possibility that some bidders will not belong to a cartel without assuming that agents are irrational or hold false beliefs in equilibrium, and we allow that more than one cartel may exist, introducing the new wrinkle that cartel members must reason about the possibility of other cartels. We model bidders’ valuations as real numbers drawn from an interval according to an arbitrary distribution (as compared, *e.g.*, to the case studied in McAfee and McMillan [1992] where valuations take one of only two discrete values), and the decision of whether or not to join a bidding ring as part of an agent’s strategy.

The recent paper by Marshall and Marx [2007] is closer to our work, but still differs significantly. In their terminology our work proposes a bid submission mechanism for single first-price auctions; thus, we contrast our work with their mechanism for the same setting. Marshall and Marx [2007] extend McAfee and McMillan [1992] in some of the same ways that we do: they identify a Bayes-Nash equilibrium of a bidding ring mechanism that is not required to involve all bidders; they consider bidders with valuations drawn from real intervals; they include bidders’ decisions about whether to join a bidding ring as part of the

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<sup>1</sup>Some of our previously published work is related to this paper. In Leyton-Brown *et al.* [2000] we considered bidding rings under the assumptions that only a single bidding ring exists, and that bidders who were not invited to join the ring are not aware that bidding rings could exist. Leyton-Brown *et al.* [2002] was a poster presentation of work that grew into this paper; we also posted a working version of this full paper on [arXiv.org](https://arxiv.org/abs/cs/0201017v1) in 2002 (arXiv:cs/0201017v1).

bidders’ strategy spaces. However, in several senses their results are more limited than ours. First and most importantly, they assume that non-cartel bidders have complete information about the existence of a cartel in the auction and about the number of bids suppressed by this cartel. We find this to be a strong assumption, since the bidding ring is a clandestine arrangement that hides its existence from the seller. Second, they do not allow for the existence of multiple cartels. It is apparently important to allow for this possibility: in an empirical analysis of forest service auctions, Marshall and Marx themselves present evidence that two cartels participated in the same first-price auction. Third, they do not allow agents to take their valuations into account when deciding whether or not to join the cartel (i.e., they require that this choice be made *ex ante*, not *ex interim*). Fourth, they are not always able to guarantee that a decision to join the bidding ring satisfies even *ex ante* individual rationality, though they do show this for some simple valuation distributions. On the other hand, in some senses Marshall and Marx [2007] present stronger results than we do. First, their protocol still works even when the auctioneer does not announce the identity of the winner; ours needs this information in order to require the winner to make a payment to the ring center. Second and most notably, they allow for bidders with asymmetric valuation distributions, while we consider only the symmetric case. Interestingly, they write “We focus on the heterogeneous IPV model, which is important for the study of collusion because, even if bidders are homogeneous, collusion creates heterogeneity among them” [Marshall and Marx 2007, page 377]. One of our main technical results is that a collusive protocol can be constructed to cancel out this latter heterogeneity, inducing all agents to bid symmetrically in the auction.

In what follows, we begin by defining an auction model and establishing some notation in Section 2. In Section 3 we give our bidder model. Some prominent features are that the number of bidders is stochastic while the number of (unsuppressed) bids placed in the auction is perfectly observed; there can be multiple cartels; all agents invited to participate in one are (privately) able to observe the number of other agents who receive invitations. We then identify a bidding ring protocol (Section 3) and show that responding truthfully to this protocol constitutes an equilibrium. Section 4 demonstrates, via an argument related to revenue equivalence, that the collusion benefits both ring and non-ring members as well as ring centers, at the auctioneer’s expense. Finally we show (as Marshall and Marx [2007] do in their own setting) that our bidding ring protocol can be disrupted by allowing agents to place undetected shill bids. In the end, our work establishes that collusion can be supported in a first-price auction setting that is in many respects less restricted than had previously been identified; at the same time, our findings support the conventional wisdom that collusion is more complex and easier to disrupt in first-price auctions than in second-price auctions.

## 2 Modeling Collusion in First-Price Auctions

In this section we consider different models of unrepeated first-price auctions, and evaluate the suitability of each for modeling collusion. The choice of information structure is very important for the study of collusion in first-price auctions, and in the end, we argue for a model in which the number of bidders is stochastic, but the number of participants in the auction is announced publicly before bids are submitted. We begin by considering simpler models both to explain why they are inappropriate and to introduce notation and state theoretical results that we will need later.

An economic environment  $E$  consists of a finite set of agents who have non-negative valuations for a good at auction, and a distinguished agent 0, the seller or center. Let  $\mathcal{T}$  be the set of possible agent types. The type  $\tau_i \in \mathcal{T}$  of agent  $i$  is the pair  $(v_i, s_i) \in V \times \mathcal{S}$ .  $v_i$  denotes an agent’s valuation, which we assume represents a purely private valuation for the good.  $v_i$  is selected independently of the  $v_j$ ’s of other agents from a known cumulative distribution,  $F$ , which is continuously differentiable with support on the interval  $[0, 1]$ ; we denote its non-cumulative distribution (density function) as  $f$ . Throughout the paper we will use upper- and lower-case symbols to respectively denote such cumulative and non-cumulative distributions. By  $s_i \in \mathcal{S}$  we denote agent  $i$ ’s private signal about the number of agents in the auction, and let  $\emptyset$  denote a null signal. We will vary the set of possible signals  $\mathcal{S}$  throughout the paper. We assume that the utility function of agent  $i$  is linear (the agent is risk-neutral), free of externalities, and normalized (the agent achieves zero utility for not winning the good and paying nothing). When asked to pay  $t$ , let the utility of agent  $i$  (having valuation  $v_i$ ) be  $v_i - t$  if  $i$  is allocated the good and  $-t$  otherwise.  $b_i : \mathcal{T} \rightarrow \mathbb{R}^+ \cup \{\bar{P}\}$  denotes agent  $i$ ’s strategy, a mapping from  $i$ ’s type  $\tau_i$  to his declaration in the auction. The declaration  $\bar{P}$  indicates that  $i$  will not participate in the auction.

We begin by defining a “classical” economic environment, which we denote  $E_c$ . In  $E_c$  let  $\mathcal{S} = \{\emptyset\}$ , meaning that agents never receive private information about the number of other agents in the auction. In this environment the number of agents,  $n$ , is a constant and hence is common knowledge among the agents.

## 2.1 Classical First-Price Auctions

Most familiar is what we call the “classical” first-price auction, where the number of participants is part of the economic environment (this is what we have called  $E_c$ ). Using standard equilibrium analysis (*e.g.*, following Riley and Samuelson [1981]) the unique symmetric equilibrium can be identified.

**Proposition 2.1** (see, *e.g.*, Riley & Samuelson, 1981) *If valuations are selected from a continuous distribution  $F$  having finite support, then the unique symmetric equilibrium is for each agent  $i$  to bid the amount<sup>2</sup>*

$$b^e(v_i, n) = v_i - F(v_i)^{-(n-1)} \int_0^{v_i} F(u)^{n-1} du. \quad (1)$$

We are interested in bidding ring protocols in which bidders with low valuations drop out of the main auction, thereby causing the ring’s expected payment to be reduced. (For example, this is the flavor of McAfee and McMillan’s protocol in which cartel non-members are assumed not to behave strategically.) It is thus important to verify that the amounts of agents’ equilibrium bids decrease as the number of participating agents decreases. It is easy to show that this always occurs in  $E_c$ .

**Proposition 2.2**  $\forall v, \forall j \geq 2, b^e(v, j+1) > b^e(v, j)$ .

**Proof.** From Equation (1), we can write

$$b^e(v, j+1) - b^e(v, j) = \int_0^v \left(1 - \left(\frac{F(u)}{F(v)}\right)\right) \left(\frac{F(u)}{F(v)}\right)^{j-1} du. \quad (2)$$

The first factor in the integrand is clearly always positive, so the right-hand side of Equation (2) is positive. Thus  $b^e(v, j)$  is strictly increasing in  $j$ . ■

Although bidders’ expected payments do decrease with the number of auction participants, collusion based on dropping bidders is nevertheless nonsensical in the classical first-price auction environment. Because bidders’ strategies depend on the number of agents in the economic environment (which is common knowledge), their bid amounts will not change if cartel members with low valuations fail to submit bids. This is a problem with our auction model rather than with collusion in first-price auctions *per se*—in practice bidders might *not* know the exact number of agents in the economic environment, and thus might adopt a strategy that depends on the number of bidders who choose to participate in the auction.

## 2.2 First-Price Auctions with a Stochastic Number of Bidders

One way of modelling agents’ uncertainty about the number of opponents they face is to say that the number of participants is drawn from a probability distribution; while the actual number of participants is not observed, the *distribution* is commonly known. First-price auctions of this kind were introduced by McAfee and McMillan [1987] (in an earlier paper that makes no mention of collusion).

To describe this setting formally, we must define a new economic environment; we denote it  $E_s$ . Let agents’ types be the same as in  $E_c$  (in particular, let all agents again receive the null signal  $\emptyset$ ). Let  $\mathcal{D}_\ell$  be the set of all probability distributions  $d : \mathbb{Z} \rightarrow \mathbb{R}$  having support on any subset of the integers greater than or equal to  $\ell$ . Denote the distribution over the number of agents in the auction as  $p \in \mathcal{D}_2$ . After nature determines the number of agents by drawing from  $p$ , let the name of each agent be selected from the uniform distribution on  $[0, 1]$ .<sup>3</sup> The unique symmetric equilibrium is as follows.

<sup>2</sup>To establish notation that will be useful in what follows, we have written the bidding strategy as a function parameterized by  $v_i$  and  $n$ . However, note that  $v_i$  and  $n$  play different roles:  $v_i$  is a bidder’s private information, while  $n$  is common knowledge among all bidders.

<sup>3</sup>This technical assumption prevents any agent from gaining information about the number of agents in the auction from his own identity and the fact of his selection. We make similar assumptions in Section 3.

**Proposition 2.3 (Harstad et al., 1990)** *If valuations are selected from a continuous (cumulative) distribution  $F$  having finite support, and the number of bidders is selected from the (non-cumulative) distribution  $p$ , then it is a unique symmetric equilibrium for each agent  $i$  to bid the amount<sup>4</sup>*

$$b^e(v_i, p) = \sum_{j=2}^{\infty} \frac{F^{j-1}(v_i)p(j)}{\sum_{k=2}^{\infty} F^{k-1}(v_i)p(k)} b^e(v_i, j). \quad (3)$$

Recall that we need the property that the amounts of agents' equilibrium bids decrease with the number of participating agents, as indeed was the case in  $E_c$ . While one might also expect that the same property holds in  $E_s$ , we show in Appendix A that simply knowing that distribution  $p$  has a smaller expected number of participants than distribution  $p'$ —or even that  $p$  stochastically dominates  $p'$ —is not enough to determine which distribution will give rise to a lower symmetric equilibrium bid for a given valuation. One class of pairs of distributions  $(p, p')$  for which it *does* hold that  $b^e(v_i, p)$  is always less than  $b^e(v_i, p')$  are those  $(p, p')$  for which  $p'$  *convolutively dominates*  $p$ . Let  $x, y \in \mathcal{D}_0$  be independent random variables, and consider the distribution of their sum  $p'$ . Since  $x$  and  $y$  are independent, the probability of their sum being  $m$  is just the sum of the product of the individual probabilities of values of  $x$  and  $y$  that sum to  $m$ ,

$$p'(m) = \sum_{j=0}^{\infty} p(m-j)q(j). \quad (4)$$

Summing independent distributions in this way corresponds to convolution, which we denote symbolically as  $p' = p * q$ . Now we can define convolutive dominance.

**Definition 2.4 (convolutive dominance)** *Given two (non-cumulative) distributions  $p, p'$ , the function  $p'$  exhibits convolutive dominance of  $p$  if there exists a probability density function  $q \in \mathcal{D}_0$  with  $q(0) \neq 1$  such that  $p' = p * q$ .*

**Lemma 2.5** *For all  $p, p' \in \mathcal{D}_2$ , if  $p'$  convolutively dominates  $p$  then  $b^e(v_i, p) < b^e(v_i, p')$ .*

**Proof.** The proof is long and quite technical, and so is deferred to Appendix A. ■

This model presumes that all bidders have the same beliefs about the number of participants in the auction. However, bidders belonging to a bidding ring should actually be better informed about the number of participants, since they know that the other members of their ring exist. In the next section, we define a final economic environment that allows us to model this phenomenon. In this environment, agents start out unsure about the number of participants, but receive additional, private information about this number corresponding to whether or not they were invited to join a bidding ring and, if so, its size.

## 2.3 Asymmetric Information about the Number of Bidders

We now define economic environment  $E_{sa}$  by extending  $E_s$  to allow agents to receive (different) *ex interim* information about the number of agents participating in the auction. Recall that we defined the type  $\tau_i \in \mathcal{T}$  of agent  $i$  as the pair  $(v_i, s_i) \in V \times \mathcal{S}$ . Define  $v_i$  as in  $E_c$ , and let  $\mathcal{S} \subseteq \mathbb{N} \cup \{\emptyset\}$ . As above, let  $p$  denote the distribution over the number of agents. Now also let  $p_{s_i}$  denote the distribution over the total number of agents conditional on  $i$ 's signal  $s_i$ . Assume that  $p_{s_i} \neq p_{s_j}$  if and only if  $s_i \neq s_j$ . Thus, each signal provides different information about the distribution over the number of bidders in the auction, and  $p_{s_i}$  is the *ex interim* belief about this distribution that  $i$  holds after observing his signal. Let  $p_{\emptyset} = p$ : the null signal is uninformative.

To our knowledge, no general methods are known for calculating the equilibria of arbitrary mechanisms in which agents have asymmetric information about the number of participants. However, here we show that it is possible to design mechanisms that impose asymmetric payment rules on agents with asymmetric signals

<sup>4</sup>Observe that  $b^e(v_i, j)$  is the amount of the equilibrium bid for a bidder with valuation  $v_i$  in a classical first-price auction setting with  $j$  bidders as described in Equation (1) above. We distinguish the two functions by whether their second argument is an integer or a probability distribution. As before, though we write the bidding strategy as a function with two arguments, note that  $p$  is common knowledge among the bidders.

in a way that gives rise to symmetric, truthful equilibria. We dub such mechanisms *symmetrizing auctions*.<sup>5</sup> Assume for this section that the mechanism (but no other agent) knows each agent’s signal about the number of participants.<sup>6</sup> Denote a bid from agent  $i$  as  $\mu_i \in \mathbb{R}^+ \cup \{\bar{P}\}$ , the tuple of bids from all agents as  $\pi \in \Pi$  and an auction’s transfer function for agent  $i$  (determining  $i$ ’s payment) as  $t_i : \mathbb{R}^+ \cup \{\bar{P}\} \times \Pi \rightarrow \mathbb{R}$ .

We say that an auction is *aligned* with signal  $s$  if, in a (hypothetical) economic environment where the number of agents is drawn from  $p_s$  and all agents receive the null signal, the auction would be efficient and incentive compatible.

**Definition 2.6 (auction aligned with a signal)** *An auction  $M_s$  is aligned with signal  $s \in S$  if  $M_s$  allocates the good to an agent  $i$  with  $\mu_i \in \max_j \mu_j$ , and  $M_s$  is a symmetric truth-revealing direct mechanism for a stochastic number of agents drawn from  $p_s$ , each of whom receives the signal  $\emptyset$ .*

Now we identify a class of asymmetric auctions (in which agents can receive different signals and are subject to potentially different transfer functions) that nevertheless have symmetric truthful equilibria. Intuitively, asymmetry is introduced into the transfer functions in a way that exactly balances the informational asymmetry among the agents. We call these auctions *symmetrizing*.

**Definition 2.7 (symmetrizing auction)**  *$\bar{M}$  is a symmetrizing auction if it allocates the good to an agent  $i$  with  $\mu_i \in \max_j \mu_j$ , and if each agent  $i$  is made to transfer  $t_{s_i}(\mu_i, \pi)$  to the center, with  $t_{s_i}$  taken from an auction  $M_{s_i}$  that is aligned with signal  $s_i$ .*

We can now prove the following lemma, which plays a crucial role in this paper’s main result. The proof is similar to that of the revelation principle.

**Lemma 2.8** *Truth revelation is an equilibrium of symmetrizing auctions.*

**Proof.** The payoff of agent  $i$  is determined (only) by the allocation rule, the transfer function  $t_{s_i}$ , the distribution over the number of agents in the auction, and all agents’ strategies. Assume that the other agents are truth revealing; then each other agent’s behavior, the allocation rule, and agent  $i$ ’s payment rule are all identical in  $\bar{M}$  and  $M_{s_i}$ . Conditioned on his private information  $s_i$ , agent  $i$ ’s posterior is that  $p_{s_i}$  is the distribution over the number of agents in the auction. Since truth-revelation is an equilibrium in  $M_{s_i}$  when the distribution of agents is  $p_{s_i}$ , truth-revelation is agent  $i$ ’s best response in  $\bar{M}$ . ■

To provide intuition for what is being claimed by Lemma 2.8, and to emphasize its generality, we use it to analyze two example domains. The first illustrates that truthful mechanisms can involve different payment rules for different agents, even when all agents receive the null signal.

**Example 2.9 (different payment rules)** *Let  $p(4) = 1$ , and for all  $i \in N$ , let  $s_i = \emptyset$ . (Thus, effectively we consider the classical environment  $E_c$ , with  $n = 4$ . Recall that  $p_\emptyset = p$ .) Let  $M_1$  be a revelation mechanism for a first-price auction in  $E_c$ : agents declare their valuations, and the winner  $i$  is charged  $b^e(v_i, 4)$ .  $M_1$  is aligned with the signal  $\emptyset$ : it allocates the good to the highest bidder, and truthful bidding is a symmetric equilibrium of  $M_1$  for a stochastic number of agents drawn from  $p_\emptyset$ , each of whom receives the signal  $\emptyset$ . Let  $M_2$  be a second-price auction.  $M_2$  is also aligned with the signal  $\emptyset$ : again, it allocates the good to the highest bidder, and truthful declaration is again an equilibrium (indeed, in weakly dominant strategies) for a stochastic number of agents drawn from  $p_\emptyset$ , each of whom receives the signal  $\emptyset$ . Now consider an auction  $\bar{M}$  where odd-numbered agents are subject to the payment rule from  $M_1$ , and even-numbered agents are subject to the payment rule from  $M_2$ . By Lemma 2.8, truth-revelation is an equilibrium of  $\bar{M}$ .*

In Example 2.9, although agents are subject to different transfer functions, they all make the same expected payments. This property no longer holds when we consider settings in which agents receive different signals about the number of participants.

<sup>5</sup>We note that symmetrizing auctions may have applications beyond the study of collusion, as illustrated by the examples that follow. Furthermore, they are even more general than presented here: e.g., they can be straightforwardly extended to inefficient allocation rules.

<sup>6</sup>While this assumption limits the application of symmetrizing auctions, it is often reasonable; again, see the following examples. In our application of symmetrizing auctions to collusion, the assumption will be justified because each ring center will formally act as part of the mechanism, and will know the number of agents invited to participate in its bidding ring.

**Example 2.10 (glass-box silent auction)** Consider a setting in which an uncertain number of agents arrive in a random sequence and place sealed bids into a glass box, and in which the high bidder wins and pays his bid. Let agents' valuations be uniformly distributed on  $[0, 1]$ . Let the number of agents be drawn from the uniform distribution over  $\{2, 3, 4\}$ ; denote this distribution  $p$ . Agent  $i$ 's signal is his order of arrival; intuitively, he infers this by seeing the number of envelopes in the box. We claim that it is an equilibrium for a bidder  $i$  with valuation  $v_i$  and signal  $s_i$  to bid  $b^e(v_i, p_{s_i})$ , where  $p_{s_i}$  is the posterior distribution obtained by conditioning  $p$  on  $s_i$ . Equivalently, consider a mechanism  $M_{gb}$  that asks each agent for his valuation, identifies the bidder  $h$  with the highest valuation  $v_h$ , and awards him the good at the price  $b^e(\mu_h, p_{s_h})$ . This is simply a revelation mechanism for the glass-box silent auction (i.e., it asks agents for their valuations and then applies the strategy we previously claimed was an equilibrium); thus, we claim that  $M_{gb}$  is truthful.

First, we claim that  $M_{gb}$  is a symmetrizing auction. To see this, consider an agent  $i$  with signal 3. Such an agent assigns ex interim probabilities of  $0, \frac{1}{2}$ , and  $\frac{1}{2}$  to there being 2, 3 and 4 agents in the auction respectively; we write  $p_3 = (0, \frac{1}{2}, \frac{1}{2})$ . Now define a mechanism aligned with signal 3. Let  $M_3$  be a revelation mechanism for a first price auction in  $E_s$ , with distribution over the number of bidders  $p^{(M_3)} = p_3$ . Agents in  $M_3$  all receive the null signal.  $M_3$  is aligned with signal 3 of  $M_{gb}$ : in equilibrium in  $M_3$ , agents truthfully (and symmetrically) declare their valuations, and the bidder with the highest valuation (and bid) wins and pays  $b^e(v, p_3)$ . Analogous to  $M_3$ , define auctions aligned with signals 1, 2, and 4.  $M_1$  uses  $p^{(M_1)} = p_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .  $M_2$  uses  $p^{(M_2)} = p_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .  $M_4$  uses  $p^{(M_4)} = p_4 = (0, 0, 1)$ .

Now consider the best response of agent  $i$  with signal 3 in both  $M_{gb}$  and  $M_3$ , assuming all other agents bid truthfully. Agent  $i$  will win in both mechanisms if and only if his bid is higher than all others' valuations. In both mechanisms,  $i$  obtains utility  $v_i - b^e(\mu_i, p_3)$  if his bid is highest, and zero otherwise. It follows that for  $i$ , the best response to truthful bidding in  $M_{gb}$  is the same as the best response to truthful bidding in  $M_3$ . Since  $M_3$  is the revelation mechanism associated with Proposition (2.3), the best response is to bid truthfully. More broadly, it follows from Lemma 2.8 that the same argument applies to agents with other signals.

v	s	$b^e(v, p_s)$	v	s	$b^e(v, p_s)$
0.50	$\emptyset, 1, 2$	0.292	0.90	$\emptyset, 1, 2$	0.567
	3	0.347		3	0.636
	4	0.375		4	0.675
0.75	$\emptyset, 1, 2$	0.461	1.00	$\emptyset, 1, 2$	0.639
	3	0.527		3	0.708
	4	0.563		4	0.750

Table 1: Equilibrium payment conditional on winning, by valuation and signal.

Table 1 gives  $b^e(v, p_s)$  for every possible signal and for various different valuations. We make two observations. First, the signals 1 and 2 provide no information about  $p$ , so they are strategically equivalent to the null signal. Second, in equilibrium an agent essentially pays a premium for appearing later in the order, and hence the auctioneer's revenue depends on the bidding order. For example, if bidders arrived in sequence with valuations 0.50, 1.00, 0.75, 0.90, the bidder with valuation 1 would win and pay 0.639. If instead the second bidder had a lower valuation, so that the sequence was 0.50, 0.10, 0.75, 0.90, the fourth bidder would win, paying a higher amount, 0.675, despite his lower valuation.

Although we will need the results we have presented so far in the analysis of our main result, the auction models offered by both  $E_s$  and  $E_{sa}$  are still insufficient for modeling collusion in a first-price auction. In both models each agent knows only the distribution of agents interested in participating in the auction, meaning he has no way of being affected by low-valued agents who drop out. We need a model in which agents know how many agents *bids* in the auction, while being uncertain about the number of agents who chose not to bid.

## 2.4 First-Price Auctions with Participation Revelation

When an auction takes place in an auction hall, no bidder can be sure about how many potential bidders stayed home, but every bidder can count the number of people in the room before placing his or her bid. We model this scenario as a *first-price auction with participation revelation*, which we define as follows:

1. Agents indicate their intention to bid in the auction.
2. The auctioneer announces  $n$ , the number of agents who registered in the first phase.
3. Agents submit bids to the auctioneer. The auctioneer will only accept bids from agents who registered in the first phase.
4. The agent who submitted the highest bid is awarded the good for the amount of his bid; all other agents are made to pay 0.

When a first-price auction with participation revelation operates in  $E_s$  or  $E_{sa}$ , the equilibrium of the corresponding *classical* first-price auction holds.

**Proposition 2.11** *In both  $E_s$  and  $E_{sa}$  it is an equilibrium of the first-price auction with participation revelation for every agent  $i$  to indicate the intention to participate, and to bid according to  $b^e(v_i, n)$ .*

**Proof.** First assume that all agents indicate an intention to participate in the first phase. Then the number of agents announced by the auctioneer is equal to the total number of agents in the economic environment. From Proposition 2.1 it is best for agent  $i$  to bid  $b^e(v_i, n)$  when it is common knowledge that the number of agents in the economic environment is  $n$ . Each agent obtains positive expected utility from bidding in this way, and would obtain zero expected utility from choosing not to participate. Thus, for every agent it is a best response to participate. ■

First-price auctions with participation revelation may often be a more realistic model than classical first-price auctions, since the former allows that bidders may not know *a priori* the number of opponents they will face. When bidders are unable to collude, there is no strategic difference between these two mechanisms, justifying the common use of the simpler classical model. For the study of bidding rings, however, there is an important difference between the mechanisms—we are now able to look for a collusive equilibrium in which bidder strategies depend only on the number of other agents who place bids in the auction. More specifically, unlike the other first-price auction models we considered, first-price auctions with participation revelation have the property that non-cartel-members' bids change when one or more cartel members choose not to participate in the auction, even if the non-cartel bidders are rational and have true beliefs about the economic environment.

### 3 Bidding Rings for First-Price Auctions

We define the economic environment  $E_{br}$  as an extension of  $E_{sa}$ , consisting of: the distinguished agent 0 who offers a good for sale; a randomly-chosen set of ring centers, who do not value the good; and a randomly-chosen set of agents, each of whom receives an invitation from exactly one ring center. Recall that the type  $\tau_i \in \mathcal{T}$  of agent  $i$  is the pair  $(v_i, s_i) \in V \times \mathcal{S}$ . Define  $v_i$  as in  $E_c$  (an independent, private valuation). Let  $\mathcal{S} \subseteq \mathbb{N} \setminus \{0\}$ ;  $s_i \in \mathcal{S}$  represents the number of agents in  $i$ 's bidding ring, which is available to  $i$  as private information. Thus, when he observes his signal  $s_i$ , an agent  $i$  learns that there are at least  $s_i$  agents in the auction, all of whom share his ring center. We model singleton bidders as bidding rings with only one invited agent; in this case we consider the ring automatically disbanded and ignore the ring center.

As before,  $p$  denotes the distribution over the total number of agents,<sup>7</sup> and  $p_{s_i}$  denotes the distribution over the total number of agents conditional on  $i$ 's signal  $s_i$ . Let  $\gamma_C(n_c) \in \mathcal{D}_2$  denote the probability that an auction will involve  $n_c$  ring centers. After a value is realized from  $\gamma_C(n_c)$ , the name of each ring center is selected from the uniform distribution on  $[0, 1]$ . The numbers of agents associated with different bidding rings are taken to be independent. Let  $\gamma_A(n) \in \mathcal{D}_1$  denote the probability that  $n$  agents will be associated with a given bidding ring. After the number of agents is determined, the name of each agent associated with a potential ring center is selected from the uniform distribution on  $[0, 1]$ .

In order to write an expression for  $p$ , we must define some notation. First, observe that convolution (previously discussed in Section 2.2) is associative and commutative. Denote repeated convolution of

<sup>7</sup>Observe that  $p$  describes with the number of agents that *exist*, not the number of agents who elect to participate.



distribution  $d$  as

$$\bigotimes_n d \equiv \overbrace{d * d * d * \dots * d}^{d \text{ repeated } n \text{ times}}. \quad (5)$$

We define the Kronecker delta (an indicator function) as

$$\delta_m(j) \equiv \begin{cases} 1 & \text{if } j = m; \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Now the following identity can be inferred from Equation (4):

$$\bigotimes_j \delta_k = \delta_{(j \cdot k)}. \quad (7)$$

We can now write

$$p = \sum_{n=2}^{\infty} \gamma_C(n) \left( \bigotimes_n \gamma_A \right), \quad (8)$$

and for  $i$ 's posterior beliefs about  $p$ , conditioned on his signal  $s_i$ ,

$$p_{s_i} = \sum_{n=2}^{\infty} \gamma_C(n) \left( \bigotimes_{n-1} \gamma_A \right) * \delta_{s_i}. \quad (9)$$

We denote by  $p_{n,s_i}$  the distribution over the number of agents, conditioned on  $i$ 's signal  $s_i$  and the additional information that there are a total of  $n$  bidding rings (and/or singleton bidders):

$$p_{n,s_i} = \left( \bigotimes_{n-1} \gamma_A \right) * \delta_{s_i}. \quad (10)$$

**Example 3.1 (no collusion; unlikely collusion)** *A broadly attended internet auction with no collusion can be modeled by setting  $\gamma_A = \delta_1$ , so that all ‘‘cartels’’ correspond to singleton bidders. If 100 potential bidders attend with 10% probability each, then  $\gamma_C$  would be a discrete binomial distribution with parameters  $N_{bin} = 100$  and  $P_{bin} = 0.10$ , which is approximately Gaussian.*

*If collusion is unlikely but not impossible, we can replace  $\gamma_A$  by  $\delta_1 * G_{0.3}$ , where  $G_{0.3}$  is a discrete Poisson distribution with parameter 0.3. The corresponding density function has most of its weight on 1, indicating a singleton bidder as before, but mean of 1.3, reflecting a small probability of a cartel of size 2 or more.*

**Example 3.2 (coordination between campuses)** *Consider an auction for a block of time on a computer cluster, where the participants are large organizations (say IBM, Microsoft, the University of California, SAP). Between 2–4 of the organizations will participate in the auction, and 1–3 of each participating organization’s different campuses want to acquire the resource. To avoid bidding against itself, each organization wants to submit only one bid; however, because the campuses are independent and self-interested, they cannot be relied upon to report truthfully to central administration. We therefore model each organization as a cartel with a ring center, and each campus as an agent. Let  $\gamma_C$  be uniform over the set  $\{2, 3, 4\}$  and let  $\gamma_A$  be uniform over the set  $\{1, 2, 3\}$ . The prior distribution,  $p(j)$ , over the number of agents in  $E_{br}$  can be calculated using Equation (8). (Observe that its support is  $\{2, 3, \dots, 12\}$ .) If  $i$  has signal  $s_i = 3$ , his ex interim belief,  $p_3(j)$ , can be calculated using Equation (9). Conditioned on observing  $n = 3$  organizations (rings) and on signal  $s_i$ , the distribution  $p_{3,s_i}(j)$  over the number of agents can be calculated using Equation (10). Table 2 shows the distributions  $p(j)$ ,  $p_3(j)$ , and  $p_{3,3}(j)$ ; we also give  $p_{3,1}(j)$  to contrast it with  $p_{3,3}(j)$ .*

### 3.1 First-Price Auction Bidding Ring Protocol

We now define the protocol followed by a ring center who approaches  $k$  agents and who operates in conjunction with a first-price auction with participation revelation in the economic environment  $E_{br}$ .

$j$	2	3	4	5	6	7	8	9	10	11	12
$p(j)$	.037	.086	.152	.165	.165	.140	.115	.078	.041	.016	.004
$p_3(j)$	0	0	0.111	.148	.198	.148	.148	.123	.074	.037	.012
$p_{3,3}(j)$	0	0	0	.111	.222	.333	.222	.111	0	0	0
$p_{3,1}(j)$	0	.111	.222	.333	.222	.111	0	0	0	0	0

Table 2: Beliefs about the number of agents in Example 3.2

1. Each agent  $i$  sends a message  $\mu_i$  to the ring center. This message either declares the agent's valuation or declines the invitation to participate in the bidding ring.
2. If all  $k$  agents declare valuations (accept the invitation), then the ring center drops all bidders except the bidder with the highest reported valuation, whom we denote as bidder  $h$ . For this bidder the ring center indicates the intention to bid in the main auction, and places a bid of  $b^e(\mu_h, p_{n,1})$ .
3. Otherwise, the ring center indicates an intention to bid in the main auction on behalf of every agent who accepted the invitation to the bidding ring. For each such bidder  $i$ , the ring center submits a bid of  $b^e(\mu_i, p_{n-k+1,k})$ , where in this case  $n$  (the number of bidders announced by the auctioneer) will include all agents invited to the bidding ring.
4. If all bidders participate in the ring, the ring center pays each member  $c_{n,k} \geq 0$ , where  $c_{n,k}$  is a commonly known function of  $n$  and  $k$ . Note  $c_{n,k}$  is independent of the outcome of the auction and the amount each bidder bid.
5. If bidder  $h$  wins in the main auction, he is made to pay  $b^e(\mu_h, p_{n,1})$  to the center and  $b^e(\mu_h, p_{n,k}) - b^e(\mu_h, p_{n,1})$  to the ring center.

Observe that if an agent declines an invitation to participate in a bidding ring, the main auction will be asymmetric in the sense that different singleton bidders will have different information about the total number of bidders in the auction. Nevertheless, we can prove the following theorem, which is our main result.

**Theorem 3.3** *It is a Bayes-Nash equilibrium for all bidding ring members to choose to participate and to truthfully declare their valuations to their respective ring centers, and for all non-bidding ring members to participate in the main auction with bids of  $b^e(v, p_{n,1})$ .*

**Proof.** We begin by partitioning the space of all possible strategies and introducing notation to describe these partitions. Since agents who are invited to join a bidding ring have a richer set of strategic choices available to them than do singleton bidders, we partition each strategy space separately. We give each set a short name which we use throughout this proof, built up of the following six symbols:  $P/\bar{P}$  (participate/do not participate) and  $T/\bar{T}$  (bid truthfully/do not bid truthfully), given  $R/\bar{R}$  (a ring bidder/a non-ring bidder).

The space of bidding ring agent strategies is partitioned as follows:

- $(\bar{P}|R)$ : the agent either chooses not to participate in the auction at all, or declines participation in the bidding ring and then bids independently in the main auction;
- $(P\bar{T}|R)$ : the agent participates in the auction, accepts the invitation to join the bidding ring, and then lies to the ring center about his valuation;
- $(PT|R)$ : the agent participates in the auction, accepts the invitation to join the bidding ring, and declares his true valuation to the ring center.

For the non-ring bidder, the space of strategies is partitioned as follows:

- $(\bar{P}|\bar{R})$ : the agent chooses not to participate in the auction at all;
- $(P\bar{T}|\bar{R})$ : the agent participates in the auction, but does not bid  $b^e(v, p_{n,1})$  in the main auction;
- $(PT|\bar{R})$ : the agent participates in the auction with a bid of  $b^e(v, p_{n,1})$ .

Given two strategy sets  $X$  and  $Y$ , and given that all agents other than agent  $i$  follow the strategy  $(PT|R)$  or  $(PT|\bar{R})$  (as appropriate) we denote the proposition that agent  $i$ 's expected utility for following some strategy  $x \in X$  is greater than his expected utility for following any strategy  $y \in Y$  as  $u(X) > u(Y)$ .

This proof consists of two main parts, the first dealing with participation and the second dealing with bidding. In Part (1a) we show that  $u(PT|\bar{R}) > u(\bar{P}|\bar{R})$ , and in Part (1b) we consider the more complex case of ring bidders and show that  $u(PT|R) > u(\bar{P}|R)$ . In Part (2) we show simultaneously that  $u(PT|\bar{R}) > u(P\bar{T}|\bar{R})$  and that  $u(PT|R) > u(P\bar{T}|R)$ .

**Part 1a:**  $u(PT|\bar{R}) > u(\bar{P}|\bar{R})$ .

Recall that we assume that all other bidders bid according to  $(PT|\bar{R})$  or  $(PT|R)$ . If non-ring bidder  $i$  also bids according to  $(PT|\bar{R})$  then all bidders follow a symmetric strategy in the main auction. Thus  $i$  has a non-zero probability of winning the good and gaining a surplus, whereas he would receive a payoff of 0 if he declined participation.

**Part 1b:**  $u(PT|R) > u(\bar{P}|R)$ .

By the argument in Part (1a) a ring bidder  $i$  should likewise opt to participate in the auction; however, we must still consider whether  $i$  is best off accepting or rejecting his bidding ring invitation. As discussed above, if  $i$  rejects the invitation then the main auction will be asymmetric; thus, this part requires a nontrivial argument. We consider the case where  $c_{n,k} = 0$ , as this is the case where  $i$  has the least incentive to accept the invitation. In this discussion let  $n$  represent the *true* number of bidding rings and singleton bidders in the economic environment (*i.e.*, the value realized from the distribution  $\gamma_c$ ).

First, consider a different setting, which we denote  $(\star)$ : a first-price auction with an uncertain number of participants in economic environment  $E_s$ , with the number of agents distributed according to  $p_{n,s_i}$ . In  $(\star)$  all bidders have the same information as  $i$  and are subject to the same payment rule. Thus, from Proposition 2.3 it is a best response for  $i$  to bid  $b^e(v_i, p_{n,s_i})$ . Bidder  $i$ 's expected utility is the same in  $(\star)$  and when following the strategy  $(PT|R)$  in the real auction, because both auctions allocate the good to the bidder who submits the highest bid, both have the same distribution over the number of agents, and both implement the same payment rule for  $i$ . Thus we need to show that  $i$ 's expected utility after rejecting his bidding ring invitation is less than his expected utility in the equilibrium of  $(\star)$ .

Given that all other bidders follow the strategies  $(PT|R)$  and  $(PT|\bar{R})$ , if the bidding ring did not alter its behavior in response to  $i$ 's deviation then there would exist some distributions  $p$  and signals  $s_i$  for which  $i$  would gain by declining the ring's invitation.<sup>8</sup> However, the bidding ring *does* change its behavior in response to deviation: if  $i$  declines the invitation, the ring center sends all the other members of the ring into the main auction, causing the auctioneer to announce  $n + s_i - 1$  participants. As a result there will be  $s_i - 1$  bidders placing bids of  $b^e(v, p_{n,s_i})$  and  $n - 1$  other bidders placing bids of  $b^e(v, p_{n+s_i-1,1})$ . We can show that these  $n - 1$  bidders will always decrease  $i$ 's expected utility by bidding too high. Recall Equation (10):  $p_{n,s_i} = (\bigotimes_{n-1} \gamma_A) * \delta_{s_i}$ , and so  $p_{n+s_i-1,1} = (\bigotimes_{n+s_i-2} \gamma_A) * \delta_1$ . We can write  $\gamma_A = g_A * \delta_1$ , where  $g_A$  is the distribution over the number of agents in a bidding ring beyond the first agent. Then

$$\begin{aligned}
p_{n+s_i-1,1} &= \left( \bigotimes_{n-1} \gamma_A \right) * \left( \bigotimes_{s_i-1} \gamma_A \right) * \delta_1 \\
&= \left( \bigotimes_{n-1} \gamma_A \right) * \left( \left( \bigotimes_{s_i-1} g_A \right) * \delta_{s_i-1} \right) * \delta_1 \\
&= p_{n,s_i} * \left( \bigotimes_{s_i-1} g_A \right). \tag{11}
\end{aligned}$$

Since  $\gamma_A$  has support on a subset of the positive integers, it follows that  $g_A$  has support on a subset of the integers greater than or equal to zero. And since  $\gamma_A(1) < 1$ ,  $g_A(0) < 1$ . Then Equation (11) expresses convolutive dominance of  $p_{n+s_i-1,1}$  over  $p_{n,s_i}$ , so it follows from Lemma 2.5 that  $b^e(v, p_{n+s_i-1,1}) > b^e(v, p_{n,s_i})$ .

<sup>8</sup>Taking into account his signal and once the auctioneer has made an announcement,  $i$  would know that the number of agents is distributed according to  $p_{n,s_i}$ ; however, if he were to deviate then all agents would bid in the main auction as though the number of agents were distributed according to  $p_{n+1,1}$ . For certain values of  $p$  and  $s_i$ ,  $i$ 's expected loss from causing the auctioneer to announce one more participant is less than his expected gain from being able to bid freely and from not having to make a payment to the ring center if he wins.

Thus if  $i$  declines the ring's invitation, the singleton bidders and other bidding rings will bid a higher<sup>9</sup> function of their valuations than the equilibrium amount in  $(\star)$ . A bidder's expected gain in a first-price auction is always reduced as other bidders' bids increase, because his probability of winning decreases while his gain in the event of winning remains constant. This is the effect of  $i$  declining the offer to join his bidding ring: the  $s_i - 1$  other bidders from  $i$ 's bidding ring bid according to the equilibrium of  $(\star)$ , but the  $n - 1$  singleton and bidding ring bidders submit bids that exceed this amount. Thus declining the offer to participate reduces  $i$ 's expected utility.

**Part 2:**  $u(PT|\bar{R}) > u(P\bar{T}|\bar{R})$  and  $u(PT|R) > u(P\bar{T}|R)$ .

Since in this part we consider only strategies in which the agent decides to participate, it is now sufficient to consider the equilibrium of a simpler, one-stage mechanism in which agents are given no choice about participation. Define the one-stage mechanism  $M$  as follows:

1. The center announces  $n$ , the number of bidders in the main auction.
2. Each bidder  $i$  submits a bid  $\mu_i$  to the mechanism.
3. The bidder with the highest bid is allocated the good and is made to pay  $b^e(\mu_i, p_{n, s_i})$ .
4. All bidders with  $s_i \geq 2$  are paid  $c_{n, s_i}$ .

$M$  has the same payment rule for bidding ring bidders as the bidding ring protocol given above, but no longer implements a first-price payment rule for singleton bidders. Observe that the original auction is efficient under the strategies stated in the theorem because each bidder  $i$  bids  $b^e(v_i, p_{n, 1})$  in the main auction. Thus, in order to prove  $u(PT|\bar{R}) > u(P\bar{T}|\bar{R})$  and  $u(PT|R) > u(P\bar{T}|R)$ , it is sufficient to show that truthful bidding is an equilibrium for all bidders in mechanism  $M$ .

Assume that all other bidders bid truthfully, and consider the strategy of bidder  $i$ . This bidder's posterior distribution over the number of other bidders he faces, given his signal  $s_i$  and the auctioneer's announcement that there are  $n$  bidders in the main auction, is  $p_{n, s_i}$ . Since agent  $i$  is made to pay  $b^e(\mu_i, p_{n, s_i})$  if he wins, and since the good is always allocated to the agent who submits the highest message,  $M$  is symmetrizing. From Lemma 2.8, agent  $i$ 's best response to truthful bidding in a symmetrizing auction is to bid truthfully. Observe that this analysis holds for both non-ring and ring bidders since it does not require  $s_i > 1$ . If  $i$  is a ring bidder then he gets the additional payment  $c_{n, s_i}$  in both scenarios, but as this payment does not depend on the amount of his bid it does not affect his strategy given his decision to participate. ■

Note that this equilibrium gives rise to an economically-efficient allocation, as was mentioned in the proof of the theorem. The highest bidder in each bidding ring always bids in the main auction, and every bidder in the main auction places a bid according to the same function, which is monotonically increasing in the bidder's valuation. Because we have included bidders' decisions about whether to participate as part of their strategy spaces, the following result follows directly from the arguments in the proof of Theorem 3.3.

**Corollary 3.4** *Participating in the bidding ring protocol according to the equilibrium from Theorem 3.3 is ex post individually rational for all bidders.*

For the sake of concreteness, we illustrate an outcome that could arise from agents following the equilibrium of Theorem 3.3 in the setting described in Example 3.2.

**Example 3.5** *Let us elaborate Example 3.2 to say that valuations are uniformly distributed on  $[0, 1]$ , and that  $c_{n, k} = 0.01 \frac{v}{k}$ . Suppose nature draws three cartels,  $n_C = 3$ , using the distribution  $\gamma_C$ . Suppose nature then draws from  $\gamma_A$  to obtain cartel sizes of 3, 2, and 1 agents. Consider agent  $i$  with type  $(v_i, s_i) = (0.9, 3)$ . Agent  $i$  is approached by a ring center, and learns from his signal that there are a total of 3 agents in his ring. In the equilibrium described by Theorem 3.3,  $i$  participates and bids truthfully; i.e., he chooses strategy  $(PT|R)$ . Under  $(PT|R)$ , in equilibrium  $i$  wins the good if his valuation is highest. In equilibrium, all agents choose to participate in their respective rings, and 3 ring centers (organizations) attend the auction. The*

<sup>9</sup>Note that this occurs because the singleton bidders and other bidding rings in the main auction follow a strategy that depends on the number of bidders announced by the auctioneer; hence they bid as though all the  $s_i - 1$  bidders from the disbanded bidding ring might each be independent bidding rings.

auctioneer announces that  $n = 3$  bidders (i.e., the ring centers) have registered for the auction. If  $i$  has the highest valuation in his ring, the ring center for  $i$ 's cartel places a bid in the main auction on  $i$ 's behalf, for  $b^e(0.9, p_{3,1}) = 0.7032$ . If it is the winning bid,  $i$  is awarded the good, and is made to pay  $b^e(v_i, p_{3,3}) = 0.7649$  to his ring center. Of that amount, the ring center is required to pay 0.7032 to the auctioneer. Based on the auctioneer's announcement that  $n = 3$ , the ring center then computes  $c_{3,3} = 0.01$  and disburses 0.01 to each of the three agents in the ring, including  $i$ . Net,  $i$  pays 0.7549 and so gains 0.1451, the other agents in  $i$ 's cartel gain 0.01,  $i$ 's ring center gains 0.0317, and the auctioneer gains 0.7032.

## 4 Are Bidding Rings Helpful?

So far we have defined a bidding ring protocol and shown that in equilibrium members will participate truthfully and never regret participating. However, this does not show that the protocol is beneficial. In this section we show that it benefits all participants except for the auctioneer. We begin by showing that *payment equivalence* holds in our setting. This result is a useful tool for establishing who gains from bidding rings. Using payment equivalence, we first show that the auctioneer's revenue is lower in the presence of bidding rings, and then that a ring center has an incentive to run the protocol. Next we consider the question of whether the bidding ring benefits the agents. We provide affirmative answers to several different versions of the question: agents are better off than they would be if their own rings did not exist, than if other agents' rings did not exist, and as compared (given the same information either *ex ante* or *ex interim*) to a world in which no bidding rings were possible. Finally, we show that the protocol is *unhelpful* in the sense that it does not offer a unique equilibrium.

### 4.1 Payment equivalence

We begin by proving that bidders make the same expected payments under the bidding ring protocol as in a standard first-price auction with the same distribution over the number of bidders.

**Lemma 4.1 (payment equivalence)** *The ex ante expected payment by an agent in  $E_{br}$  who follows the equilibrium of Theorem 3.3 (i.e., the sum of net amounts paid by this bidder to the auctioneer and the ring center) is the same as the expected equilibrium payment of an agent in a first-price auction in  $E_s$ , where  $p$  and  $F$  are held constant between the two environments and where  $c_{n,k} = 0$  in  $E_{br}$ .*

**Proof.** The proof has two parts. First we introduce a transformed mechanism  $M'$  in  $E_s$  that is payment equivalent to the bidding ring mechanism. Then we argue that with  $c_{n,k} = 0$ ,  $M'$  is payment equivalent to a first-price auction in  $E_s$ . Combination of these two parts completes the proof.

Mechanism  $M'$  in  $E_s$  is constructed to mimic the bidding ring protocol in  $E_{br}$ , but with the ring centers and signals absorbed into the mechanism, and without decisions about participation.  $M'$  is defined as follows:

1. Each agent  $i$  sends a bid  $\mu_i$  to the center.
2. From the number of bids received the center learns  $j$ , the total number of agents in the economic environment  $E_s$ . The center artificially allocates agents to bidding rings, choosing the number of rings and the populations of the rings according to  $\gamma_c$  and  $\gamma_A$  respectively, conditioning both distributions on the observed total number of agents  $j$ . Combined with nature's distribution over  $j$ , Equation (8), this implements the same distribution over the cartel numbers and populations as is implemented by  $E_{br}$ .
3. The good is allocated to the agent  $h$  with the highest bid,  $\mu_h$ . This winning agent is made to pay  $b^e(\mu_h, p_{n,k})$  where  $n$  is the number of rings and  $k$  is the size of the ring containing agent  $h$ .
4. Each agent  $i$  receives a payment  $c_{n,k_i}$ , where  $k_i$  is the number of agents in the ring containing agent  $i$ .

As a function of the agents' bids,  $M'$  implements the same allocation rule and same *ex ante* expected payments as the bidding ring mechanism. Since equilibria are specified in terms of these functions, and since it is an equilibrium in the bidding ring mechanism for agents to truthfully declare their valuations, it follows that it is an equilibrium for agents to bid truthfully in  $M'$ . Thus as a function of agents' valuations, in equilibrium

both  $M'$  and the bidding ring mechanism induce the same *ex ante* expected payments and allocation of the good at auction.

Now we must show that with  $c_{n,k} = 0$ ,  $M'$  is payment-equivalent to a first-price auction in  $E_s$ . Observe that the revenue equivalence theorem holds in  $E_s$ . That is, any auction mechanism in which in equilibrium, both (1) the good is allocated to the agent with the highest valuation; and (2) any agent with valuation zero has expected utility zero, induces the same expected payment for an agent with valuation  $v$ . Standard proofs of this result (see, e.g., Klemperer [1999]) begin by showing that every agent makes the same expected payment under any such mechanism. The same argument (not repeated here) suffices to show that for  $c_{n,k} = 0$ , the *ex ante* expected payments of agents in  $M'$  and a first price auction are the same in  $E_s$ ; the requirement  $c_{n,k} = 0$  ensures that agents with valuation zero have expected utility zero. ■

**Example 4.2** *Let us modify Example 3.5 to say that  $c_{n,k} = 0$ . If agents follow the equilibrium identified in Theorem 3.3, they make ex ante expected payments of 0.123. This is just the expectation with respect to  $j$  and  $v_i$  of the probability that  $i$  has the highest valuation multiplied by  $b^e(v_i, p)$ .*

## 4.2 Ring centers gain at the expense of the auctioneer

Next, we show that *revenue* equivalence does not hold in  $E_{br}$ , because some of the revenue that would be earned by the auctioneer in an auction in  $E_s$  is instead captured by the ring centers.

**Theorem 4.3 (revenue inequivalence)** *The expected revenue from a first-price auction in  $E_{br}$  where bidders follow the equilibrium of Theorem 3.3 is less than the expected revenue of a first-price auction in  $E_s$ , where  $p$  and  $F$  are held constant between the two environments.*

**Proof.** First consider  $c_{n,k} = 0$ . From Lemma 4.1, the expected *ex ante* payments in the two auctions are the same. But whereas the center is the sole recipient of payments in the auction with a stochastic number of participants, in the bidding ring mechanism agents' payments are partially diverted from the center to the ring centers. We need to establish that the expected amount diverted is positive. Since the distribution  $p_{n,k}$  is just  $p_{n,1}$  with  $k - 1$  singleton agents added,  $p_{n,k} = p_{n,1} * \delta_{k-1}$ . Since  $k \geq 2$ , it follows that  $p_{n,k}$  convolutively dominates  $p_{n,1}$ . It then follows from Lemma 2.5 that  $b^e(v_i, p_{n,k}) > b^e(v_i, p_{n,1})$ . This proves that a ring center always receives a positive payment when a ring member wins; since every auction has a winner, a positive amount is diverted from the center to the ring centers.

Next consider the case in which  $c_{n,k}$  is positive for some  $(n, k)$ . The value of  $c_{n,k}$  does not affect the allocation rule, and since the size of the side payment  $c_{n,k}$  is independent of agents' bids, it does not affect agents' strategies either. Since this side payment is made between ring centers and agents, it has no effect on the expected revenue of the auctioneer. Therefore the auctioneer's expected revenue for nonzero  $c_{n,k}$  is the same as it is for  $c_{n,k} = 0$ . ■

We can also show that ring centers experience a net gain on expectation from running bidding rings as long as the unconditional payment  $c$  is small enough.

**Theorem 4.4** *Under the equilibrium of Theorem 3.3, the ring center gains on expectation if it pays agents  $c_{n,k} = \frac{1}{k}(g_{n,k} - c'_{n,k})$  with  $0 < c'_{n,k} \leq g_{n,k}$  and*

$$g_{n,k} = k \int_0^\infty f(v_i) \sum_{j=2}^\infty p_{n,k}(j) F^{j-1}(v_i) (b^e(v_i, p_{n,k}) - b^e(v_i, p_{n,1})) dv_i,$$

*and is budget-balanced on expectation when  $c'_{n,k} = 0$ .*

**Proof.** Recall from the proof of Theorem 4.3 that a ring center always receives a positive payment when a ring member wins.  $g_{n,k}$  is the ring center's *ex ante* expected gain if all  $k$  invited agents behave according to the equilibrium in Theorem 3.3, the auctioneer announces  $n$  participants, and the ring center makes no payment to the agents. Thus the ring center will gain on expectation if each ring member's unconditional payment is less than  $\frac{1}{k}g_{n,k}$ , and will budget-balance on expectation when each ring member's payment is exactly  $\frac{1}{k}g_{n,k}$ . ■

The payment of  $c$  to all bidders follows an idea from Graham and Marshall [1987] for returning a ring center's profits to bidders without changing incentives. In equilibrium the ring center will have an expected profit of  $c'_{n,k}$ , though it will lose  $kc_{n,k}$  whenever the winner of the main auction does not belong to its ring. If a ring center wants to budget balance on expectation, it can set  $c'_{n,k} = g_{n,k}$ .

### 4.3 Bidders gain as compared to a world without bidding rings

There are several ways of asking whether *bidders* gain by being invited to join bidding rings. One natural question is whether bidders prefer a world with bidding rings to a world without. We consider two such worlds without bidding rings: an auction with participation revelation in  $E_s$  and an auction with a stochastic number of bidders in  $E_s$ .<sup>10</sup> First we compare the three environments *ex ante*, asking which environment an agent would prefer if he knew the distribution over types but did not know what type he would receive. Second, we compare the environments *ex interim*, asking which environment an agent would prefer given knowledge of his own type. (Recall that we have defined an agent's type to include his signal  $s_i$  about the number of agents in the economic environment.)

We first consider the *ex ante* case. Observe that in this case an agent does not know whether or not he will be invited to a ring, as this is part of his type.

**Theorem 4.5 (*ex ante*)** *For all  $n \geq 2$ , as long as  $\exists n, \exists k, \gamma_c(n) > 0$  and  $\gamma_a(k) > 0$  and  $c_{n,k} > 0$ , agent  $i$  obtains greater expected utility by*

**Case (1)** *participating in  $E_{br}$  and following the equilibrium from Theorem 3.3 than by*

**Case (2)** *participating in a first-price auction with participation revelation in  $E_s$  with number of bidders distributed according to  $p$ ; or by*

**Case (3)** *participating in a first-price auction with a stochastic number of bidders in  $E_s$  with number of bidders distributed according to  $p$ .*

*When  $\forall n, \forall k, c_{n,k} = 0$ , agent  $i$  obtains the same expected utility in all three cases.*

**Proof.** Our claim for  $c_{n,k} = 0$  follows immediately from Lemma 4.1 and the fact that all three mechanisms are efficient. Now consider  $c_{n,k} \geq 0$ . The value of  $c_{n,k}$  does not affect the allocation rule, nor does it ever matter to an agent's decision about which strategy to prefer. Thus in Case (1) an agent's utility is higher than in Cases (2) and (3) by exactly the expectation (over signals) of  $c_{n,k}$ . It follows that agent  $i$  prefers Case (1) as long as there exists a pair  $(n, k)$  that is realized with positive probability (*i.e.*,  $(n, k)$  for which  $\gamma_c(n) > 0$  and  $\gamma_a(k) > 0$ ) and for which  $c_{n,k} > 0$ ; otherwise,  $i$  is indifferent between the three cases. ■

We now consider the *ex interim* case.

**Theorem 4.6 (*ex interim, ring members*)** *For all  $\tau_i \in \mathcal{T}$ , for all  $k \geq 2$ , for all  $n \geq 2$ , for all  $c_{n,k} > 0$ , agent  $i$  obtains greater expected utility by:*

**Case (1)** *participating in a bidding ring of size  $k$  in  $E_{br}$  and following the equilibrium from Theorem 3.3 than by*

**Case (2)** *participating in a first-price auction with participation revelation in  $E_s$  with number of bidders distributed according to  $p_{n,k}$ ; or by*

**Case (3)** *participating in a first-price auction with a stochastic number of participants in  $E_s$  with number of bidders distributed according to  $p_{n,k}$ .*

*When  $c_{n,k} = 0$ , agent  $i$  obtains the same expected utility in all three cases.*

<sup>10</sup>The two worlds without bidding rings that we consider are revenue equivalent, so it would have sufficed to consider just one of them. We nevertheless discuss both because doing so doesn't appreciably complicate the argument.

**Proof.** For an efficient first-price auction, an agent  $i$ 's expected utility  $EU_i$  is  $\sum_{j=2}^{\infty} p(j)F^{j-1}(v_i)(v_i - b)$ , where  $p(j)$  is the probability that there are a total of  $j$  agents in the economic environment,  $F^{j-1}(v_i)$  is the probability that  $i$  has the high valuation among these  $j$  agents, and  $b$  is the amount of  $i$ 's bid.

First, we consider Case (1). Let  $EU_{i,bc}$  denote agent  $i$ 's expected utility in  $E_{br}$  as a member of a bidding ring of size  $k$ , in the equilibrium from Theorem 3.3. Recall that in this equilibrium the bidder with the globally highest valuation always wins, and if bidder  $i$  wins he is made to pay  $b^e(v_i, p_{n,k})$ . In any case  $i$  receives an unconditional positive payment of  $c_{n,k}$ . Thus,

$$EU_{i,bc} = \sum_{j=2}^{\infty} p_{n,k}(j)F^{j-1}(v_i)(v_i - b^e(v_i, p_{n,k})) + c_{n,k}. \quad (12)$$

We now consider Case (2). From Proposition 2.11 it is an equilibrium for agent  $i$  in economic environment  $E_s$  to bid  $b^e(v_i, j)$  in a first-price auction with participation revelation, where  $j$  is the number of bidders announced by the auctioneer. Since the number of agents is distributed according to  $p_{n,k}$ , agent  $i$ 's expected utility in a first-price auction with participation revelation, which we denote  $EU_{i,pr}$ , is

$$\begin{aligned} EU_{i,pr} &= \sum_{j=2}^{\infty} p_{n,k}(j)F^{j-1}(v_i)(v_i - b^e(v_i, j)) \quad (13) \\ &= \frac{\sum_{\ell=2}^{\infty} p_{n,k}(\ell)F^{\ell-1}(v_i)}{\sum_{\ell'=2}^{\infty} p_{n,k}(\ell')F^{\ell'-1}(v_i)} \sum_{j=2}^{\infty} p_{n,k}(j)F^{j-1}(v_i)(v_i - b^e(v_i, j)) \\ &= \sum_{\ell=2}^{\infty} p_{n,k}(\ell)F^{\ell-1}(v_i) \left( v_i - \sum_{j=2}^{\infty} \frac{p_{n,k}(j)F^{j-1}(v_i)}{\sum_{\ell'=2}^{\infty} p_{n,k}(\ell')F^{\ell'-1}(v_i)} b^e(v_i, j) \right) \\ &= \sum_{\ell=2}^{\infty} p_{n,k}(\ell)F^{\ell-1}(v_i)(v_i - b^e(v_i, p_{n,k})). \quad (14) \end{aligned}$$

Observe that we make use of the definition of  $b^e(v_i, p)$  from Equation (3). Equation (14) is agent  $i$ 's expected utility in Case (3), so  $i$ 's expected utility is equal in Cases (2) and (3).

Combining equations (12) and (14), we obtain  $EU_{i,bc} - EU_{i,pr} = c_{n,k}$ . Thus, when  $c_{n,k} > 0$ , agent  $i$ 's expected utility is strictly greater in Case (1) than in Cases (2) and (3); when  $c_{n,k} = 0$  he has the same expected utility in all three cases. ■

What about agents who do not belong to bidding rings? We can show in the same way that they are not harmed by the existence of bidding rings: they are neither better nor worse off in the bidding ring economic environment than facing the same distribution of opponents in the two cases described above.

**Corollary 4.7 (*ex interim*, singleton bidders)** *For all  $\tau_i \in \mathcal{T}$ , for all  $n \geq 2$ , agent  $i$  obtains the same expected utility in each of the following cases:*

**Case (1)** *participating as a singleton bidder in  $E_{br}$  and following the equilibrium from Theorem 3.3;*

**Case (2)** *participating in a first-price auction with participation revelation in  $E_s$  with number of bidders distributed according to  $p_{n,1}$ ;*

**Case (3)** *participating in a first-price auction with a stochastic number of participants in  $E_s$  with number of bidders distributed according to  $p_{n,k}$ .*

**Proof.** We follow the same argument as in Theorem 4.6, except that  $k = 1$  and  $EU_{i,bc}$  does not include  $c_{n,k}$ . Thus we get  $EU_{i,bc} = EU_{i,pr}$ . ■



## 4.4 Bidders gain vs. a world where one ring is replaced by singletons

At this point, we have showed that bidding rings help agents in two ways: it is always better for agents to join a ring than to decline the invitation (the proof of Theorem 3.3) and all agents are always at least weakly better off both *ex ante* and *ex interim* in a world with bidding rings than in a world without (Section 4.3). The final way we will ask whether bidding rings are helpful is by considering the effect of disbanding a single ring on both ring and non-ring bidders.

First, we demonstrate that bidders are better off being invited to a bidding ring than being sent to the auction as singleton bidders.

**Theorem 4.8 (ring members)** *An agent  $i$  has higher expected utility<sup>11</sup> in a bidding ring of size  $k$  bidding as described in Theorem 3.3 than he does if the bidding ring does not exist and  $k$  additional agents (including  $i$ ) participate directly in the main auction as singleton bidders, again bidding as described in Theorem 3.3, even when  $c_{n,k} = 0$ .*

**Proof.** Consider the counterfactual case where agent  $i$ 's bidding ring does not exist, and all the members of this bidding ring are replaced by singleton bidders in the main auction. Clearly  $i$  gains the least from the bidding ring when  $c_{n,k} = 0$ ; we consider this case and show that  $i$  is still strictly better off as a member of the bidding ring. If there were  $n$  potential ring centers in the original auction and  $k$  agents in  $i$ 's bidding ring, then the auctioneer would announce  $n + k - 1$  as the number of participants in the new auction. In both cases the auction is economically efficient, which means  $i$  is better off in the auction that requires him to pay a smaller amount when he wins. Under the equilibrium from Theorem 3.3, as a singleton bidder  $i$  will pay  $b^e(v_i, p_{n+k-1,1})$  when he wins. If he belonged to the bidding ring and followed the same equilibrium  $i$  would pay  $b^e(v_i, p_{n,k})$  when he wins. As argued in the proof of Theorem 3.3, Lemma 2.5 shows that  $\forall k \geq 2, \forall n \geq 2, \forall v, b^e(v, p_{n+k-1,1}) > b^e(v, p_{n,k})$ , and so our result follows. ■

Intuitively, an agent gains by not having to consider the possibility that other bidders who would otherwise have belonged to his bidding ring might themselves be bidding rings.

We can also show that singleton bidders and members of other bidding rings benefit from the existence of each bidding ring in the same sense. Following an argument similar to the one in Theorem 4.8, other bidders gain from not having to consider the possibility that additional bidders might represent bidding rings. Paradoxically, whenever  $c'_{n,k} > 0$  (i.e., the ring center is profitable on expectation), other bidders' gain from the existence of a given bidding ring is greater than the gain of that ring's members.

**Corollary 4.9 (ring non-members)** *In the equilibrium described in Theorem 3.3, singleton bidders and members of other bidding rings have higher expected utility when  $k \geq 2$  agents form a bidding ring than when  $k$  additional agents participate directly in the main auction as singleton bidders.*

**Proof.** Consider a singleton bidder  $i$  in the first case, where the ring of  $k$  agents does exist. (It is sufficient to consider a singleton bidder, since other bidding rings bid in the same way as singleton bidders.) Following the equilibrium from Theorem 3.3,  $i$  would submit the bid  $b^e(v_i, p_{n,1})$ . In the second case, following the equilibrium from Theorem 3.3,  $i$  would bid  $b^e(v_i, p_{n+k-1,1})$ . In both cases, the auction is economically efficient, so  $i$  is better off in the case where he makes the smaller bid. From the argument in Theorem 4.4 we know that  $\forall k \geq 2, b^e(v_i, p_{n,1}) < b^e(v_i, p_{n,k})$ ; from the argument in Theorem 3.3 Part (1b) we know that  $\forall k \geq 2, b^e(v, p_{n+k-1,1}) > b^e(v, p_{n,k})$ . Thus  $\forall k \geq 2, b^e(v_i, p_{n,1}) < b^e(v_i, p_{n+k-1,1})$ . ■

## 4.5 Another Equilibrium

So far we have considered whether agents benefit under the equilibrium from Theorem 3.3. However, we can also show that this equilibrium is not unique. There is another equilibrium under which no agents accept bidding ring invitations, and they instead bid according to the equilibrium for first-price auctions with participation revelation given in Proposition 2.11.

<sup>11</sup>This is weakly higher (i.e., equal) expected utility for agents with the lowest possible valuation, and strictly higher expected utility otherwise. The same caveat also holds below.

**Proposition 4.10** *It is a Bayes-Nash equilibrium for each bidding ring invitee to decline his bidding ring invitation, and for each agent  $i$  to bid  $b^e(v_i, n)$ .*

**Proof.** If at least one agent declines the invitation to join a bidding ring, other invitees of that bidding ring are at least as well off if they decline as well. (If they decline then they can bid freely, rather than being made to submit bids of a particular form.) If no agents join bidding rings then agents' signals contain no useful information. Thus the argument from Proposition 2.11 applies, and it is a Bayes-Nash equilibrium for each bidder to submit a bid of  $b^e(v_i, n)$ . ■

The theorems and corollaries in Section 4 allow us to compare our first equilibrium (from Theorem 3.3) with this new equilibrium (from Proposition 4.10).

**Corollary 4.11** *When  $c_{n,k} > 0$ , all bidders prefer the equilibrium from Theorem 3.3 to the equilibrium from Proposition 4.10 ex ante; ex post bidding ring invitees prefer the equilibrium from Theorem 3.3 to the equilibrium from Proposition 4.10, while singleton bidders are indifferent between the equilibria. When  $c_{n,k} = 0$ , all bidders are indifferent between the equilibria both ex ante and ex post.*

**Proof.** A bidder's expected utility under the equilibrium from Proposition 4.10 in economic environment  $E_{br}$  is the same as his expected utility from an auction with participation revelation in economic environment  $E_s$  with the same distribution over the number of bidders, since (as given by Propositions 2.11 and 4.10) in both cases each bidder  $i$  follows the strategy  $b^e(v_i, n)$ . Then the result is immediate from Theorem 4.6, Corollary 4.7 and Corollary 4.5. ■

Since both bidders (and, trivially, ring centers) prefer the equilibrium from Theorem 3.3 to the equilibrium from Proposition 4.10, it follows that auctioneers have opposite preferences. It turns out that auctioneers can disrupt bidding rings by slightly changing the rules of the auction so that the strategies described in Theorem 3.3 no longer constitute an equilibrium while the equilibrium from Proposition 4.10 is preserved. This can be achieved by making it possible for bidders to participate in their bidding rings and also place shill bids in the main auction without detection by the ring center.<sup>12</sup> If all agents but  $i$  followed the strategies specified in Theorem 3.3,  $i$  could declare a low valuation to the ring center but also place a competitive bid in the main auction, gaining all the benefits of the cartel without having to make any payments to the ring center and without causing the ring to change its behavior because its invitation had been declined. Note however that this defense may not be available to all auctioneers; for example, the auctioneer might be required to verify and announce the winner's identity.

## 5 Conclusions

We have presented a formal model of bidding rings in first-price auctions that in many ways extends models traditionally used in the study of collusion. Most importantly, in our model all agents behave strategically and take into account the possibility that groups of other agents will collude. We have argued that the right information structure to support this kind of bidder reasoning is an auction with an initially uncertain number of participants, revelation by the auctioneer of the number of bidders choosing to participate (or, equivalently, observation by the bidders of the number of agents attending the auction), but bidders' inability to detect the suppression of additional bids by one or more bidding rings. Our model treats agents' decisions about whether or not to join bidding rings as endogenous, but their possible invitations to participate in bidding rings as exogenous.

We presented a bidding ring protocol for first-price auctions that leads to a (globally) efficient allocation in equilibrium. In this equilibrium all invited agents choose to participate, even when the bidding ring operates in a single auction as opposed to a sequence of auctions. This means that the protocol's stability does not rely on the threat of an agent being denied future opportunities to collude. Bidding rings make money on expectation, and can optionally be configured so they never lose money.

We asked the question of whether agents gain by participating in bidding rings in first-price auctions in three different ways:

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<sup>12</sup>As mentioned earlier, despite studying a fairly different setting, Marshall and Marx [2007] found that their own bidding ring protocol for unrepeated first-price auctions could also be disrupted by allowing bidders to place shill bids.

1. Could any agent gain by deviating from the protocol?
2. Would any agent be better off if his bidding ring did not exist?
3. Would any agent would be better off (either *ex interim* or *ex ante*) in an economic environment that did not include bidding rings at all?

We have shown that agents are strictly better off in all three senses. (In the third sense, the gain is only strict when ring centers make an otherwise optional side-payment to agents.) We have also shown that each bidding ring causes *non-members* to gain in the second sense, and does not hurt them in the third sense.

Our work provides many opportunities for further study. These include the following questions, all of which refer to first-price auctions in economic environment  $E_{br}$ .

- What is the optimal bidding ring protocol? (We conjecture that our protocol, with the largest possible  $c$ , gives agents the highest possible *ex ante* expected utility, as compared to all efficient protocols in which the ring center does not lose on expectation.)
- Is it possible to construct bidding ring protocols that play best responses to each other in the main auction? (Observe that under our protocol, ring centers do not behave strategically.)
- Can a bidding ring protocol be made to budget-balance *ex post*, e.g., using ideas similar to those that Mailath and Zemsky [1991] applied to second-price auctions?
- What is a full characterization of the symmetric equilibria of our protocol? (We conjecture that beyond the equilibria we described, it is an equilibrium for agents in one cartel to continue to bid truthfully when another cartel disbands.)
- Can our protocol be extended to allow agents to receive multiple invitations to join different bidding rings? If so, will agents always want to join the largest possible coalition? (We conjecture that they will.)
- Can any bidding ring protocol for first-price auctions withstand shill bidding?
- Can a protocol be identified that works with risk-averse bidders, affiliated values, asymmetric valuation distributions, or arbitrary joint distributions over the numbers of bidders in different cartels?

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## A Relating $p$ to $b^e$

In Proposition 2.2 we observed that in  $E_c$ , the amounts of agents' equilibrium bids increase with the number of participating agents. In Section 2.2, however, we stated that the situation is more complicated in  $E_s$ . In this appendix we provide a more in-depth analysis of this relationship between the distribution over the number of agents,  $p$ , and the equilibrium bid amount,  $b^e$ . We begin with an example.

**Example A.1** Consider a distribution  $p$  over the number of agents that assigns nonzero probability only to the integers 2 and  $j$ , and furthermore let  $p(2) = p(j)$ . Now consider the strategy of agent  $i$ . The classical case is recovered if  $j = 2$ , in which case  $i$ 's equilibrium bid will be simply  $b^e(v_i, 2)$ . If  $j$  is increased to 3, the equilibrium bid increases by a finite amount to some  $b^e(v_i, p) \in (b^e(v_i, 2), b^e(v_i, 3))$ , as determined by Equation (3) from Section 2.2. As  $j$  is increased to an arbitrarily high value,  $F(v_i)^{j-1}$ , the probability that agent  $i$  has the highest valuation when there are  $j$  agents involved, approaches zero. With arbitrarily close to unit probability, there will be 2 agents involved when agent  $i$  has the highest valuation, and Equation (3) indicates that  $i$ 's bid will be arbitrarily close to the  $j = 2$  result. Thus while the  $j \rightarrow \infty$  distribution has a higher expected number of participants than the  $j = 3$  distribution, it elicits a lower equilibrium bid.

This phenomenon also occurs among distributions of practical interest. So in a first-price auction with a stochastic number of participants, simply knowing that distribution  $p$  has a smaller expected number of participants than distribution  $p'$  is not enough to know which distribution gives rise to a lower symmetric equilibrium bid for a given valuation. The same holds for stochastic dominance. For example, in Example A.1, the distribution with very large  $j$  stochastically dominates the distribution with  $j = 3$  but elicits a lower equilibrium bid. Lemma 2.5 identifies a class of pairs of distributions  $(p, p')$  for which it does hold that  $b^e(v_i, p) < b^e(v_i, p')$ : those for which  $p'$  convolutively dominates  $p$ . Before we can prove this lemma, we must define additional notation that was not given in Section 2.2.

Let  $r_j(F, v_i, p)$  denote the probability that  $j$  agents participate, conditional on agent  $i$  having the highest valuation. This is equal to the probability that  $j$  agents participate and agent  $i$  has the highest valuation among these agents, normalized by the unconditional probability that agent  $i$  has the highest valuation. Let  $Z(F, v_i, p)$  be the probability that agent  $i$  has the highest valuation given that his valuation is  $v_i$ . Then

$$Z(F, v_i, p) \equiv \sum_{k=2}^{\infty} F(v_i)^{k-1} p(k); \quad (15)$$

$$r_j(F, v_i, p) \equiv \frac{F(v_i)^{j-1} p(j)}{Z(F, v_i, p)}. \quad (16)$$

Observe<sup>13</sup> that Equation (3) for the equilibrium bid in a stochastic first-price auction can be written in terms of the distribution  $r(F, v_i, p)$ :

$$b^e(v_i, p) = \sum_{j=2}^{\infty} r_j(F, v_i, p) b^e(v_i, j). \quad (17)$$

The cumulative distribution  $R_m(F, v_i, p)$  for the distribution  $r$ , denoting the probability that  $m$  or fewer agents participate conditional on  $i$  having the highest valuation, is simply

$$R_m(F, v_i, p) \equiv \sum_{j=2}^m r_j(F, v_i, p). \quad (18)$$

**Lemma 2.5**  $\forall p, p' \in \mathcal{D}_2$ , if  $p'$  convolutively dominates  $p$  then  $b^e(v_i, p) < b^e(v_i, p')$ .

**Proof.** The proof has two parts. First we show that for every  $j$ , the probability that no more than  $j$  bidders participate conditional on bidder  $i$  having the highest valuation is at least as high when the number of agents

<sup>13</sup>We can use 2 rather than  $-\infty$  as the lower limit of the sum in Equation (15) because  $p(k)$ 's support is a subset of  $\{2, 3, \dots\}$ . While  $Z(F, v_i, p)$  is undefined when  $F(v_i) = 0$ , this technicality is of no practical interest.

is drawn from  $p$  as when it is drawn from  $p'$ , and that for some  $j$  this probability is higher in  $p$  than in  $p'$ . Next, we show that this relationship between conditional probabilities implies that the equilibrium bid is smaller under  $p$  than under  $p'$ .

**Step 1:**  $\forall j, R_j(F, v_i, p') \leq R_j(F, v_i, p)$ , and  $\exists j, R_j(F, v_i, p') < R_j(F, v_i, p)$ .

Consider the difference between the cumulative distributions:

$$\begin{aligned} \Delta R_j &\equiv R_j(F, v_i, p) - R_j(F, v_i, p') \\ &= \sum_{m=-\infty}^j \left( \frac{F(v_i)^{m-1} p(m)}{Z(F, v_i, p)} - \frac{F(v_i)^{m-1} p'(m)}{Z(F, v_i, p')} \right). \end{aligned} \quad (19)$$

The denominators can be related as follows:

$$\begin{aligned} Z(F, v_i, p') &= \sum_{k=-\infty}^{\infty} F(v_i)^{k-1} \sum_{j=0}^{\infty} p(k-j)q(j) \\ &= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} (F(v_i)F(v_i)^{j-1}F(v_i)^{k-j-1}) p(k-j)q(j) \\ &= F(v_i) \sum_{j=0}^{\infty} F(v_i)^{j-1} q(j) \sum_{k=-\infty}^{\infty} F(v_i)^{k-j-1} p(k-j) \\ &= F(v_i) Z(F, v_i, q) Z(F, v_i, p). \end{aligned} \quad (20)$$

$$\quad (21)$$

Substituting Equation (21) into Equation (19), and using Equation (4),

$$\Delta R_j = \frac{1}{Z(F, v_i, p')} \sum_{m=-\infty}^j \left( Z(F, v_i, q) F(v_i)^m p(m) - F(v_i)^{m-1} \sum_{k=0}^{\infty} p(m-k)q(k) \right) \quad (22)$$

$$\begin{aligned} &= \frac{F(v_i)}{Z(F, v_i, p')} \left( \sum_{k=0}^{\infty} q(k) F(v_i)^{k-1} \sum_{m=-\infty}^j F(v_i)^{m-1} p(m) \right. \\ &\quad \left. - \sum_{k=0}^{\infty} q(k) F(v_i)^{k-1} \sum_{m=-\infty}^j F(v_i)^{m-k-1} p(m-k) \right) \end{aligned} \quad (23)$$

$$= \frac{F(v_i)}{Z(F, v_i, p')} \sum_{k=0}^{\infty} q(k) F(v_i)^{k-1} \left( \sum_{m=-\infty}^j F(v_i)^{m-1} p(m) - \sum_{m=-\infty}^{j-k} F(v_i)^{m-1} p(m) \right). \quad (24)$$

To obtain Equation (23), we have reordered the sums, made use of Equation (15) and performed factoring like that done to obtain Equation (20). To obtain Equation (24), we have factored the bracketed expression in Equation (23) and shifted the dummy indices of the second sum.

When  $k = 0$ , the bracketed expression in Equation (24) is zero, so that term can be dropped from the sum. The bracketed sums can then be combined, yielding

$$\Delta R_j = \frac{F(v_i)}{Z(F, v_i, p')} \sum_{k=1}^{\infty} \left( F(v_i)^{k-1} q(k) \sum_{m=j-k+1}^j F(v_i)^{m-1} p(m) \right). \quad (25)$$

Since  $k \in [1, \infty)$  in Equation (25), the lower summand of the second sum is always less than or equal to the upper summand, so that sum is well defined. Furthermore, all of the factors in Equation (25) are non-negative, so it remains only to be established whether  $\Delta R_j > 0$  or  $\Delta R_j = 0$ . Since  $p \in \mathcal{D}_2$ , there exists some least element in the support of  $p$ ; call this value  $m^*$ . For values of  $j < m^*$  the second sum in Equation (25) gives exactly 0, and  $\Delta R_j = 0$ . Similarly, for all values of  $j \geq m^*$ , the second sum is nonzero, and since by assumption  $\exists k > 0$  such that  $q(k) > 0$ , we have that  $\Delta R_j > 0$ . Thus for all  $j$ ,  $R_j(F, v_i, p') \leq R_j(F, v_i, p)$ , and for some  $j$ ,  $R_j(F, v_i, p') < R_j(F, v_i, p)$ .

**Step 2:**  $(\forall j, R_j(F, v_i, p') \leq R_j(F, v_i, p))$  and  $\exists j, R_j(F, v_i, p') < R_j(F, v_i, p)$  implies  $b^e(v_i, p) < b^e(v_i, p')$ .  
We must show that  $\Delta b > 0$ , where we use Equation (17) to write

$$\begin{aligned}\Delta b &\equiv b^e(v_i, p') - b^e(v_i, p) \\ &= \sum_{m=2}^{\infty} (r_m(F, v_i, p') - r_m(F, v_i, p)) b^e(v_i, m).\end{aligned}$$

We rewrite this sum using summation by parts (the discrete analog of integration by parts). This yields

$$\Delta b = \sum_{m=2}^{\infty} (b^e(v_i, m+1) - b^e(v_i, m)) \sum_{j=2}^m (r_j(F, v_i, p) - r_j(F, v_i, p')) \quad (26)$$

$$= \sum_{m=2}^{\infty} (b^e(v_i, m+1) - b^e(v_i, m)) (R_m(F, v_i, p) - R_m(F, v_i, p')). \quad (27)$$

To obtain Equation (26), we have also used the fact that both  $r(F, v_i, p)$  and  $r(F, v_i, p')$  are normalized. Lemma 2.2 tells us that  $b^e(v_i, m)$  is strictly increasing in  $m$ ; clearly it is always positive. Thus  $\forall m, b^e(v_i, m+1) - b^e(v_i, m) > 0$ . From Step 1,  $R_m(F, v_i, p) - R_m(F, v_i, p')$  is non-negative, and for all  $m \geq m^*$  it is greater than zero. The right-hand side of Equation (27) is therefore a sum of products of non-negative factors, of which at least one is a product of strictly positive factors. Thus  $\Delta b > 0$ , or  $b^e(v_i, p) < b^e(v_i, p')$ . ■

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