Finding a Hamiltonian cycle in the dual graph of Right-Triangulations

Viann W. Chan
Department of Computer Science
University of British Columbia
Vancouver, B.C. V6T 1Z4
vchan@cs.ubc.ca

William S. Evans
Department of Computer Science
University of British Columbia
Vancouver, B.C. V6T 1Z4
will@cs.ubc.ca

January 27, 2006

Abstract

In this paper, we describe a method for refining a class of balanced bintree triangulations which maintains a hamiltonian cycle in the dual graph. We also introduce a method for building refinable balanced bintree triangulations using two types of tiles, a diamond tile and a triangular tile.

1 Introduction

A popular representation of a terrain surface whose elevation has been sampled at regularly spaced intervals is a triangulation of a subset of the sample points that consists of axis-aligned, right-angled isoceles triangles [4]. The family of such triangulations has a hierarchical structure, much like a quadtree [8]. We will call such triangulations *bintree* triangulations. We insist that the triangulation be composed of triangles that have only three vertices on their boundary. Such *balanced* triangulations insure that the surface obtained by linear interpolation has no "gaps", i.e., discontinuities (see Figure 1).

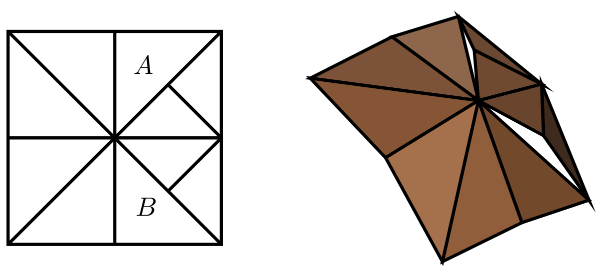


Figure 1: A bintree triangulation that isn't balanced (A and B have four vertices on their boundaries) and the gaps that result in the linearly interpolated surface

A terrain visualization algorithm selects a triangulation from the bintree hierarchy, according to some accurracy or complexity constraint, and renders it as a set of triangles. One technique used to speed the rendering is to construct *triangle strips* from the triangulation. Each triangle strip is a sequence of adjacent triangles. If the rendering system knows the coordinates of the vertices of one triangle in the strip, it knows two of the vertices of the next triangle in the strip and only needs the coordinates of one vertex (rather than three vertices) in order to render the next triangle. This decreases the amount of data that needs to be sent to the graphics hardware. Hamiltonicity of the dual graph of the triangulation implies that the triangulation can be represented using one triangle strip.

Several algorithms for terrain visualization construct triangle strips for all balanced bintree triangulations of terrains that have been sampled at all grid points in a square grid with dimension $2^k \times 2^k$.[1] In this paper, we consider (generalized) balanced bintree triangulations for axis-aligned simple polygons whose vertices lie on integer coordinates. The points with integer coordinates contained within the simple polygon (and on its boundary) represent sampled elevation points. A bintree triangulation of these more general shapes is again a triangulation of a subset of the sample points consisting of axis-aligned, right-angled isoceles triangles. The family of such triangulations no longer forms a bintree hierarchy, but we can still define a partial order on the triangulations where one balanced bintree triangulation is *finer* than another if it is a Steiner triangulation of the other.

We describe a class of balanced bintree triangulations of simple polygons such that the triangulation *and* all finer balanced bintree triangulations of it have a hamiltonian cycle in their dual graph. The class contains the standard $2^k \times 2^k$ grid triangulation as a special case, but includes triangulations of other shapes as well.

We show that a hamiltonian cycle exists in the dual graph of all refinements of an initial balanced bintree triangulation if the initial triangulation contains a hamiltonian cycle with certain properties. These properties are outlined in Section 4.

2 Splitting

A finer triangulation may be obtained by *splitting* triangles in a coarse triangulation. A triangle is split by dividing the triangle in half through its right-angled vertex. The resulting two triangles remain right-angled and isosceles.

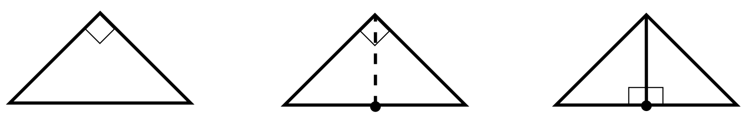


Figure 2: Splitting a triangle.

Splitting a triangle may introduce a vertex on the perimeter of one of its neighbors. In order that all faces remain triangles, we are forced to split this triangular quadrilateral. This, in turn, may force other splits until no quadrilaterals remain in the triangulation. The series of splits instigated by (and including) the original split is called a *chain* of triangle splits.

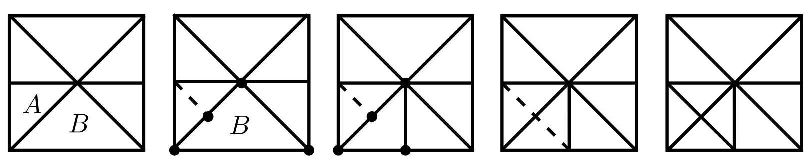


Figure 3: A chain of triangle splits instigated by the split of A.

Exactly those balanced bintree triangulations finer than a given triangulation T can be obtained by applying, perhaps several, chains of triangle splits to T.

In a finite balanced bintree triangulation, only a finite number of triangle splits occur in any chain since at most two triangles of any given size are split in the chain.

[1] We will assume throughout this paper that k is a non-negative integer.

3 Related Work

A surface approximation of a desired accuracy may be obtained by repeatedly splitting the most inaccurate triangle. Several algorithms for rapidly displaying bintree triangulations employ this top-down method of constructing a surface approximation that is sensitive to the position of a moving observer [4]. Bottom-up techniques [5] and view-coherent techniques [3] have also been used. In all these algorithms, the displayed surface is bintree triangulation. Bintrees produce meshes that are also known as right-triangulated networks [4], restricted quadtree triagulations [6], and 4-k meshes [11].

In the dual graph of an balanced bintree triangulation, each triangle represents a vertex, a neighbouring triangle pair represnts an edge. A hamiltonian cycle in the dual graph of an balanced bintree triangulation is a (circular) path which visits every vertex exactly once. In the balanced bintree triangulation, the hamiltonian cycle forms a triangle strip.

Triangle strips are used to model geometric surfaces. The surface approximations are refined through a series of triangle splits. In previous work have focused on efficient data structures for storing triangle strips, and algorithms for retrieving and viewing surface approximations.

Bell et. al. [2] discussed eleven types of distinct isohedral tilings. The tiling which occurs at a uniform level in our triangulation forms a $[4.8^2]$ Laves net. A $[4.8^2]$ partition is a uniform triangulated grid (see Figure 4). It is one of the tilings capable of generating unlimited tilings.

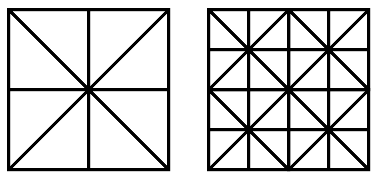


Figure 4: Examples of $[4.8^2]$ tilings.

The hamiltonian cycle we find in our balanced bintree triangulations correspond to a Sierpiński space-filling curve [7] (see Figure 5). Sierpiński's curve is a map from $I = [0, 1]$ onto right isosoceles triangles.

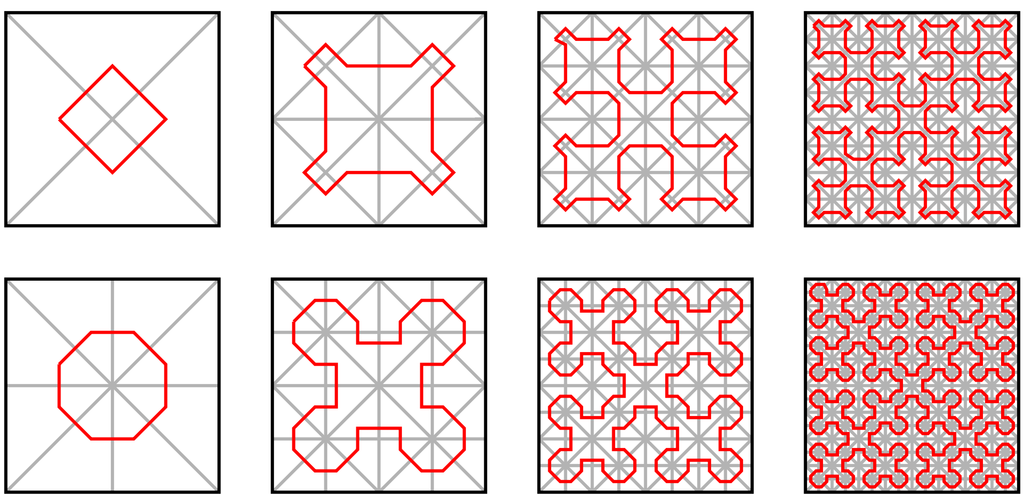


Figure 5: Examples of Sierpiński's space-filling curves.

Velho et. al. [10] considered two types of refinable subdivision templates: a uniform templated and an adaptive template. For each template, they showed how to maintain a triangle strip when a refinement occurs. Their subdivisions allow for cracks to be introduced onto the surface.

Evans et. al. [4] proposed using right-triangulated irregular networks to represent a height field. They implemented this using a top-down approach with a thresholding refinement scheme.

Duchaineau et. al. [3] introduced real-time optimally adapting meshes (ROAM) and implemented this using the triangle bintree datastructure. They used two priority queues to perform the merge and split operations. These operations are used to refine and update the triangulation.

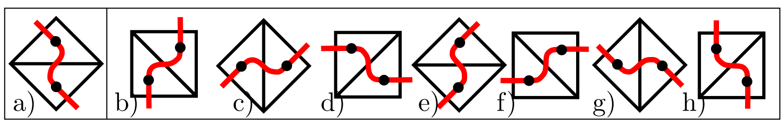
Pajarola used a restricted quadtree to represent a height field. In a restricted quadtree hierarchy, adjacent quadtree blocks may only differ by at most one level. This dependency may cause a cascade of other triangle splits to occur when the triangulation is being refined. They describe two methods to construct the restricted quadtree hierarchy: a top-down recursive approach, and an iterative bottom-up approach.

Lindstrom and Pascucci [5] represented the triangulation as a directed acyclic graph and proposed a recursive scheme to construct a triangle strip. They used a binary tree representation for triangle strips and alternated labeling of left and right children at each split level. The preprocessing enabled a quick retrieval of a triangle strip.

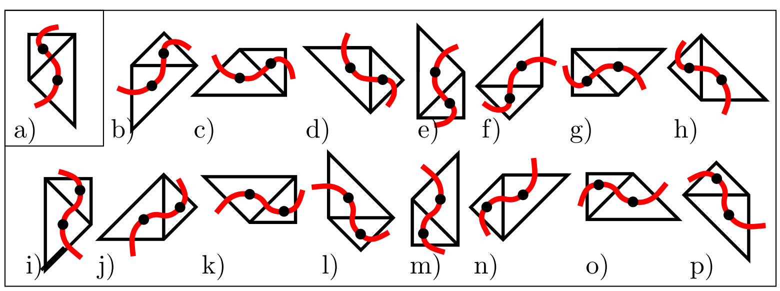
In the following sections, we will show that a hamiltonian cycle exists in the dual graph of a special class of balanced bintree triangulation.

4 Refinable Hamiltonian cycles

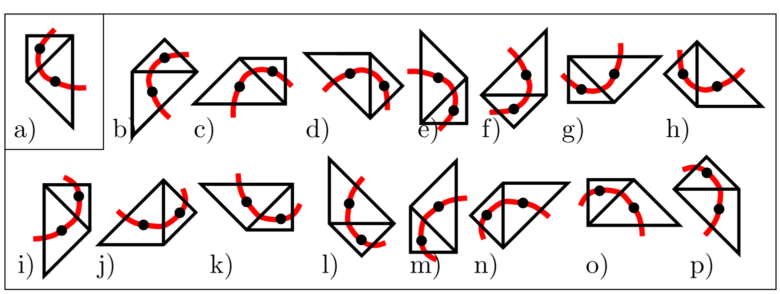
A hamiltonian cycle is called *refinable* if it does not contain a *forbidden* path. There are three types of forbidden paths: Z-paths, S-paths, and C-paths. They are illustrated in the following figures.



Forbidden Z-path (and rotations and reflections)



Forbidden S-path (and rotations and reflections)



Forbidden C-path (and rotations and reflections)

Z, S, and C-paths are forbidden because each path contains a triangle whose split will prevent the hamiltonian cycle from being locally updated (see Figure 6).

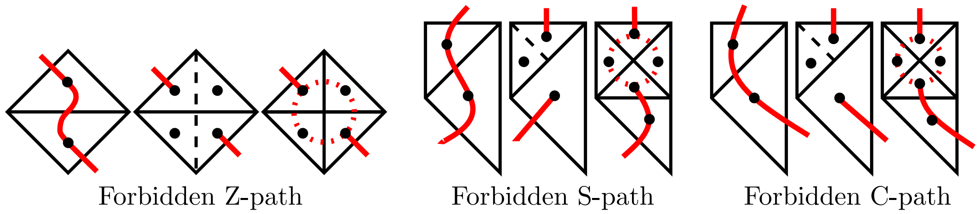


Figure 6: Triangle splits prevent a hamiltonian cycle from being locally updated if it contains a forbidden path

Though the existance of a refinable hamiltonian cycle insures the existance of a triangle strip, it is not a necessary condition. It may be the case that a non-refinable hamiltonian cycle exists in the dual graph when a refinable hamiltonian cycle does not. In fact, a split that prevents the hamiltonian cycle from being locally updated may, nonetheless, create a triangulation with a hamiltonian dual (see Figure 7).

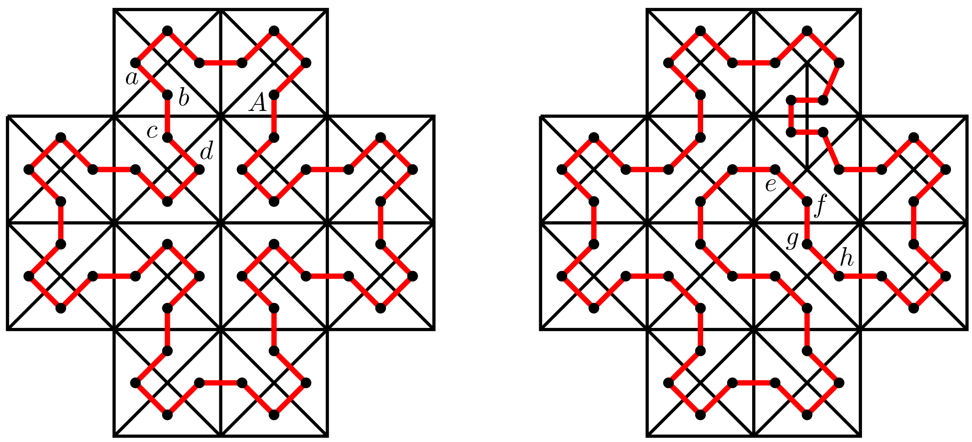


Figure 7: Left: A triangulation that contains a non-refinable hamiltonian cycle in its dual graph ($\langle a, b, c, d\rangle$ is a Z-path). Right: After a chain of triangle splits (instigated by splitting A), the triangulation's dual graph is still hamiltonian, but again non-refinable ($\langle e, f, g, h\rangle$ is a Z-path).

We focus on refinable hamiltonian cycles to show that certain balanced bintree triangulations (and all finer triangulations) support a triangle strip. However, the lack of a refinable hamiltonian cycle does not imply the complement.

Our first goal is to show that any balanced bintree triangulation that contains a refinable hamiltonian cycle in its dual graph will contain a (refinable) hamiltonian cycle after any chain of triangle splits.

Any chain of triangle splits can be decomposed into a sequence of simple operations. In a chain of triangle splits, there is either a unique largest triangle split (its hypotenuse is part of the boundary) or there are two largest triangles split that share a hypotenuse. Every chain of triangle splits could be performed by splitting this largest triangle or triangles, then splitting the second largest triangles, and repeating this process until the original instigator of the chain is split. Breaking a chain of splits into these pieces allows us to update the hamiltonian cycle for any chain of triangle splits by using three update operations: simple triangle split, diamond split 1, and diamond split 2. Figure 8 shows each update operation. Each operation applies to all rotations and reflections for each illustration. Figure 9 gives an example of how to update a refinable hamiltonian cycle in response to a chain of triangle splits.

One condition that we must place on balanced bintree triangulations in order to show that the existance of a refinable hamiltonian cycle in the dual graph implies the existance of a (refinable) hamiltonian cycle

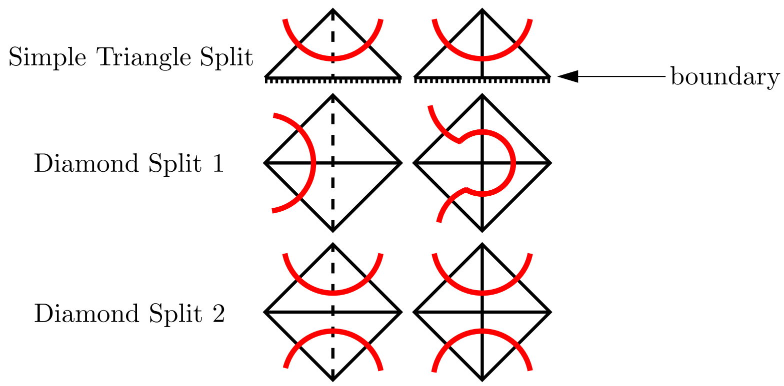


Figure 8: Hamiltonian cycle local update operations.

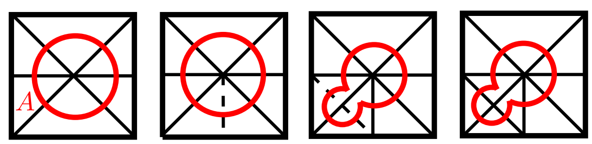


Figure 9: Locally updating a refinable hamiltonian cycle after a chain of splits instigated by splitting A. The second frame shows a simple triangle split and the third shows a diamond split 1.

in all finer balanced bintree triangulations, is that the balanced bintree triangulation contains no *skewed* triangle pair. A skewed triangle pair is a pair of right-triangles that share a non-hypotenuse, boundary edge, and do not share a right-angled vertex (see Figure 10).

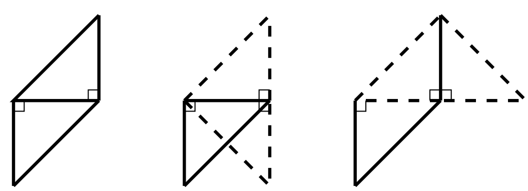


Figure 10: A skewed triangle pair (a). Middle & Right:

Fortunately, if a balanced bintree triangulation contains no skewed triangle pair then no finer balanced bintree triangulation contains a skewed triangle pair.

Lemma 4.1. *A skewed triangle pair cannot be created from a balanced bintree triangulation by a triangle split.*

Proof. A right angle can only be created by splitting the midpoint of the hypotenuse of a larger triangle. Consider the right angle of the top triangle in Figure 10(a). The right angle cannot be formed by splitting the large (dashed) triangle in Figure 10 (b) since the bottom triangle is fixed. The right angle cannot be formed by splitting the large triangle in Figure 10 (c) since the large triangle will have two neighbours on the bottom edge, a contradiction to the balanced property of the original triangulation. ☐

Theorem 4.1. *If a balanced bintree triangulation with no skewed triangle pair contains a refinable hamiltonian cycle in its dual graph then the triangulation obtained by any chain of triangle splits contains a refinable hamiltonian cycle.*

Proof. We show that each update operation introduces no forbidden paths provided the original triangulation has no skewed triangle pairs and its hamiltonian cycle contains no forbidden paths. Lemma 4.1 insures that the new triangulation will not contain a skewed triangle pair.

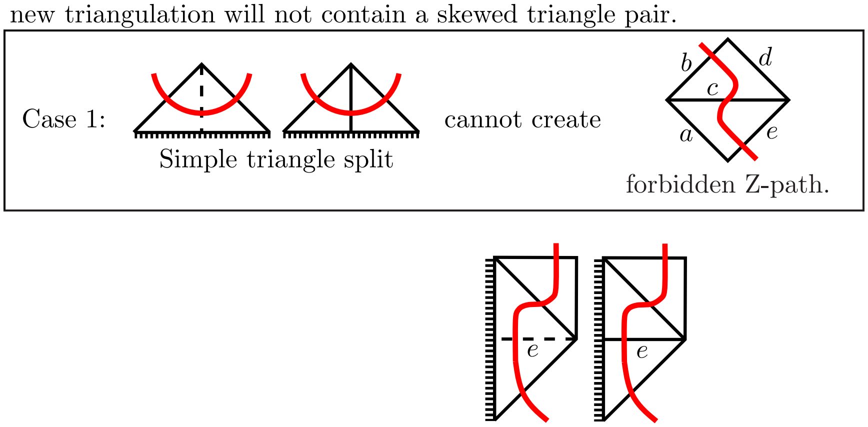


Figure 11: Simple triangle split does not form forbidden Z-path.

In order to create a forbidden Z-path in a hamiltonian cycle that has no forbidden paths, the simple triangle split must create one of the edges a, b, c, d, or e. The split creates only one edge and that edge is crossed by the path (so it can't be edge a or d) and is not a hypotenuse (so it can't be edge c). If the split created e, then a forbidden S-path must have existed before the split (see figure 11) contradicting our assumption. By symmetry, edge b also cannot be created.

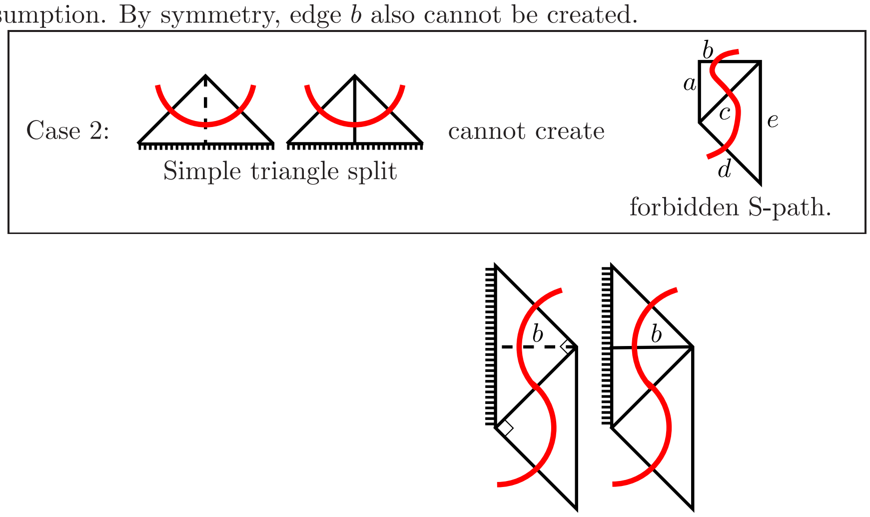
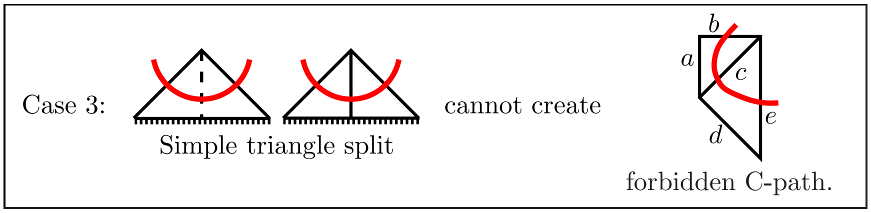


Figure 12: Simple triangle split does not form forbidden S-path.

Again, the edge created by the simple triangle split cannot be a or e (because they are not crossed by the path) or c or d (because they are hypotenuses). If b were created by the split, the balanced bintree triangulation must have contained an illegal triangle pair (figure 12) contradicting our assumption.



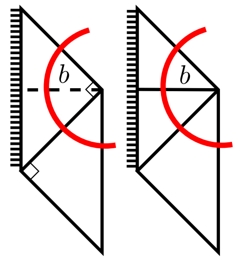
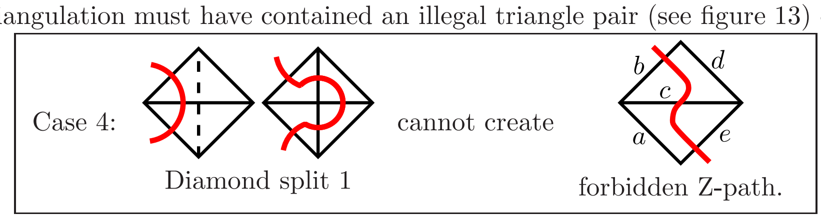


Figure 13: Simple triangle split does not form forbidden C-path.

As in case 2, the edge created by the simple triangle split cannot be a or d (because they are not crossed by the path) or c or e (because they are hypotenuses). If b were created by the split, the balanced bintree triangulation must have contained an illegal triangle pair (see figure 13) contradicting our assumption.



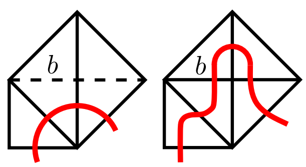
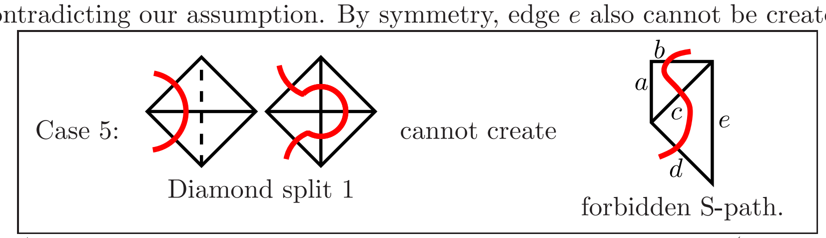


Figure 14: Diamond split 1 does not form forbidden Z-path.

Diamond split 1 creates two new edges, one of which must be edge a, b, c, d, or e in order to form a forbidden Z-path. Both new edges are crossed by the path (ruling out a and d) and are not a hypotenuse (ruling out c). If the split created b, then a forbidden C-path must have existed before the split (see figure 14) contradicting our assumption. By symmetry, edge e also cannot be created.



Again, the edges created by diamond split 1 cannot be a or e (because they are not crossed by the path) or c or d (because they are hypotenuses). If b were created by the split, the balanced bintree triangulation must have contained an illegal triangle pair (figure 15) contradicting our assumption.

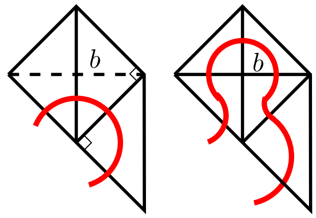
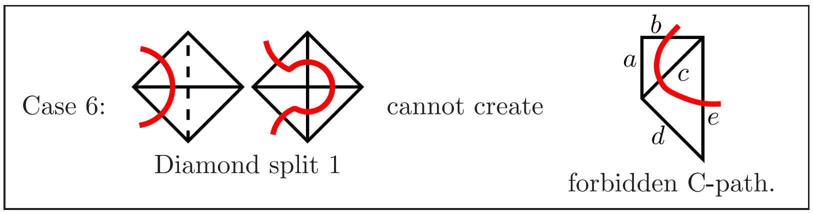


Figure 15: Diamond split 1 does not form forbidden S-path.



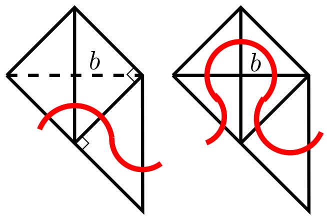
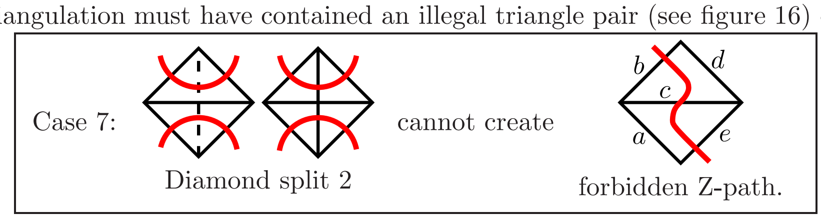


Figure 16: Diamond split 1 does not form forbidden C-path.

As in case 5, the edges created by diamond split 1 cannot be a or d (because they are not crossed by the path) or c or e (because they are hypotenuses). If b were created by the split, the balanced bintree triangulation must have contained an illegal triangle pair (see figure 16) contradicting our assumption.



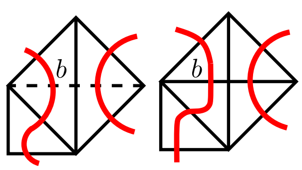


Figure 17: Diamond split 2 does not form forbidden Z-path.

Diamond split 2 creates two new edges, one of which must be edge a, b, c, d, or e in order to form a forbidden Z-path. Both new edges are crossed by the path (ruling out a and d) and are not a hypotenuse (ruling out c). If the split created b, then a forbidden S-path must have existed before the split (see figure 17) contradicting our assumption. By symmetry, edge e also cannot be created.

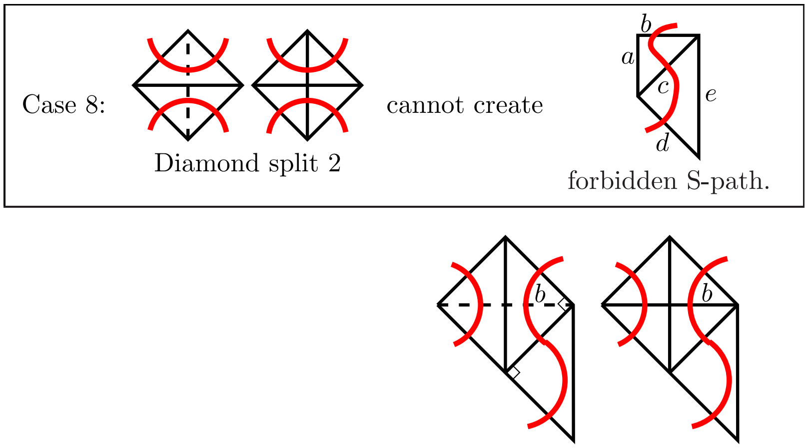


Figure 18: Diamond split 2 does not form forbidden S-path.

Again, the edges created by diamond split 2 cannot be a or e (because they are not crossed by the path) or c or d (because they are hypotenuses). If b were created by the split, the balanced bintree triangulation must have contained an illegal triangle pair (figure 18) contradicting our assumption.

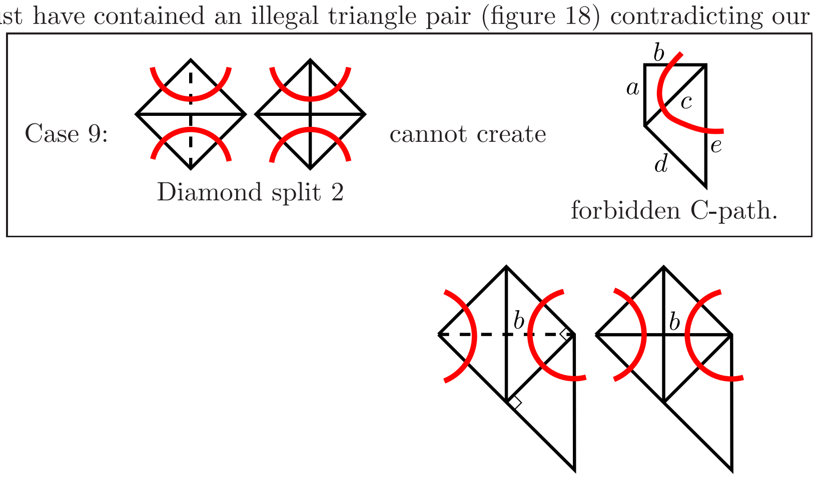


Figure 19: Diamond split 2 does not form forbidden C-path.

As in case 8, the edges created by diamond split 2 cannot be a or d (because they are not crossed by the path) or c or e (because they are hypotenuses). If b were created by the split, the balanced bintree triangulation must have contained an illegal triangle pair (see figure 19) contradicting our assumption. $\square$

5 Classifying Refinable Triangulations

We would like to classify simple polygons into two categories; those which permit balanced bintree triangulations with refinable hamiltonian cycles in their dual graphs (good polygons) and those which do not permit balanced bintree triangulations with refinable hamiltonian cycles in their dual graphs (bad polygons).

5.1 Good Polygons

A polygon is *good* if some balanced bintree triangulation of the shape has a hamiltonian dual graph. An example of a good shape is that of a square with dimensions $2^k \times 2^k$. The proof that all balanced bintree triangulations in the bintree hierarchy of the $2^k \times 2^k$ square contain (refinable) hamiltonian cycles is simple given Lemma 4.1.

Theorem 5.1. *Any balanced bintree triangulation in the bintree hierarchy of the $2^k \times 2^k$ square with at least four triangles contains a refinable hamiltonian cycle in its dual graph.*

Proof. The following figure shows a refinable hamiltonian cycle in the dual graph of the balanced bintree triangulation of four triangles. The theorem then follows from Lemma 4.1.

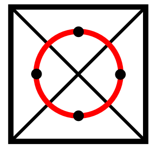


Figure 20: Four triangle balanced bintree triangulation and hamiltonian cycle in its dual graph.

□

This is, however, just a special case of a more general theorem. Given a simple, axis-aligned polygon whose vertices have integer coordinates, tile the polygon with squares of size $2^k \times 2^k$ for maximum $k \geq 1$. Create a triangulation from this tiling by triangulating every square using the four-triangle balanced bintree triangulation (shown in Figure 20). The dual graph of the resulting triangulation contains four-cycles (around degree four vertices in the triangulation) and eight-cycles (around the degree eight vertices). Direct each four-cycle clockwise and bi-direct the remaining edges. The resulting directed graph is called the *base digraph* of the simple polygon (see Figure 21).

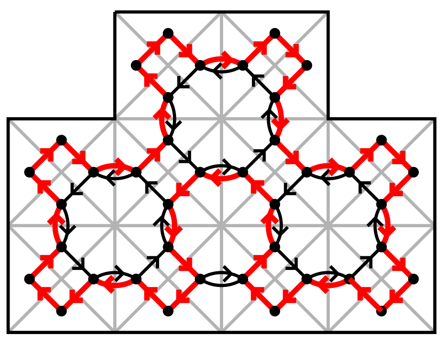


Figure 21: The base digraph of a simple polygon and a directed hamiltonian cycle in it.

Theorem 5.2. *A simple, axis-aligned polygon P has a balanced bintree triangulation with a refinable hamiltonian cycle in its dual graph if and only if P's base digraph has a directed hamiltonian cycle.*

Proof. We will show that the directed hamiltonian cycle contains no forbidden (undirected) path. Since all triangles in the triangulation are the same size, the only possible forbidden path is a Z-path. A Z-path contains three edges: one from one four-cycle, one from an adjacent four-cycle, and one that connects the two. Since both four-cycles are oriented clockwise, the two edges from the two fourcycles in the Z-path are directed in such a way that they cannot be part of a common directed hamiltonian cycle. See Figure 22.

□

A more constructive approach to specifying the class of good polygons is to use two special "tiles" to build the polygon and its hamiltonian cycle at the same time. In addition to squares with dimensions $2^k \times 2^k$,

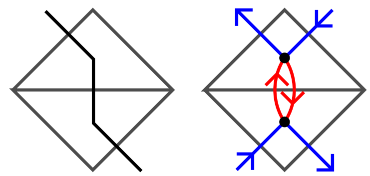


Figure 22: (a) Shows an example of a forbidden Z-path; (b) Shows an example of directed edges forming an undirected forbidden Z-path

and polygons containing Sierpiński curves, there are other polygons which contain a good hamiltonian cycle. Other such shapes include squares of dimension $2^k \times 2^k$ with the removal of internal Sierpiński curves where the removal does not form a 1-degree triangle (so that a cycle remains in the graph).

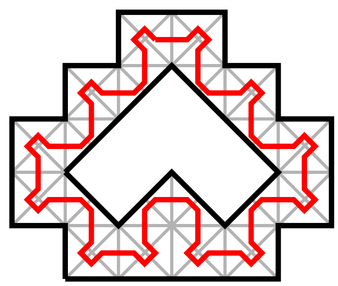


Figure 23: An example of a good polygon containing a hole.

Let's consider how to build polygons containing a good hamiltonian cycle. For simplicity, we'll consider triangulations at the same level of refinement.

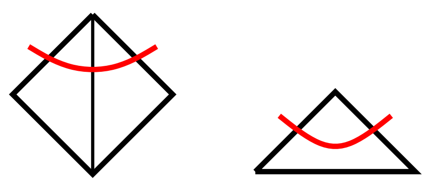


Figure 24: (a) diamond tile, (b) triangle tile. These tiles are the building blocks for good cycles.

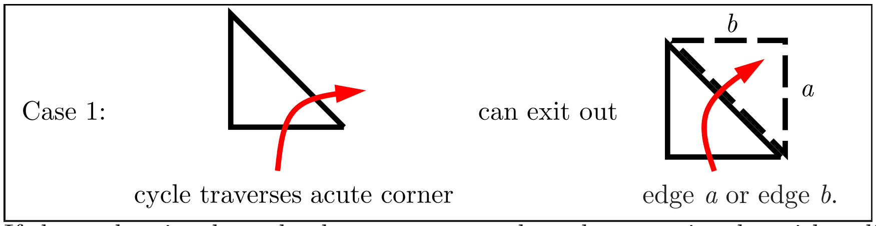
Theorem 5.3. *Given that no skewed triangles are permitted. At a fixed depth of refinement all shapes contain a refinable hamiltonian cycle if and only if the hamiltonian cycle can be formed using diamond tiles and triangle tiles.*

Proof. Assume skewed triangles are not permitted.

$\Rightarrow$ Given a refinable hamiltonian cycle at a fixed depth, show that it can be formed from diamond and triangle tiles.

The tiling process maintains the invariant that there is no untiled triangle, whose hypotenuse borders a tiled triangle (or the polygon's exterior) and is crossed by the hamiltonian cycle.

If we start from any triangle, the hamiltonian cycle will either traverse around the acute corner or it will traverse around the right angle. After we determine if it falls into Case 1 or Case 2, we repeat and continue to tile the neighbouring triangles until all the triangles in the triangulation are tiled.



If the path exits through edge a, we can replace the two triangles with a diamond tile. If the path exits through edge b, it forms the forbidden Z-path. This cannot occur since we were given a refinable hamiltonian cycle.

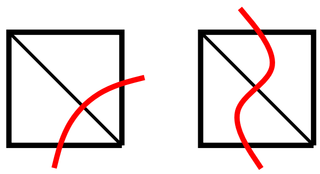
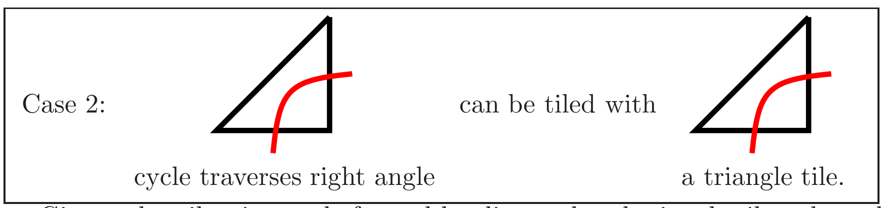


Figure 25: A path traversing around an acute corner can be tiled using a diamond tile.



$\Leftarrow$ Given a hamiltonian cycle formed by diamond and triangle tiles, show that it is refinable.

The only forbidden path that occurs at a fixed depth refinement is the forbidden Z-path. We need to show that no forbidden Z-paths can be formed from diamond-diamond, diamond-triangle, and triangle-triangle tilings. The only subpaths which can be formed without introducing skewed triangles are shown in Figure 26. We observe that none of the paths formed by these tilings will the forbidden Z-path.

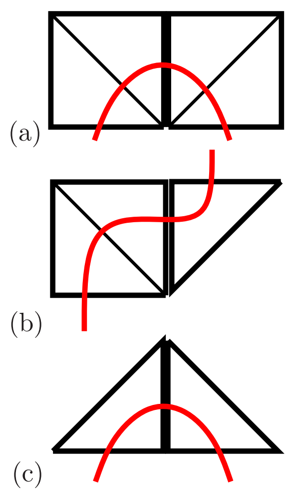


Figure 26: (a) diamond-diamond tiling, (b) diamond-triangle tiling, (c) triangle-triangle tiling.

Since no forbidden paths can be formed from the tiles, a hamiltonian cycle created from these tiles will be refinable. $\square$

5.2 Bad Polygons

A polygon is *bad* if some balanced bintree triangulation (with at least four triangles) has a non-hamiltonian dual graph. An example of a bad shape is a rectangle that is not of dimensions $2^k \times 2^k$.

Theorem 5.4. *Any rectangle (except for squares of size 2^k) is bad.*

Proof. Let $\mathcal{T}$ be a balanced bintree triangulation of P without a skewed triangle pair such that it and every finer triangulation of it contain a hamiltonian cycle in their dual graph. By repeatedly splitting the largest triangle in $\mathcal{T}$, we can obtain a balanced bintree triangulation $\mathcal{U}$ in which all triangles are the same size and they all have non-vertical and non-horizontal hypotenuses. Since $\mathcal{T}$ has no skewed triangle pair, $\mathcal{U}$ has no skewed triangle pair.

Suppose, without loss of generality, that the number of triangles with a boundary on the top edge of P is odd. The orientation of the leftmost of these triangles must be ◹ and the orientation of the rightmost must be ◸ , otherwise the dual graph contains a vertex of degree one. By a parity argument, some pair of successive triangles that have the top edge of P on their boundary must form a skewed triangle pair. ☐

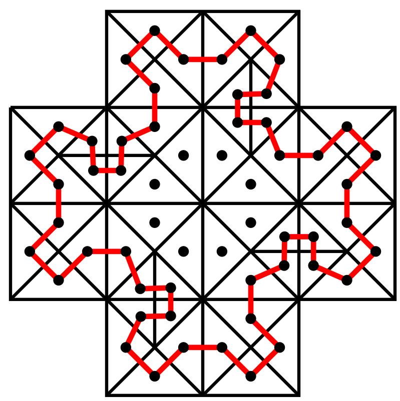


Figure 27: A graph containing no hamiltonian cycle in its dual.

We can expand on the class of bad polygons with the addition of Sierpiński curves. We define a generalized Sierpiński curve as Sierpiński corner additions to a triangulation.

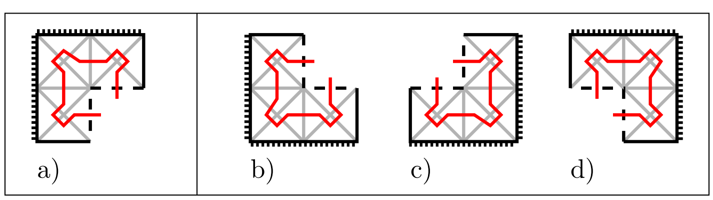


Figure 28: Example of Sierpiński corners (under rotations).

Sierpiński corners can only be attached to corners of a triangulation if no overlap occurs.

Lemma 5.1. *Given a hamiltonian cycle, subpaths in the cycle must follow these properties:*
Path rules
(1) Create a sub-path through each degree 2 triangle (i.e. a triangle with two adjacent triangles).
(2) Don't form loops/cycles of size less than the entire triangulation.

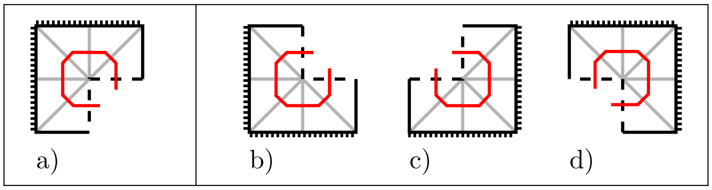


Figure 29: Example of Sierpiński corners (under rotations).

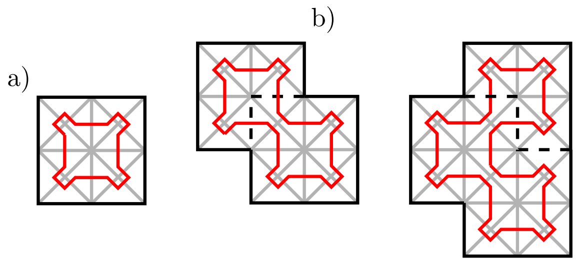


Figure 30: Example of Sierpiński corner addition to corner a) and corner b).

Proof. Rule 1 is true since a hamiltonian cycle must cover all triangles, in order to cover a degree 2 triangle, a path must go through the corresponding 2 available edges.

Rule 2 is obviously true since a hamiltonian cycle is a single connected path which covers each triangle only once. □

Lemma 5.2. *Given a "bad" triangulation, any subsequent Sierpiński corner additions will result in a "bad" triangulation. (In other words, badness is preserved.)*

Proof. In order to cover all the triangles, rule 1 must be followed. This will force the path in the Sierpiński addition.

When this piece gets added to a bad triangulation, the triangles *not* previously covered cannot be reached through this addition. This is because the path through the Sierpiński addition can only enter and exit through the two edges a and b.

Suppose that the path can exit through edge c instead. This would not form a hamiltonian cycle since a 1-degree triangle is created. A symmetric argument can be made for edge d.

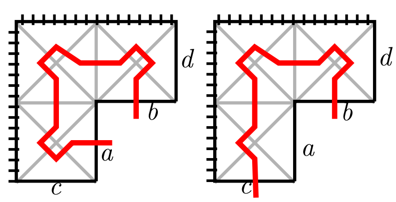


Figure 31: No forbidden Z-paths are formed.

□

6 Future Work

Given that a hamiltonian cycle exists in the dual graph of some types of balanced bintree triangulations, an interesting problem would be finding a way of determining the next triangle in a triangle strip of the balanced bintree triangulation with a constant number of operations.

If were are given a shape, can we determine if a refinable hamiltonian cycle exists. Obviously this can be determined in exponential time by brute force, but is there a faster algorithm?

References

[1] Laurent Balmelli. *Rate-distortion optimal mesh simplification for communications.* PhD thesis, École Polytechnique Fédérale de Lausanne, 2000.

[2] S. M. B. Bell, B. M. Diaz, F. Holroyd, and M. J. Jackson. Spatially referenced methods of processing raster and vector data. *Image and Vision Computing*, 1(4):211–220, 1983.

[3] Mark A. Duchaineau, Murray Wolinsky, David E. Sigeti, Mark C. Miller, Charles Aldrich, and Mark B. Mineev-Weinstein. ROAMing terrain: real-time optimally adapting meshes. In *IEEE Visualization*, pages 81–88, 1997.

[4] William S. Evans, David G. Kirkpatrick, and G. Townsend. Right-triangulated irregular networks. *Algorithmica*, 30(2):264–286, 2001.

[5] Peter Lindstrom and Valerio Pascucci. Terrain simplification simplified: A general framework for view-dependent out-of-core visualization. Technical report, U.S. Department of Energy, 2002.

[6] Renato B. Pajarola. Large scale terrain visualization using the restricted quadtree triangulation. In David Ebert, Hans Hagen, and Holly Rushmeier, editors, *IEEE Visualization '98*, pages 19–26, 1998.

[7] Hans Sagan. *Space-filling curves.* Springer-Verlag, New York, 1994.

[8] Hanan Samet. *The design and analysis of spatial data structures.* Addision-Wesley, 1990.

[9] Bettina Speckmann and Jack Snoeyink. Easy triangle strips for tin terrain models. In *Proceedings of the Canadian Conference on Computational Geometry*, pages 239–244, 1997.

[10] Luiz Velho, Luiz Henrique de Figueiredo, and Jonas Gomes. Hierarchical generalized triangle strips. *The Visual Computer*, 15(1):21–35, 1999.

[11] Luiz Velho and Jonas Gomes. Variable resolution 4-k meshes: concepts and applications. *Computer Graphics Forum*, 19(4):195–212, 2000.