Quantum Signal Propagation in Depolarizing Channels

Nicholas Pippenger*
(nicholas@cs.ubc.ca)

Department of Computer Science
The University of British Columbia
Vancouver, British Columbia V6T 1Z4
CANADA

Abstract: Let X be an unbiassed random bit, let Y be a qubit whose mixed state depends on X, and let the qubit Z be the result of passing Y through a depolarizing channel, which replaces Y with a completely random qubit with probability p. We measure the quantum mutual information between X and Y by $T(X;Y) = S(X) + S(Y) - S(X,Y)$, where $S(\cdots)$ denotes von Neumann's entropy. (Since X is a classical bit, the quantity $T(X;Y)$ agrees with Holevo's bound $\chi(X;Y)$ to the classical mutual information between X and the outcome of any measurement of Y.) We show that $T(X;Z) \leq (1-p)^2\, T(X;Y)$. This generalizes an analogous bound for classical mutual information due to Evans and Schulman, and provides a new proof of their result.

* The work reported here was supported by an NSERC Research Grant.

1. Introduction

Let X be an unbiassed random bit, let Y be a random bit depending on X, and let the bit Z be the result of passing Y through a binary symmetric channel, which complements the value of Y with probability ε. The binary symmetric channel can be viewed as replacing Y by an unbiassed random bit with probability $p = 2\varepsilon$. Evans and Schulman [E1, E2] have established the inequality

$$I(X;Z) \le (1-p)^2\, I(X;Y), \tag{1.1}$$

where $I(X;Y) = H(X) + H(Y) - H(X,Y)$ is the mutual information between X and Y, and $H(\cdots)$ denotes Shannon's entropy [S].

Our goal is to establish a quantum analogue of (1.1). As before we let X be an unbiassed random bit, but now we let Y be a qubit whose mixed state depends on X, and we let Z be the result of passing Y through a quantum depolarizing channel, which replaces Y by a completely random qubit with probability p. (See the survey of Bennett and Shor [B] for all quantum information-theoretic notions used in this paper.) To measure the information that Y contains about X, we define the quantum mutual information

$$T(X;Y) = S(X) + S(Y) - S(X,Y),$$

where $S(\cdots)$ denotes von Neumann's entropy [N]. Since X is a classical bit, $T(X;Y)$ agrees with Holevo's upper bound $\chi(X;Y)$ to the classical mutual information between X and the outcome of any measurement of Y (see Holevo [H]). Our result is

$$T(X;Z) \le (1-p)^2\, T(X;Y). \tag{1.2}$$

Since von Neumann's entropy is a generalization of Shannon's entropy, (1.2) is a generalization of (1.1), and our proof of (1.2) provides a new proof of (1.1).

2. Density Matrices

Let the joint mixed state of X and Y be described by the 4×4 density matrix $\varrho_{XY} = \frac{1}{2}\begin{pmatrix} \varrho_0 & 0 \\ 0 & \varrho_1 \end{pmatrix}$, where ϱ_0 and ϱ_1 are the 2×2 density matrices describing Y when $X = 0$ and $X = 1$, respectively. We then have $\varrho_X = \mathrm{Tr}_Y(\varrho_{XY}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\varrho_Y = \mathrm{Tr}_X(\varrho_{XY}) = \frac{1}{2}(\varrho_0 + \varrho_1)$, and the quantum mutual information between X and Y is

$$\begin{aligned} T(X;Y) &= S(\varrho_X) + S(\varrho_Y) - S(\varrho_{XY}) \\ &= 1 + S\big(\tfrac{1}{2}(\varrho_0 + \varrho_1)\big) - \big(1 + \tfrac{1}{2}S(\varrho_0) + \tfrac{1}{2}S(\varrho_1)\big) \\ &= S\big(\tfrac{1}{2}(\varrho_0 + \varrho_1)\big) - \tfrac{1}{2}S(\varrho_0) - \tfrac{1}{2}S(\varrho_1). \end{aligned}$$

Let Y be the input to a depolarizing channel, whose output Z is a completely random qubit (described by the density matrix $\tau = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) with probability p and is the intact qubit Y with probability $1-p$. The X and Z are described by the density matrix

$$\varrho_{XZ} = \tfrac{1}{2}\begin{pmatrix} (1-p)\varrho_0 + p\tau & 0 \\ 0 & (1-p)\varrho_1 + p\tau \end{pmatrix},$$

and the quantum mutual information between X and Z is

$$T(X;Z) = S\left(\tfrac{1}{2}(1-p)(\varrho_0+\varrho_1)\right) - S\big((1-p)\varrho_0 + p\tau\big) - S\big((1-p)\varrho_1 + p\tau\big).$$

Our goal is to establish the inequality

$$T(X;Z) \leq (1-p)^2\, T(X;Y). \tag{2.1}$$

A 2×2 density matrix ϱ can be expressed as

$$\varrho = \tfrac{1}{2}(I + \boldsymbol{\pi}\cdot\boldsymbol{\sigma}),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 identity matrix, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector whose components are the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ o & -1 \end{pmatrix}$, and $\boldsymbol{\pi} = (\pi_x, \pi_y, \pi_z)$ is a real polarization vector in the Bloch sphere: $\boldsymbol{\pi}\cdot\boldsymbol{\pi} \leq 1$.

For a 2×2 density matrix ϱ, the von Neumann entropy is given by

$$S(\varrho) = -\lambda_0 \log\lambda_0 - \lambda_1 \log\lambda_1,$$

where λ_0 and λ_1 are the eigenvalues of ϱ and the logarithms are to base 2. Since the von Neumann entropy is invariant under a unitary transformation $\varrho \mapsto U^\dagger \varrho U$ (where $U \in \mathbf{SU}(2)$), it depends for a 2×2 matrix ϱ only on the length $r = \|\boldsymbol{\pi}\| = (\pi_x^2 + \pi_y^2 + \pi_z^2)^{1/2}$ of the polarization vector. Specifically, the eigenvalues of of ϱ are then $\frac{1+r}{2}$ and $\frac{1-r}{2}$, so

$$\begin{aligned} S(\varrho) &= -\frac{1+r}{2}\log\frac{1+r}{2} - \frac{1-r}{2}\log\frac{1-r}{2} \\ &= 1 - \tfrac{1}{2}(1+r)\log(1+r) - \tfrac{1}{2}(1-r)\log(1-r). \end{aligned}$$

To apply this formula to our situation, we need the lengths r_0 and r_1 of the polarization vectors $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$ of ϱ_0 and ϱ_1, respectively, as well as the length r_2 of the polarization vector $\frac{1}{2}(\boldsymbol{\pi}_0 + \boldsymbol{\pi}_1)$ of $\frac{1}{2}(\varrho_0 + \varrho_1)$. Again using unitary invariance, we may assume that $\boldsymbol{\pi}_0$ is along the positive x-axis, and that $\boldsymbol{\pi}_1$ is in the (x, y)-plane and at angle ϑ to $\boldsymbol{\pi}_0$. Then

$$r_2 = \tfrac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1t)^{1/2},$$

where $t = \cos\vartheta$. We can now write

$$T(X;Y) = A\left(\tfrac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1t)^{1/2}\right) - \tfrac{1}{2}A(r_0) - \tfrac{1}{2}A(r_1),$$

where

$$A(r) = -\tfrac{1}{2}(1+r)\log(1+r) - \tfrac{1}{2}(1-r)\log(1-r).$$

The effect of a depolarizing channel with depolarizing probability p is to reduce the polarization vector by a factor of $q = 1 - p$. Thus

$$T(X;Y) = A\left(\tfrac{1}{2}q(r_0^2 + r_1^2 + 2r_0r_1t)^{1/2}\right) - \tfrac{1}{2}A(qr_0) - \tfrac{1}{2}A(qr_1).$$

The inequality (2.1) that we want to prove is therefore equivalent to

$$\begin{aligned} A\left(\tfrac{1}{2}q(r_0^2 + r_1^2 + 2r_0r_1t)^{1/2}\right) - \tfrac{1}{2}A(qr_0) - \tfrac{1}{2}A(qr_1) \\ \le q^2\left(A\left(\tfrac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1t)^{1/2}\right) - \tfrac{1}{2}A(r_0) - \tfrac{1}{2}A(r_1)\right), \end{aligned}$$

or

$$B(qr_0, qr_1, t) \le q^2\, B(r_0, r_1, t)$$

for $-1 \le t \le 1$ and $0 \le r_0, r_1, r_2 \le 1$, where

$$B(r_0, r_1, t) = A\left(\tfrac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1t)^{1/2}\right) - \tfrac{1}{2}A(r_0) - \tfrac{1}{2}A(r_1).$$

Since r_0 and r_1 appear symmetrically, we may assume that $r_0 \ge r_1$ and set $r_1 = sr_0$. If we now set $C(r, s, t) = B(r, sr, t)$, the inequality (2.1) then becomes

$$C(qr, s, t) \le q^2\, C(r, s, t), \tag{2.2}$$

where $-1 \le t \le 1$ and $0 \le r, s \le 1$.

3. Convexity

To show that a function $f(x)$ satisfying $f(0) = 0$ also satisfies

$$f(qx) \le q^2\, f(x) \tag{3.1}$$

for $0 \leq q \leq 1$ and $x > 0$, it will suffice to show that $f(x^{1/2})$ is convex in x. For then we will have

$$\begin{aligned} f(qx) &= f\big(((1-q^2)\cdot 0 + q^2 x^2)^{1/2}\big) \\ &\leq (1-q^2)\, f(0^{1/2}) + q^2\, f\big((x^2)^{1/2}\big) \\ &= q^2\, f(x). \end{aligned}$$

To show that a function $f(x^{1/2})$ is convex in x for $x > 0$, it is sufficient to show that

$$\frac{d^2}{dx^2} f(x^{1/2}) \geq 0. \tag{3.2}$$

Since

$$\frac{d^2}{dx^2} f(x^{1/2}) = \frac{1}{4x} f''(x^{1/2}) - \frac{1}{4x^{3/2}} f'(x^{1/2}),$$

multiplying though by $4x^2 > 0$ and substituting $y = x^{1/2}$ yields that (3.2) is equivalent to

$$y^2 f''(y) - y f'(y) \geq 0.$$

Thus if we define the operator

$$\Delta_y = y^2 \frac{d^2}{dy^2} - y \frac{d}{dy},$$

then to prove (3.1), it will suffice to show that

$$\Delta_y\, f(y) \geq 0.$$

In particular, to prove (2.2), it will suffice to show that

$$\Delta_r\, C(r,s,t) \geq 0. \tag{3.3}$$

Define

$$E(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x).$$

Then

$$\begin{aligned} E'(x) &= \ln(1+x) - \ln(1-x) \\ &= 2\sum_{k\geq 1} \frac{1}{2k-1}\, x^{2k-1} \end{aligned}$$

and

$$E''(x) = \frac{1}{1+x} + \frac{1}{1-x}$$
$$= \frac{2}{1-x^2}$$
$$= 2\sum_{k\geq 1} x^{2k-2}.$$

Thus if we define

$$D(x) = \Delta_x E(x),$$

we have

$$D(x) = x^2 E''(x) - xE'(x)$$
$$= 2\sum_{k\geq 1}\left(1 - \frac{1}{2k-1}\right) x^{2k}.$$

Since each term in this sum is non-decreasing and convex, $D(x)$ is non-decreasing and convex.

Since $A(x) = -\frac{1}{2}\log e\, E(x)$, we have

$$C(r,s,t) = A\big(u(s,t)\, r\big) - \tfrac{1}{2}A(r) + \tfrac{1}{2}A(sr)$$
$$= \tfrac{1}{2}\log e\left(\tfrac{1}{2}E(r) + \tfrac{1}{2}E(sr) - E\big(u(s,t)\, r\big)\right),$$

where

$$u(s,t) = \tfrac{1}{2}(1+s^2+2st)^{1/2}.$$

Thus

$$\Delta_r\, C(r,s,t) = \tfrac{1}{2}\log e\left(\tfrac{1}{2}D(r) + \tfrac{1}{2}D(sr) - D\big(u(s,t)\, r\big)\right).$$

Since $D(x)$ is convex in x, we have

$$\Delta_r\, C(r,s,t) \geq \tfrac{1}{2}\log e\left(D\left(\tfrac{1}{2}(1+s)r\right) - D\big(u(s,t)\, r\big)\right).$$

Since $t \leq 1$, we have

$$u(s,t) = \tfrac{1}{2}\left(1+s^2+2st\right)^{1/2}$$
$$\leq \tfrac{1}{2}(1+s).$$

Thus since $D(x)$ is non-decreasing in x, we have

$$\Delta_r C(r,s,t) \geq \tfrac{1}{2}\log e\left(D\left(\tfrac{1}{2}(1+s)\, r\right) - D\big(u(s,t)\, r\big)\right)$$
$$\geq 0,$$

which completes the proof of (3.3).

4. References

[B] C. H. Bennett and P. W. Shor, "Quantum Information Theory", *IEEE Trans. Inform. Theory*, 44 (1998) 2724–2742.

[E1] W. Evans and L. J. Schulman, "Signal Propagation, with Application to a Lower Bound on the Depth of Noisy Formulas", *Proc. IEEE Symp. on Foundations of Comp. Sci.*, 34 (1993) 594–603.

[E2] W. S. Evans and L. J. Schulman, "Signal Propagation and Noisy Circuits", *IEEE Trans. Inform. Theory*, 45 (1999) 2367–2373.

[H] A. S. Holevo, "Bounds for the Quantity of Information Transmitted by a Quantum Communcation Channel", *Probl. Inform. Transm.*, 9 (1973) 177–183.

[N] J. von Neumann, "Thermodynamik quantenmechanischer Gesamtheiten", *Gött. Nachr.*, (1927) 273–291.

[S] C. E. Shannon, "A Mathematical Theory of Communication", *Bell Syst. Tech. J.*, 27 (1948) 379–423, 623–655.