Quantum Signal Propagation in Depolarizing Channels

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Abstract: Let $X$ be an unbiased random bit, let $Y$ be a qubit whose mixed state depends on $X$, and let the qubit $Z$ be the result of passing $Y$ through a depolarizing channel, which replaces $Y$ with a completely random qubit with probability $p$. We measure the quantum mutual information between $X$ and $Y$ by $T(X; Y) = S(X) + S(Y) - S(X, Y)$, where $S(\cdot \cdot \cdot)$ denotes von Neumann’s entropy. (Since $X$ is a classical bit, the quantity $T(X; Y)$ agrees with Holevo’s bound $\chi(X; Y)$ to the classical mutual information between $X$ and the outcome of any measurement of $Y$.) We show that $T(X; Z) \leq (1 - p)^2 T(X; Y)$. This generalizes an analogous bound for classical mutual information due to Evans and Schulman, and provides a new proof of their result.

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1. Introduction

Let $X$ be an unbiased random bit, let $Y$ be a random bit depending on $X$, and let the bit $Z$ be the result of passing $Y$ through a binary symmetric channel, which complements the value of $Y$ with probability $\varepsilon$. The binary symmetric channel can be viewed as replacing $Y$ by an unbiased random bit with probability $p = 2\varepsilon$. Evans and Schulman [E1, E2] have established the inequality

$$I(X; Z) \leq (1 - p)^2 I(X; Y),$$

(1.1)

where $I(X; Y) = H(X) + H(Y) - H(X, Y)$ is the mutual information between $X$ and $Y$, and $H(\cdots)$ denotes Shannon’s entropy [S].

Our goal is to establish a quantum analogue of (1.1). As before we let $X$ be an unbiased random bit, but now we let $Y$ be a qubit whose mixed state depends on $X$, and we let $Z$ be the result of passing $Y$ through a quantum depolarizing channel, which replaces $Y$ by a completely random qubit with probability $p$. (See the survey of Bennett and Shor [B] for all quantum information-theoretic notions used in this paper.) To measure the information that $Y$ contains about $X$, we define the quantum mutual information

$$T(X; Y) = S(X) + S(Y) - S(X, Y),$$

where $S(\cdots)$ denotes von Neumann’s entropy [N]. Since $X$ is a classical bit, $T(X; Y)$ agrees with Holevo’s upper bound $\chi(X; Y)$ to the classical mutual information between $X$ and the outcome of any measurement of $Y$ (see Holevo [H]). Our result is

$$T(X; Z) \leq (1 - p)^2 T(X; Y).$$

(1.2)

Since von Neumann’s entropy is a generalization of Shannon’s entropy, (1.2) is a generalization of (1.1), and our proof of (1.2) provides a new proof of (1.1).

2. Density Matrices

Let the joint mixed state of $X$ and $Y$ be described by the $4 \times 4$ density matrix $\varrho_{XY} = \frac{1}{2} \begin{pmatrix} \varrho_0 & 0 \\ 0 & \varrho_1 \end{pmatrix}$, where $\varrho_0$ and $\varrho_1$ are the $2 \times 2$ density matrices describing $Y$ when $X = 0$ and $X = 1$, respectively. We then have $\varrho_X = \text{Tr}_Y(\varrho_{XY}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\varrho_Y = \text{Tr}_X(\varrho_{XY}) = \frac{1}{2}(\varrho_0 + \varrho_1)$, and the quantum mutual information between $X$ and $Y$ is

$$T(X; Y) = S(\varrho_X) + S(\varrho_Y) - S(\varrho_{XY})$$

$$= 1 + S\left(\frac{1}{2}(\varrho_0 + \varrho_1)\right) - (1 + \frac{1}{2}S(\varrho_0) + \frac{1}{2}S(\varrho_1))$$

$$= S\left(\frac{1}{2}(\varrho_0 + \varrho_1)\right) - \frac{1}{2}S(\varrho_0) - \frac{1}{2}S(\varrho_1).$$
Let $Y$ be the input to a depolarizing channel, whose output $Z$ is a completely random qubit (described by the density matrix $\tau = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) with probability $p$ and is the intact qubit $Y$ with probability $1 - p$. The $X$ and $Z$ are described by the density matrix

$$
\rho_{XZ} = \frac{1}{2} \begin{pmatrix} (1 - p)g_0 + p\tau & 0 \\ 0 & (1 - p)g_1 + p\tau \end{pmatrix},
$$

and the quantum mutual information between $X$ and $Z$ is

$$T(X; Z) = S\left(\frac{1}{2}(1 - p)(g_0 + g_1)\right) - S(1 - p)g_0 + p\tau - S((1 - p)g_1 + p\tau).
$$

Our goal is to establish the inequality

$$T(X; Z) \leq (1 - p)^2 T(X; Y). \tag{2.1}
$$

A $2 \times 2$ density matrix $\rho$ can be expressed as

$$
\rho = \frac{1}{2}(I + \sigma \cdot \pi),
$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the $2 \times 2$ identity matrix, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a vector whose components are the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\pi = (\pi_x, \pi_y, \pi_z)$ is a real polarization vector in the Bloch sphere: $\pi \cdot \pi \leq 1$.

For a $2 \times 2$ density matrix $\rho$, the von Neumann entropy is given by

$$S(\rho) = -\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1,
$$

where $\lambda_0$ and $\lambda_1$ are the eigenvalues of $\rho$ and the logarithms are to base 2. Since the von Neumann entropy is invariant under a unitary transformation $\rho \mapsto U^\dagger \rho U$ (where $U \in SU(2)$), it depends for a $2 \times 2$ matrix $\rho$ only on the length $r = \|\pi\| = (\pi_x^2 + \pi_y^2 + \pi_z^2)^{1/2}$ of the polarization vector. Specifically, the eigenvalues of of $\rho$ are then $\frac{1+r}{2}$ and $\frac{1-r}{2}$, so

$$S(\rho) = -\frac{1+r}{2} \log \frac{1+r}{2} - \frac{1-r}{2} \log \frac{1-r}{2}
$$

$$= 1 - \frac{1}{2}(1+r) \log(1+r) - \frac{1}{2}(1-r) \log(1-r).$$

To apply this formula to our situation, we need the lengths $r_0$ and $r_1$ of the polarization vectors $\pi_0$ and $\pi_1$ of $g_0$ and $g_1$, respectively, as well as the length $r_2$ of the polarization
vector $\frac{1}{2}(\pi_0 + \pi_1)$ of $\frac{1}{2}(q_0 + q_1)$. Again using unitary invariance, we may assume that $\pi_0$ is along the positive $x$-axis, and that $\pi_1$ is in the $(x,y)$-plane and at angle $\vartheta$ to $\pi_0$. Then
\[ r_2 = \frac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1 t)^{1/2}, \]
where $t = \cos \vartheta$. We can now write
\[ T(X;Y) = A\left(\frac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1 t)^{1/2}\right) - \frac{1}{2}A(r_0) - \frac{1}{2}A(r_1), \]
where
\[ A(r) = -\frac{1}{2}(1+r)\log(1+r) - \frac{1}{2}(1-r)\log(1-r). \]

The effect of a depolarizing channel with depolarizing probability $p$ is to reduce the polarization vector by a factor of $q = 1 - p$. Thus
\[ T(X;Y) = A\left(\frac{1}{2}q(r_0^2 + r_1^2 + 2r_0r_1 t)^{1/2}\right) - \frac{1}{2}A(qr_0) - \frac{1}{2}A(qr_1). \]
The inequality (2.1) that we want to prove is therefore equivalent to
\[ A\left(\frac{1}{2}q(r_0^2 + r_1^2 + 2r_0r_1 t)^{1/2}\right) - \frac{1}{2}A(qr_0) - \frac{1}{2}A(qr_1) \]
\[ \leq q^2 \left( A\left(\frac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1 t)^{1/2}\right) - \frac{1}{2}A(r_0) - \frac{1}{2}A(r_1) \right), \]
or
\[ B(qr_0, qr_1, t) \leq q^2 B(r_0, r_1, t) \]
for $-1 \leq t \leq 1$ and $0 \leq r_0, r_1, r_2 \leq 1$, where
\[ B(r_0, r_1, t) = A\left(\frac{1}{2}(r_0^2 + r_1^2 + 2r_0r_1 t)^{1/2}\right) - \frac{1}{2}A(r_0) - \frac{1}{2}A(r_1). \]
Since $r_0$ and $r_1$ appear symmetrically, we may assume that $r_0 \geq r_1$ and set $r_1 = sr_0$. If we now set $C(r, s, t) = B(r, sr, t)$, the inequality (2.1) then becomes
\[ C(qr, s, t) \leq q^2 C(r, s, t), \quad (2.2) \]
where $-1 \leq t \leq 1$ and $0 \leq r, s \leq 1$.

3. Convexity

To show that a function $f(x)$ satisfying $f(0) = 0$ also satisfies
\[ f(qx) \leq q^2 f(x) \quad (3.1) \]
for $0 \leq q \leq 1$ and $x > 0$, it will suffice to show that $f(x^{1/2})$ is convex in $x$. For then we will have

$$f(qx) = f\left((1 - q^2) \cdot 0 + q^2 x^2\right)^{1/2} \leq (1 - q^2) f(0^{1/2}) + q^2 f\left((x^2)^{1/2}\right) = q^2 f(x).$$

To show that a function $f(x^{1/2})$ is convex in $x$ for $x > 0$, it is sufficient to show that

$$\frac{d^2}{dx^2} f(x^{1/2}) \geq 0. \quad (3.2)$$

Since

$$\frac{d^2}{dx^2} f(x^{1/2}) = \frac{1}{4x} f''(x^{1/2}) - \frac{1}{4x^{3/2}} f'(x^{1/2}),$$

multiplying though by $4x^2 > 0$ and substituting $y = x^{1/2}$ yields that (3.2) is equivalent to

$$y^2 f''(y) - y f'(y) \geq 0.$$

Thus if we define the operator

$$\Delta_y = y^2 \frac{d^2}{dy^2} - y \frac{d}{dy},$$

then to prove (3.1), it will suffice to show that

$$\Delta_y f(y) \geq 0.$$

In particular, to prove (2.2), it will suffice to show that

$$\Delta_r C(r, s, t) \geq 0. \quad (3.3)$$

Define

$$E(x) = (1 + x) \ln(1 + x) + (1 - x) \ln(1 - x).$$

Then

$$E'(x) = \ln(1 + x) - \ln(1 - x)$$

$$= 2 \sum_{k \geq 1} \frac{1}{2k - 1} x^{2k-1}$$

$$= 2 \sum_{k \geq 1} \frac{1}{2k - 1} x^{2k-1}$$
and
\[ E''(x) = \frac{1}{1+x} + \frac{1}{1-x} \]
\[ = \frac{2}{1-x^2} \]
\[ = 2 \sum_{k \geq 1} x^{2k-2}. \]

Thus if we define
\[ D(x) = \Delta_x E(x), \]
we have
\[ D(x) = x^2 E''(x) - x E'(x) \]
\[ = 2 \sum_{k \geq 1} \left( 1 - \frac{1}{2k-1} \right) x^{2k}. \]

Since each term in this sum is non-decreasing and convex, \( D(x) \) is non-decreasing and convex.

Since \( A(x) = -\frac{1}{2} \log e E(x) \), we have
\[ C(r, s, t) = A(u(s, t) r) - \frac{1}{2} A(r) + \frac{1}{2} A(s r) \]
\[ = \frac{1}{2} \log e \left( \frac{1}{2} E(r) + \frac{1}{2} E(s) - E(u(s, t) r) \right), \]
where
\[ u(s, t) = \frac{1}{2}(1 + s^2 + 2st)^{1/2}. \]

Thus
\[ \Delta_r C(r, s, t) = \frac{1}{2} \log e \left( \frac{1}{2} D(r) + \frac{1}{2} D(s) - D(u(s, t) r) \right). \]

Since \( D(x) \) is convex in \( x \), we have
\[ \Delta_r C(r, s, t) \geq \frac{1}{2} \log e \left( \frac{1}{2} (1 + s) r - D(u(s, t) r) \right). \]

Since \( t \leq 1 \), we have
\[ u(s, t) = \frac{1}{2}(1 + s^2 + 2st)^{1/2} \]
\[ \leq \frac{1}{2}(1 + s). \]

Thus since \( D(x) \) is non-decreasing in \( x \), we have
\[ \Delta_r C(r, s, t) \geq \frac{1}{2} \log e \left( \frac{1}{2} (1 + s) r - D(u(s, t) r) \right) \]
\[ \geq 0, \]
which completes the proof of (3.3).
4. References


