Toward the Rectilinear Crossing Number of K_n: New Drawings, Upper Bounds, and Asymptotics

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Abstract

Scheinerman and Wilf [SW94] assert that "an important open problem in the study of graph embeddings is to determine the rectilinear crossing number of the complete graph K_n." A *rectilinear drawing of* K_n is an arrangement of n vertices in the plane, every pair of which is connected by an edge that is a line segment. We assume that no three vertices are collinear, and that no three edges intersect in a point unless that point is an endpoint of all three. The *rectilinear crossing number of* K_n is the fewest number of edge crossings attainable over all rectilinear drawings of K_n.

For each n we construct a rectilinear drawing of K_n that has the fewest number of edge crossings and the best asymptotics known to date. Moreover, we give some alternative infinite families of drawings of K_n with good asymptotics. Finally, we mention some old and new open problems.

keywords crossing number, rectilinear, complete graph

1 Introduction and History

Given an arbitrary graph G, determining a drawing of G in the plane that produces the fewest number of edge crossings is NP-Complete [GJ83]. The complexity is not known for an arbitrary graph when the edges are assumed to be line segments [Bie91]. Recent exciting work on the general crossing number problem (where edges are simply homeomorphs of the unit interval $[0,1]$ rather than line segments) has been accomplished by Pach, Spencer, and Tóth [PST99], who give a tight lower bound for the crossing number of families of graphs with certain forbidden subgraphs. We study the specific instance of determining the **rectilinear crossing number of** K_n, denoted $\overline{cr}(K_n)$, and we offer drawings with "few" edge crossings. The difficulty of determining the exact value of $\overline{cr}(K_n)$, even for small values of n, manifests itself in the sparsity of literature [Guy72, EG73, Sin71, BDG00a]. Other contributions are given as general constructions [Jen71, Hay87] that yield upper bounds and asymptotics, none of which lead to exact values of $\overline{cr}(K_n)$ for all n. Finally, there is an elegant and surprising connection between the asymptotics of the rectilinear crossing number of K_n and Sylvester's four point problem of geometric probability [SW94, Wil97].

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Much of the information regarding progress of any kind has been disseminated by personal communication, and now in this era of "the information highway," some revealing sources of the unfolding story can be found on the web [Fin00, Arc95].

In this paper we offer new constructions, upper bounds, and asymptotics, which we motivate and explain by the interesting and nondeterministic historical progress of the problem and its elusive solution.

2 Recursive Construction of K_n

2.1 Introduction

Upon examining different configurations of vertices in the plane, one quickly realizes that drawings that minimize crossings tend to have vertices aligned along three axes, forming a triangular structure of nested concentric triangles; such configurations are "opposite" in flavour to placing vertices on a convex hull. Two nested triangles t_1 and t_2 are **concentric** if and only if any edge with endpoints in t_1 and t_2 does not intersect any edge of t_1 or t_2 (see Figure 1). In K_4 through K_9, for which optimal drawings are known [Guy72, WB78], the tripartite pattern is evident. The same pattern exists in generalized constructions presented by Jensen [Jen71] and Hayward [Hay87] for any K_n.

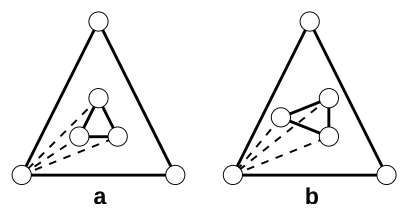

Figure 1: concentric versus non-concentric triangles

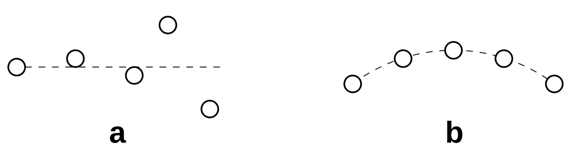

Figure 2: positioning vertices using Jensen's [Jen71] and Hayward's [Hay87] constructions

Various schemes are possible for positioning vertices within each of the three parts. In Jensen's construction, vertices along an axis are positioned by alternating above and below the axis (see Figure 2a). In Hayward's construction, vertices along an axis are are positioned on a concave curve (see Figure 2b). Alternatively, the collection of vertices along each axis could be arranged to minimize crossings within the collection, while maintaining concentricity of the triangles. We examine a construction and variations, originally suggested by Singer [Sin71], that positions vertices along each axis by recursive definition of similarly constructed smaller graphs.

2.2 Definitions

We identify specific sets of edges, sets of vertices, and subgraphs, within the larger construction of K_n. Those components of the graph that are recursively defined form **clustervertices**. Each clustervertex is itself a complete graph K_a, where $a < n$; a clustervertex with a vertices is said to have **order** a. If both endpoints of an edge uw are contained within clustervertex c, then uw is **internal** to c. Similarly, a vertex w contained within a clustervertex c is internal to c. Given two clustervertices c_1 and c_2, the set of all edges that have one endpoint in each of c_1 and c_2 form a **clusteredge**. Finally, if q clusteredges meet at clustervertex c, then c has **clusterdegree** q.

Recursively constructed clustervertices are **flattened** by an affine transformation [Mar82, Ch. 15]. Vertices appear as a sequence of nearly collinear vertices. Of course, no three vertices in the graph can be collinear, thus the flattened clustervertex has some height $\epsilon > 0$ (see Figure 3) and its edge crossings are unaltered by the scaling. When a clustervertex c is flattened, an incident clusteredge e is said to **dock** at c. Given a flat clustervertex c, two clusteredges e_1 and e_2 may dock at c from opposite sides such that no edge crossings are created between e_1 and e_2. When two clusteredges e_1 and e_2 dock on the same side of a clustervertex c, we say e_1 and e_2 **merge** at c (see Figure 5).

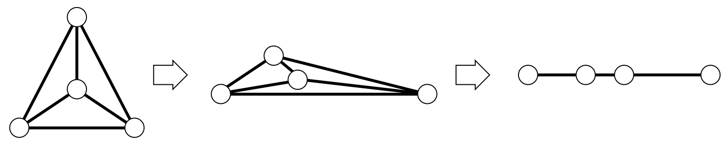

Figure 3: flattening a clustervertex

2.3 Counting Toolbox

Given a generalized definition for graph construction involving clustervertex interconnection, the following functions count edge crossings for the various types of edge intersections.

2.3.1 $f(k)$: Single Vertex Docked at a Clustervertex

When a new vertex u is created, new edges are added from u to all other existing vertices. Specifically, given a clustervertex c of order k, an edge must be added from u to every vertex in c. An edge from u to a vertex w in c may cross some internal edges of c. If w is the ith vertex in the sequence of vertices of c, $i-1$ vertices lie on one side of w in c and $k-i$ vertices lie on the other side (see Figure 4a). Thus, edge uw will be required to cross at most $(i-1)(k-i)$ edges of c. If we add edges from u to every vertex in c, the number of new edge crossings within c will be at most

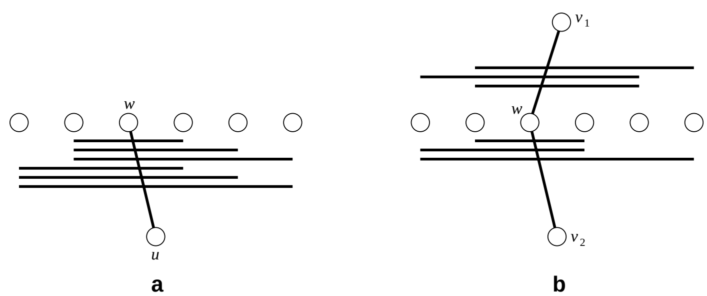

Figure 4: Edge uw crosses at most six internal edges.

$$f(k) = \sum_{i=1}^{k}(i-1)(k-i) = \frac{k^3}{6} - \frac{k^2}{2} + \frac{k}{3}\ . \tag{1}$$

If we add two vertices v_1 and v_2 on opposite sides of a clustervertex c, then for every internal vertex w of c, the internal edges that span w will be crossed exactly once, either by edge v_1w or by edge v_2w but not both (see Figure 4b). The number of new edge crossings among vertices of c and v_1 and v_2 will be exactly $f(k)$.

2.3.2 $i(p,k)$: Internal Clusteredge Intersections

Given two clustervertices c_k and c_p of orders k and p, and a clusteredge e between them that docks completely on one side of each clustervertex, selecting two vertices from each clustervertex forms a quadrilateral that contributes one edge crossing. The number of edge crossings within e is given by

$$i(p,k) = \binom{p}{2}\binom{k}{2} = \frac{p(p-1)k(k-1)}{4} \ . \tag{2}$$

2.3.3 $e(k,p,j)$: Two Clusteredges Merge at a Clustervertex

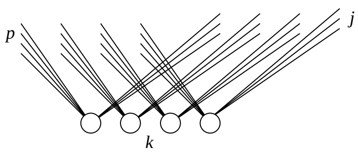

Figure 5: two clusteredges merge at a clustervertex

When two clusteredges originate from clustervertices of orders p and j and merge at a clustervertex of order k (see Figure 5) the number of crossings between edges of the two clusteredges (ignoring crossings with edges internal to the clustervertex) is given by

$$e(k,p,j) = \sum_{i=0}^{k-1} ipj = \frac{pjk(k-1)}{2} \ . \tag{3}$$

If the two clusteredges intersect away from a clustervertex, then the number of crossings is simply $p \cdot j \cdot k \cdot l$, where the clusteredge crossing is between four clustervertices of orders p, j, k, and l.

2.4 Recursive Definitions

The following constructions of K_n involve recursive definition by connecting q clustervertices K_k of order k, where $n = q \cdot k$. Scheinerman and Wilf show that $\overline{cr}(K_n) = \Theta(n^4)$ [SW94]. In a worst case drawing, where edge crossings are maximized, every subset of four vertices contributes one edge crossing. This occurs when all vertices lie on a convex hull, creating $\binom{n}{4}$ crossings. Thus, when a better drawing is found, we examine what fraction of the crossings remain by taking the limit of $g(n)/\binom{n}{4}$ as $n \to \infty$, where $g(n)$ is a count of the crossings in the new drawing.

2.4.1 Triangular Definition

Singer suggests a recursive construction [Sin71, Wil97] where, given $n = 3^j$, we draw K_n by taking three flat instances of $K_{n/3}$ and adding new edges (see Figure 6). Each instance of $K_{n/3}$ is drawn recursively. K_3 gives a base case.

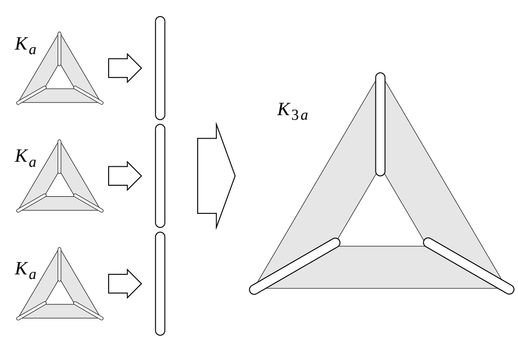

Figure 6: K_n defined by three $K_{n/3}$

Let $k = n/3$ and let $C_3(n)$ represent the total number of crossings in K_n under the drawing defined by this recursive construction. There are $C_3(k)$ crossings internal to each of the clustervertices, $k \cdot f(k)$ crossings for each clustervertex corresponding to clusteredge to clustervertex dockings, and $i(k,k)$ crossings internal to each clusteredge.

Given that $C_3(3) = 0$, the total number of crossings is given by

$$C_3(n) \quad = \quad 3C_3(k) + 3k \cdot f(k) + 3i(k,k) = \frac{5}{312}n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{19}{104}n \tag{4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{C_3(n)}{\binom{n}{4}} = \frac{15}{39} \approx 0.3846 \ . \tag{5}$$

2.4.2 Recursive Definitions Using a Larger K_a

Just as we do for K_3, we may use any optimal drawing of K_a as a recursive template. Given $n = a^j$, we apply an analogous procedure where clustervertices are defined recursively. In addition to counting recursive terms, $C_a(k)$, internal clusteredge crossings, $i(k,k)$, and clusteredge-clustervertex crossings, $k \cdot f(k)$, we must also count pairs of clusteredges that merge, $e(k,k,k)$, and clusteredge crossings away from a clustervertex, k^4. Using K_4 as a basis and $C_4(4) = 0$, we derive

$$C_4(n) = 4C_4(k) + 6i(k,k) + 6k \cdot f(k) + 4e(k,k,k) = \frac{1}{56}n^4 - \frac{2}{15}n^3 + \frac{7}{24}n^2 - \frac{37}{210}n \qquad (6)$$

$$\Rightarrow \lim_{n\to\infty} \frac{C_4(n)}{\binom{n}{4}} = \frac{3}{7} \approx 0.4286\ . \qquad (7)$$

Using K_5 as a basis and $C_5(5) = 1$, we derive

$$C_5(n) = 5C_5(k) + 10i(k,k) + 10k \cdot f(k) + 10e(k,k,k) + k^4 = \frac{61}{3720}n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{227}{1240}n \quad (8)$$

$$\Rightarrow \lim_{n\to\infty} \frac{C_5(n)}{\binom{n}{4}} = \frac{227}{155} \approx 0.3935\ . \qquad (9)$$

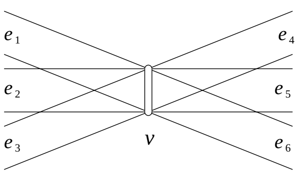

Figure 7: balanced clusteredge dockings

Similarly, we derive limits using K_7 and K_9 as templates (see Table 1). As one would expect, the limit for K_9 is equal to that for K_3, since both are powers of three. For any odd a, we derive a generalized exact count using a recursive K_a construction. We require a count for the number of crossings in K_a, both for our base case, $C_a(a) = \overline{cr}(K_a)$, and for recursively-defined clusteredge to clusteredge crossings.

The count breaks down as follows. Let $k = n/a$. We take a recursive instances of K_k which contribute $a \cdot C_a(k)$ crossings. We add crossings for every pair of clusteredges that merge at a clustervertex. Each clustervertex has clusterdegree $a-1$. To minimize crossings, clusteredges must be split evenly on either side of a flattened clustervertex (see Figure 7). Thus, clusteredge dockings contribute $2a\binom{(a-1)/2}{2}e(k,k,k)$ crossings. Pairs of dockings on opposite sides of a clustervertex contribute exactly $\binom{a}{2}k \cdot f(k)$ crossings. Clusteredges have internal crossings that add another $\binom{a}{2}i(k,k)$. Finally, we must account for

	a	$\lim_{n\to\infty} \frac{g(n)}{\binom{n}{4}}$	comment
Singer [Sin71]	3	0.3846	$n = 3^j$, $C_3(3) = 0$
Brodsky-Durocher-Gethner	4	0.4286	$n = 4^j$, $C_4(4) = 0$
Brodsky-Durocher-Gethner	5	0.3935	$n = 5^j$, $C_5(5) = 1$
Brodsky-Durocher-Gethner	7	0.3885	$n = 7^j$, $C_7(7) = 9$
Brodsky-Durocher-Gethner	9	0.3846	$n = 9^j$, $C_9(9) = 36$
Jensen [Jen71]	–	0.3888	any n
Hayward [Hay87]	–	0.4074	any n
Scheinerman-Wilf [SW94]	–	0.2905	lower bound
Guy [Guy60]	–	0.3750	conjectured $cr(K_n)$ (non-rectilinear)

Table 1: asymptotics for $C_a(n)$ compared with known bounds

clusteredge to clusteredge crossings that occur in K_a itself; thus we add $\overline{cr}(K_a) \cdot k^4$. This gives

$$C_a(n) = a \cdot C_a(k) + \binom{a}{2} k \cdot f(k) + 2a \binom{\frac{a-1}{2}}{2} e(k,k,k) + \binom{a}{2} i(k,k) + \overline{cr}(K_a) \cdot k^4 \ . \tag{10}$$

We can solve for a non-recursive closed form of $C_a(n)$ by simplifying

$$\frac{n}{a}\overline{cr}(K_a) + \sum_{j=1}^{\log_a n - 1} a^{j-1} \left[\binom{a}{2} k \cdot f(k) + \binom{a}{2} i(k,k) + 2a \binom{\frac{a-1}{2}}{2} e(k,k,k) + \overline{cr}(K_a) \cdot k^4 \right], \tag{11}$$

where $k = n/a^j$.

Out of all recursive constructions for which $\overline{cr}(K_a)$ is known, the best results are achieved by $C_3(n)$ (see Table 1). The construction can easily be generalized by dividing n into three parts of sizes $\lfloor \frac{n}{3} \rfloor$, $\lceil \frac{n}{3} \rceil$, and $n - \lfloor \frac{n}{3} \rfloor - \lceil \frac{n}{3} \rceil$. Since two of the three parts will always have the same size, $f(k)$ always gives an exact count. By induction, one can show that $C_{3g}(n) < jen(n)$ for $n \geq 24$, where $C_{3g}(n)$ is a count of the crossings in the generalized construction and $jen(n)$ is the number of crossings in K_n using Jensen's construction[1][Jen71]. Thus, asymptotically, $C_{3g} < 3 \cdot [jen(k) + k \cdot f(k) + i(k,k)]$, with $k = n/3$, and we get an upper bound of 0.3848 for a general n. In the next section we offer some improvements.

3 Asymptotic Improvements

Within the recursive constructions presented thus far, edges arriving at a flattened clustervertex are balanced; if q edges arrive at clustervertex c of degree p, then exactly $q/2$ edges arrive at c from each side and $\frac{q}{2} f(p)$ crossings are added. However, depending on the side of entry, the number of edges crossed when entering a clustervertex differs. Thus, it may be advantageous to have an imbalance in the number of edges docking on each side of a clustervertex.

Most of the crossings in $C_3(n)$ occur at the top level of the recurrence, as is shown by

$$\lim_{n \to \infty} \frac{C_3(n) - 3C_3(n/3)}{C_3(n)} = \frac{26}{27} \ . \tag{12}$$

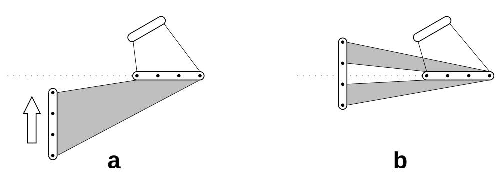

Figure 8: sliding a clustervertex

Improving the top level of the construction while slightly compromising on recursive constructions could reduce the total crossings. Improvements at the top-level can be achieved by moving clustervertices to alter the number of edges that reach a neighbouring clustervertex from above and from below (see Figure 8). In doing so, however, new crossings are created at the merging of clusteredges. Thus, there exists a point of balance that minimizes total crossings lost and gained by the translation.

[1] $jen(n) = \left\lfloor \frac{7n^4 - 56n^3 + 128n^2 + 48n \lfloor \frac{n-7}{3} \rfloor + 108}{432} \right\rfloor$

3.1 Maximally Asymmetric Internal Clustervertices

In the extreme case, we construct each of the three partitions by taking a convex K_k (see Figures 9 and 11). Crossings from above are minimized and crossings from below are maximized to form a maximally asymmetric drawing.

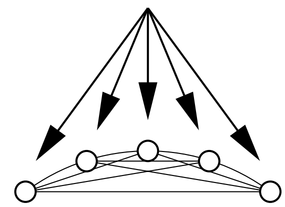

Figure 9: minimizing crossings from above

Let $k = n/3$ and let $a + b = k$ determine how much to slide the clustervertex, where b is a measure of how many vertices in one clustervertex change position relative to the other two. Assuming each clustervertex is moved by the same amount, the top-level graph will appear as in Figure 11. Accounting using the usual tools gives the following count of crossings

$$\begin{aligned} C_m(n,a) &= 3\Big[\binom{k}{4} + a \cdot f(k) + i(a,a) + i(b,b) + 2i(a,b) \\ &\quad + e(a,b,b) + 2e(a,a,b) + e(b,b,b) + 2e(b,a,b) + ab^3 + a^2b^2\Big] \\ &= \frac{19}{648}n^4 - \frac{5}{54}n^3a + \frac{1}{6}n^2a^2 - \frac{5}{36}n^3 + \frac{1}{6}n^2a - \frac{1}{2}na^2 + \frac{17}{72}n^2 + \frac{1}{3}na - \frac{1}{4}n. \end{aligned} \tag{13}$$

$C_m(n,a)$ is a quadratic polynomial in a and is minimized when $a_0 = 5n/18 + 1/3$. This gives

$$C_m(n,a_0) = \frac{4}{243}n^4 - \frac{85}{648}n^3 + \frac{67}{216}n^2 - \frac{7}{36}n \tag{14}$$

$$\Rightarrow \lim_{n\to\infty} \frac{C_m(n,a_0)}{\binom{n}{4}} = \frac{32}{81} \approx 0.3951\ . \tag{15}$$

$C_3(n)$ still performs better than $C_m(n,a)$ for any a. Thus, using convex K_k as first-level clustervertices overcompensates the savings of the recursive structure in $C_3(n)$. Therefore, we define a new construction that maintains the recursive structure of $C_3(n)$ for clustervertices.

3.2 Retaining $C_3(n)$ as Internal Clustervertices

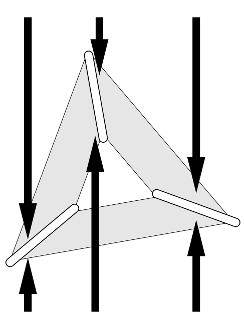

Figure 10: docking above versus below

Previously, $f(k)$ counted access into an internal clustervertex c of order k, where dockings were balanced on both sides of c. For imbalanced access, we derive a separate count of edge crossings entering c from above and from below where c is recursively defined by $C_3(k)$ and $k = 3^j$. In the base cases, $n = 3$, no crossings occur above and a single crossing occurs below. Thus, we define $f_{top}(3) = 0$ and $f_{bot}(3) = 1$. Assume the triangles are arranged recursively to point upwards. We count crossings as follows. Assume $k = n/3$. If the new point is positioned above the clustervertex, $3 \cdot f_{top}(k)$ edges are crossed recursively and $3 \cdot e(k,k,1)$ are crossed at the top-level. If the new point is positioned below the clustervertex, then k^3 additional crossings occur (see Figure 10). Thus, we derive the following recurrences:

$$f_{top}(n) = 3[f_{top}(k) + e(k,k,1)] = \frac{n^3}{16} - \frac{n^2}{4} + \frac{3n}{16} \tag{16}$$

$$f_{bot}(n) = 3[f_{bot}(k) + e(k,k,1)] + k^3 = \frac{5n^3}{48} - \frac{n^2}{4} + \frac{7n}{48}\ . \tag{17}$$

As expected, $f(n) = f_{top}(n) + f_{bot}(n)$. The difference between $f_{top}(n)$ and $f_{bot}(n)$ is significant as is shown by,

$$\lim_{n\to\infty} \frac{f_{top}(n)}{f_{bot}(n)} = \frac{3}{5} \tag{18}$$

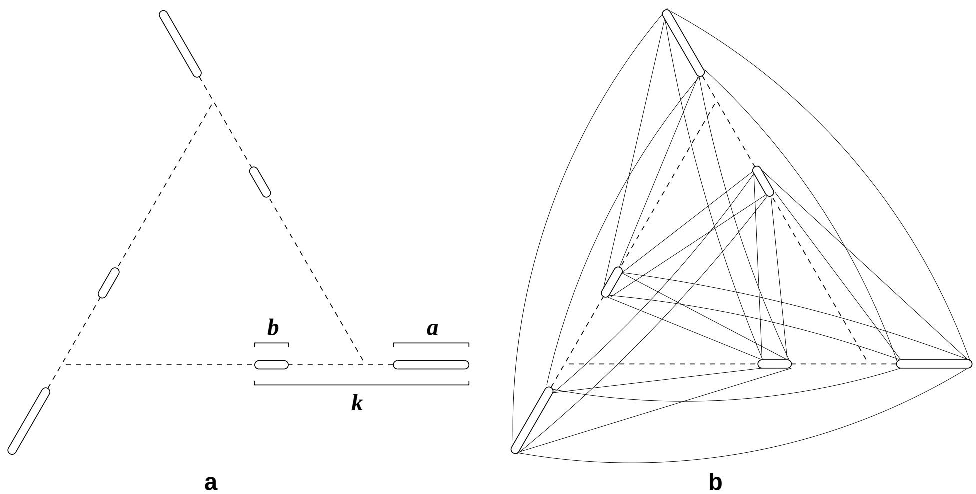

Figure 11: Clustervertices are not actually broken, only translated; they are drawn as two parts for counting. Clusteredges are drawn as arcs to reduce clutter.

Sliding a clustervertex creates new crossings at the merging of two clusteredges and at the crossing of new clusteredges (see Figure 11b). We count the cost of sliding one, two, or three clustervertices. These counts are given by $C_{s1}(n,a)$, $C_{s2}(n,a)$, and $C_{s3}(n,a)$, respectively. For each, a represents the portion of the affected clustervertex that still docks on the same side of incident clustervertices. a is defined in terms of n. When more than one clustervertex is moved, both or all three being moved are moved by the same amount.

	graph	internal	top-level	total	minimizing a_0
Singer[Sin71]	$C_3(n)$	0.0142	0.3704	0.3846	
Brodsky-Durocher-Gethner	$C_m(n,a)$	0.0370	0.3580	0.3951	$a_0 = 5n/18 + 1/3$
Brodsky-Durocher-Gethner	$C_{s1}(n,a)$	0.0142	0.3701	0.3843	$a_0 = 23n/72 - 1/24$
Brodsky-Durocher-Gethner	$C_{s2}(n,a)$	0.0142	0.3699	0.3841	$a_0 = 23n/72 - 1/24$
Brodsky-Durocher-Gethner	$C_{s3}(n,a)$	0.0142	0.3696	0.3838	$a_0 = 23n/72 - 1/24$

Table 2: asymptotic improvements on $C_3(n)$

Using a counting argument identical to that for $C_m(n,a)$, we derive the following:

$$C_{s1}(n,a) = \frac{137}{6318}n^4 - \frac{23}{648}n^3a + \frac{1}{18}n^2a^2 - \frac{31}{216}n^3 + \frac{1}{9}n^2a - \frac{1}{6}na^2 + \frac{8}{27}n^2 - \frac{1}{72}na - \frac{19}{104}n, \tag{19}$$

$$C_{s2}(n,a) = \frac{691}{25272}n^4 - \frac{23}{324}n^3a + \frac{1}{9}n^2a^2 - \frac{35}{216}n^3 + \frac{2}{9}n^2a - \frac{1}{3}na^2 + \frac{65}{216}n^2 - \frac{1}{36}na - \frac{19}{104}n, \tag{20}$$

$$C_{s3}(n,a) = \frac{139}{4212}n^4 - \frac{23}{216}n^3a + \frac{1}{6}n^2a^2 - \frac{13}{72}n^3 + \frac{1}{3}n^2a - \frac{1}{2}na^2 + \frac{11}{36}n^2 - \frac{1}{24}na - \frac{19}{104}n. \tag{21}$$

Again, each count is quadratic with respect to a and each is minimized when $a_0 = 23n/72 - 1/24$. The value a represents the number of vertices in a clustervertex that dock on the bottom of the clustervertex on its (counter-clockwise) right side. Thus, we require a to be an integer. One observes, however, that $a_0 = 23n/72 - 1/24$ is never an integer for $n = 3^i$, but an induction argument shows that $\lceil 23n/72 - 1/24 \rceil$ is the integer nearest a_0. Let $a_1(j) = 3^j \cdot 23/72 - 1/24$ and let $a_2(j) = \lceil 3^j \cdot 23/72 - 1/24 \rceil$. Asymptotically, $C_{s3}(n,a)$ remains unaffected since

$$\forall \epsilon > 0,\ \exists i \in \mathbf{Z}\ s.t.\ \forall j > i \ \left| \frac{C_{s3}(3^j, a_1(j))}{\binom{3^j}{4}} - \frac{C_{s3}(3^j, a_2(j))}{\binom{3^j}{4}} \right| < \epsilon \ . \tag{22}$$

To obtain the number of edge crossings for a given $n = 3^i$ and $a_0 = 23n/72 - 1/24$, simply evaluate $C_{s3}(n, \lceil a_0 \rceil)$. Thus

$$\overline{cr}(K_n) \leq C_{s3}(n, \lceil 23n/72 - 1/24 \rceil) \ . \tag{23}$$

Asymptotically, this value approaches $C_{s3}(n, a_0)$, which gives

$$C_{s3}(n, a_0) = \frac{6467}{404352} n^4 - \frac{1297}{10368} n^3 + \frac{1009}{3456} n^2 - \frac{2723}{14976} n \ . \tag{24}$$

A similar argument holds for $C_{s1}(n,a)$ and $C_{s2}(n,a)$. Thus, we derive the following limits:

$$\lim_{n\to\infty} \frac{C_{s1}(n,a_0)}{\binom{n}{4}} = \frac{19427}{50544} \approx 0.3846, \tag{25}$$

$$\lim_{n\to\infty} \frac{C_{s2}(n,a_0)}{\binom{n}{4}} = \frac{9707}{25272} \approx 0.3841, \tag{26}$$

$$\lim_{n\to\infty} \frac{C_{s3}(n,a_0)}{\binom{n}{4}} = \frac{6467}{16848} \approx 0.3838 \ . \tag{27}$$

3.3 Generalized Upper Bounds

Theorem 1

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leq \frac{6467}{16848} \approx 0.3838 \tag{28}$$

Proof. Scheinerman and Wilf show that $\overline{cr}(K_n)/\binom{n}{4}$ is a nondecreasing function [SW94]. We know $\overline{cr}(K_n) \leq C_{s3}(n,a_0)$ for all $n = 3^i$. Therefore,

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leq \lim_{n\to\infty} \frac{C_{s3}(n,a_0)}{\binom{n}{4}} = \frac{6467}{16848} \ . \blacksquare \tag{29}$$

As we did for $C_3(n)$, our construction for $C_{s3}(n,a)$ can be generalized by dividing n into three partitions of sizes p_1, p_2, and p_3 such that $\max_{i,j} |p_i - p_j| \leq 1$. Each partition then forms a clustervertex defined recursively by $C_{3g}(p_i)$. Clustervertices are translated by an appropriate a_i that is the integer nearest $23p_i/72 - 1/24$. We conjecture that such constructions produce asymptotics close to those achieved in Theorem 1.

We also mention recent work on a new lower bound in equation (29) based on work accomplished in [BDG00a]. That is, $\overline{cr}(K_{10}) = 62$, from which it follows that $.3001 \leq \lim_{n\to\infty} \frac{C_{s3}(n,a_0)}{\binom{n}{4}}$. In summary we have

$$.3001 \le \lim_{n\to\infty} \frac{C_{s3}(n,a_0)}{\binom{n}{4}} \le .3838. \tag{30}$$

3.4 Example: K_{81}

In Figure 12, we give two rectilinear drawings of K_{81}. The first drawing is based on Singer's construction [Sin71, Wil97] and has $625,320$ edge crossings. The second drawing[2] is based on the construction given by the strategy corresponding to $C_{s1}(81,26) = 624,852$.

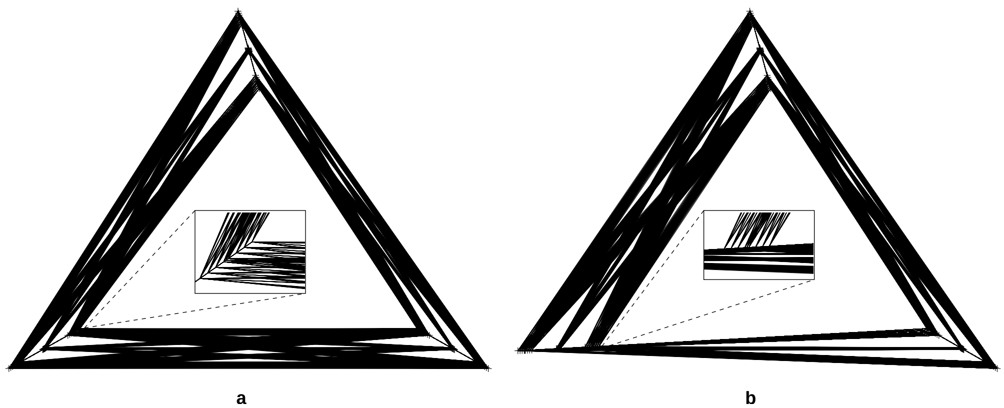

Figure 12: two instances of K_{81}

strategy	$C_{s3}(81,26)$	$C_{s2}(81,26)$	$C_{s1}(81,26)$	$C_3(81)$[Sin71]	[Jen71]	[Hay87]	$\binom{81}{4}$
count	$623,916$	$624,384$	$624,852$	$625,320$	$630,786$	$659,178$	1,663,740

Table 3: drawings of K_{81} that count

The largest number of edge crossings in a rectilinear drawing of K_{81} is $\binom{81}{4} = 1,663,740$ and occurs when all 81 vertices are placed on a convex hull. The fewest number of edge crossings of K_{81} known to date is $C_{s3}(81,26) = 623,916$.

4 Summary and Future Work

In summary, most forward progress toward determining $\overline{cr}(K_n)$ has been accomplished by producing a good rectilinear drawing of K_n for each n. A "good" rectilinear drawing of K_n has relatively few edge crossings and avails itself of an exact count of said crossings. Throughout the history of the problem, drawings that have produced the best asymptotic results amount to iteratively producing three clustervertices, which upon examination of the whole graph, yield a configuration of nested concentric triangles. Our best closed form and asymptotics arose from a break in tradition by

[2]These calculations were verified by an arbitrary precision edge-crossing counter.

yielding a graph with three clustervertices forming a set of nested triangles, but whose triangles are *not* pairwise concentric.

We offer the following open question: can one extend the technique given in Section 3 to produce a graph with more than three clustervertices that will yield better upper bounds and asymptotics for $\overline{cr}(K_n)$? Singer's rectilinear drawing of K_{10} with 62 edge crossings [Gar86, Sin71] was the first successful recorded instance of this break with tradition. Additionally, can the technique given in Section 3 be applied successfully to other families of interesting graphs? See, for example, the work of Bienstock and Dean [BD93, BD92].

Our second open question is based on the current rapidly changing status of computing, which makes feasible the use of brute-force techniques in extracting information about small graphs. In particular, it is possible to determine the exact value of $\overline{cr}(K_n)$ for small values of n beyond what is presently known [Guy72, WB78, BDG00a]. For example, a complete catalogue of inequivalent drawings is available through $n = 6$ for both rectilinear and non-rectilinear drawings of K_n [HT96, GH90]. As the catalogue grows, exact values for $\overline{cr}(K_n)$ will be found. The catalogue is being extended computationally by Applegate, Dash, Dean, and Cook [Dea00]. Additionally, Harris and Thorpe [TH96] have accomplished a randomized search and produced drawings of K_{12} and K_{13} with 155 and 229 edge crossings respectively. Both drawings have fewer edge crossings than the drawings given by Jensen [Jen71]. Our question is the following: how many inequivalent drawings of K_n produce a number of edge crossings equal to $\overline{cr}(K_n)$? Experimental work leads us to believe that the answer to this question is nontrivial. As more concrete information becomes available, we will be better able to investigate this question. Lastly, we note that Brodsky, Durocher, and Gethner [BDG00a] have given a combinatorial proof that $\overline{cr}(K_{10}) = 62$. We know of only one drawing of K_{10} with 62 edge crossings.

Our third and final open question concerns a problem addressed by Hayward [Hay87] and Newborn and Moser [NM80] and is the following: find a rectilinear drawing of K_n that produces the largest possible number of crossing-free Hamiltonian cycles. Hayward, building on the work in [NM80] has asymptotics based on a generalized rectilinear drawing of K_n, as mentioned in Section 3, Table 1. Our construction given in Section 3 improves Hawyard's result [BDG00b]. A related open problem is: does some rectilinear drawing of K_n with the minimum number of edge crossings necessarily produce the optimal number of crossing-free Hamiltonian cycles? Hayward conjectures that the answer is "yes," as do we, but as of yet, no proof is known.

Crossing number problems are rich and numerous with much work to be done. For an excellent exposition of further diverse open questions, see [PT00].

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