The Rectilinear Crossing Number of K_{10} is 62

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"Oh what a tangled web we weave..."
Sir Walter Scott

Abstract

A drawing of a graph G in the plane is said to be a rectilinear drawing of G if the edges are required to be line segments (as opposed to Jordan curves). We assume no three vertices are collinear. The rectilinear crossing number of G is the fewest number of edge crossings attainable over all rectilinear drawings of G. Thanks to Richard Guy, exact values of the rectilinear crossing number of K_n, the complete graph on n vertices, for $n = 3, \ldots, 9$, are known [Guy72, WB78, Fin00, Slo00]. Since 1971, thanks to the work of David Singer [Sin71, Gar86], the rectilinear crossing number of K_{10} has been known to be either 61 or 62, a deceptively innocent and tantalizing statement. The difficulty of determining the correct value is evidenced by the fact that Singer's result has withstood the test of time. In this paper we use a purely combinatorial argument to show that the rectilinear crossing number of K_{10} is 62. Moreover, using this result, we improve an asymptotic lower bound for a related problem. Finally, we close with some new and old open questions that were provoked, in part, by the results of this paper, and by the tangled history of the problem itself.

1 Introduction and History

Mathematicians and Computer Scientists are well acquainted with the vast sea of crossing number problems, whose 1944 origin lies in a scene described by Paul Turán. The following delightful excerpt, taken from [Guy69], has appeared numerous times in the literature over the years, and is now known as "Turán's brick factory problem."

> [sic.]*In 1944 our labor cambattation had the extreme luck to work—thanks to some very rich comrades—in a brick factory near Budapest. Our work was to bring out bricks from the ovens where they were made and carry them on small vehicles which run on rails in some of several open stores which happened to be empty. Since one could never be sure which store will be available, each oven was connected by rail with each store. Since we had to settle a fixed amount of loaded cars daily it was our interest to finish it as soon as possible. After being loaded in the (rather warm) ovens the vehicles run smoothly with not much effort; the only trouble arose at the crossing of two rails. Here the cars jumped out, the bricks fell down; a lot of extra work and loss of time arose. Having this experience a number of times it occurred to me why on earth did they build the rail system so uneconomically; minimizing the number of crossings the production could be made much more economical.*

And thus the crossing number of a graph was born. The original concept of the crossing number of the complete bipartite graph $K_{m,n}$, as inspired by the previous quotation, was addressed by Kövari, Sós, and Turán in [KST54]. Following suit, Guy [Guy60] initiated the hunt for the crossing number of K_n.

Precisely,

Definition 1.1 *Let* G *be a graph drawn in the plane such that the edges of* G *are Jordan curves, no three vertices are collinear, no vertex is contained in the interior of any edge, and no three edges may intersect*

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in a point, unless the point is a vertex. The **crossing number of G**, *denoted* **cr(G)**, *is the minimum number of edge crossings attainable over all drawings of* G *in the plane. A drawing of* G *that achieves the minimum number of edges crossings is called* **optimal**.

In this paper we are interested in drawings of graphs in the plane in which the edges are line segments.

Definition 1.2 *Let* G *be a graph drawn in the plane with the requirement that the edges are line segments, no three vertices are collinear, and no three edges may intersect in a point, unless the point is a vertex. Such a drawing is said to be a* **rectilinear drawing of G**. *The* **rectilinear crossing number of G**, *denoted* $\overline{\mathbf{cr}}(\mathbf{G})$, *is the fewest number of edge crossings attainable over all rectilinear drawings of* G. *Any such a drawing is called* **optimal**.

1.1 A Few General Results

We mention a small variety of papers on crossing numbers problems for graphs drawn in the plane that merely hint at the proliferation of available (and unavailable!) results. Other important results will be highlighted in Section 6.

Garey and Johnson [GJ83] showed that the problem of determining the crossing number of an arbitrary graph is NP-complete. Leighton [Lei84] gave an application to VLSI design by demonstrating a relationship between the area required to design a chip whose circuit is given by the graph G and the rectilinear crossing number of G. Bienstock and Dean [BD93] produced an infinite family of graphs $\{G_m\}$ with $\mathrm{cr}(G_m) = 4$ for every m but for which $sup_m\{\overline{\mathrm{cr}}(G_m)\} = \infty$. Kleitman [Kle70, Kle76] completed the very difficult task of determining the exact value of $\mathrm{cr}(K_{5,n})$ for any $n \in \mathbb{Z}^+$. Finally, a crucial method of attack for both rectilinear crossing number and crossing number problems has been that of determining the parity (i.e., whether the crossing number is even or odd). See, for example, [Har76, Kle70, Kle76, AR88, HT96].

Crossing number problems are inherently rich and numerous, and have captured the attention of a diverse community of researchers. For a nice exposition of current open questions as well as a plethora of references, see the recent paper of Pach and Tóth [PT00].

1.2 Closer to Home: $\overline{\mathrm{cr}}(K_n)$

Many papers, dating back as far as 1954 [KST54], have addressed the specific problem of determining $\mathrm{cr}(K_{m,n})$ and $\mathrm{cr}(K_n)$. For a nice overview see Richter and Thomassen [RT97]. For those who are tempted by some of the problems mentioned in this paper, it is imperative to read [Guy69] for corrections and retractions in the literature.

Our present interest is that of finding $\overline{\mathrm{cr}}(K_n)$ whose notion was first introduced by Harary and Hill [HH63]. As promised in the abstract, the small values of $\overline{\mathrm{cr}}(K_n)$ are known through $n = 9$, which can be found in [Guy72, WB78, Fin00] and [Slo00, sequence A014540]; see Table 1. Ultimately, the $n = 10$ entry [Sin71, Gar86] will be the focus of this paper.

K_n	$\overline{\mathrm{cr}}(K_n)$
K_3	0
K_4	0
K_5	1
K_6	3
K_7	9
K_8	19
K_9	36
K_{10}	*61 or 62*

Table 1: $\overline{\mathrm{cr}}(K_n)$

Asymptotics have played an important role in deciphering some of the mysteries of $\overline{\mathrm{cr}}(K_n)$. To this end, it is well known (see for example [SW94]) that $\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}$ exists and is finite; let

$$\overline{\nu}^* = \lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}. \tag{1}$$

H.F. Jensen [Jen71] produced a specific rectilinear drawing of K_n for each n, which availed itself of a formula, denoted $j(n)$, for the exact number of edge crossings. In particular,

$$j(n) = \left\lfloor \frac{7n^4 - 56n^3 + 128n^2 + 48n\left\lfloor\frac{n-7}{3}\right\rfloor + 108}{432} \right\rfloor, \tag{2}$$

from which it follows that $\overline{\mathrm{cr}}(K_n) \leq j(n)$ and that $\overline{\nu}^* \leq .3\overline{8}$. Moreover, it follows from work in [Sin71] as communicated in [Wil97, BDG00] that

$$\frac{61}{210} = .290476 \leq \overline{\nu}^* \leq .3846. \tag{3}$$

In Section 4.1, for completeness of exposition we reproduce the argument in [Sin71] that $\overline{\mathrm{cr}}(K_{10}) > 60$, which is required to obtain the lower bound in equation (3).

In the recent past, Scheinerman and Wilf [SW94, Wil97, Fin00] have made an elegant connection between $\overline{\nu}^*$ and a variation on Sylvester's four point problem. In particular, let R be any open set in the plane with finite Lebesgue measure, and let $q(R)$ be the probability of choosing four points uniformly and independently at random in R such that all four points are on a convex hull. Finally, let $q_* = \inf_R\{q(R)\}$. Then it is shown that $q_* = \overline{\nu}^*$.

Most recently, Brodsky, Durocher, and Gethner [BDG00] have reduced the upper bound in equation (3) to .3838. In the present paper, as a corollary to our main result, that $\overline{\text{cr}}(K_{10}) = 62$, we increase the lower bound in equation (3) to approximately .30.

2 Outline of the proof that $\overline{\text{cr}}(K_{10}) = 62$

As mentioned in the abstract, the main purpose of this paper is to settle the question of whether $\overline{\text{cr}}(K_{10}) = 61$ or 62. Our conclusion, based on a combinatorial proof, is that $\overline{\text{cr}}(K_{10}) = 62$. The following statements, which will be verified in the next sections, constitute an outline of the proof. As might be expected, given the long history of the problem and its variants, there are many details of which we must keep careful track.

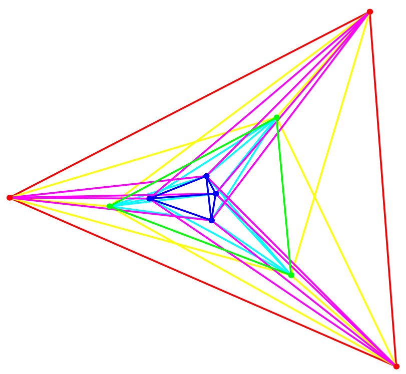


Figure 1: **The reader is invited to count the number of edge crossings in this optimal drawing of K_{10}.**

1. Any optimal rectilinear drawing of K_9 consists of three nested triangles: an outer, middle, and inner triangle. For purposes of both mnemonic and combinatorial considerations, we colour the vertices of the outer triangle *red*. Similarly, the vertices of the middle triangle will be coloured *green* and the vertices of the inner triangle will be coloured *blue*. For those who are accustomed to working with computers, the mnemonic is that the vertices of the outer, middle, and inner triangles correspond to *RGB*.

   Continuing in this vein, each of the edges of the K_9 drawing are coloured by way of the colour(s) of the two vertices on which they are incident. For example, an edge incident on a red vertex and a green vertex will naturally be coloured yellow. An edge incident on two red vertices (i.e., an edge of the outer triangle) will be coloured red, and so on. This step is done purely for purposes of visualization. For examples, see Figures 10, 11, and 12.

   Combinatorially, an edge crossing has a label identified by the four (not necessarily distinct) colours of the two associated edges, wx×yz, where w,x,y,z∈ $\{r, g, b\}$.

2. A drawing of K_{10} with 61 crossings must contain a drawing of K_9 with 36 crossings and must have a convex hull that is a triangle.

3. In any pair of nested triangles with all of the accompanying edges (i.e., a K_6), we exploit a combinatorially invariant: the subgraph induced by a single outer vertex together with the three vertices of the inner triangle is a K_4. There are exactly two rectilinear drawings of K_4. That is, the convex hull of rectilinear drawing of K_4 is either a triangle or a quadrilateral. If the former, since the drawing is rectilinear, there are no edge crossings. If the latter, there is exactly one edge crossing, namely that of the two inner diagonals.

4. With the above machinery in place, we enumerate the finitely many cases that naturally arise. In each case we find a lower bound for the number of edge crossings. In all cases, the result is at least 62.

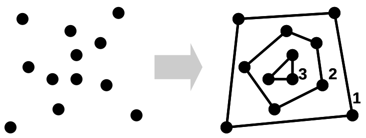
5. Singer [Sin71] produced a rectilinear drawing of K_{10} with 62 edge crossings, which is exhibited in [Gar86, p. 142]. This together with the work in step 4 implies that $\overline{\text{cr}}(K_{10}) = 62$; see Figure 1.

The remainder of this paper is devoted to the details of the outline just given, the improvement of the lower bound in equation (3), and finally, a list of open problems and future work.

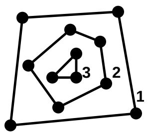
3 Edge Crossing Toolbox

3.1 Definitions

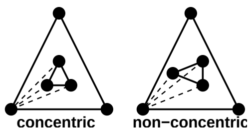
We assume that all drawings are in general position, i.e., no three vertices are collinear. A rectilinear drawing of a graph is decomposable into a set of convex **hulls**. The **first hull** of a drawing is the convex hull. The i**th hull** is the convex hull of the drawing of the subgraph strictly contained within the $(i-1)$st hull.



The **responsibility** of a vertex in a rectilinear drawing, defined in [Guy72], is the total number of crossings on all edges incident on the vertex.



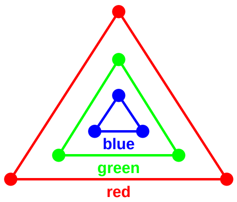
A **polygon of size** k is a rectilinear drawing of a non-crossing cycle on k vertices. A polygon is **contained** within another polygon if all the vertices of the former are strictly contained within the boundaries of the latter; the former is termed the **inner** polygon and the latter, the **outer** polygon. We say that n polygons are **nested** if the $(i+1)$st polygon is contained within the ith polygon for all $1 \leq i < n$. A triangle is a polygon of size three and every hull is a convex polygon.



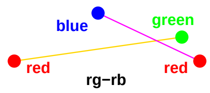
A rectilinear drawing of K_n is called a **nested triangle drawing** if any pair of hulls of the drawing are nested triangles.

Two polygons are **concentric** if one polygon contains the other polygon and any edge between the two polygons intersects neither the inner nor the outer polygon. Given two nested polygons, if the inner polygon is not a triangle then the two polygons a priori cannot be concentric. A crossing of two edges is called a **non-concentric crossing** if one of its edges is on the inner hull and the other has endpoints on the inner and outer hulls.

We know that the first hull of an optimal rectilinear drawing of K_9 must be a triangle [Guy72]. Furthermore, in Subsection 4.2 we will reproduce a theorem from [Sin71], that the outer two hulls of a rectilinear drawing of K_9 must be triangles.



For clarity, we colour the outer triangle red, the second triangle green, and the inner triangle blue. The vertices of a triangle take on the same colour as the triangle, and an edge between two vertices is labeled by a colour pair, e.g., red-blue (rb). A crossing of two edges is labeled by the colours of the comprising edges, e.g., red-blue×red-green (rb×rg). A crossing is called **2-coloured** if only two colours are involved in the crossing. This occurs when both edges are incident on the same two triangles, e.g., rg×rg, or when one of the edges belongs to the triangle that the other edge is incident on, e.g., rg×gg. A **3-coloured** crossing is one where the two edges that are involved are incident on three different triangles, e.g., rb×rg. A **4-coloured** crossing is defined similarly.



Crossings may be referred to by their full colour specification, the colours of an edge comprising the crossing, or the colour of a vertex comprising the crossing. For example, an rg×rb crossing is fully specified by the four colours, two per edge; the crossing is also a red-blue crossing and a red-green crossing because one of the edges is coloured red-blue and the other is coloured red-green. Since the edges of the crossing are incident on the red, green and blue vertices, the crossing may also be called red, green or blue; a rg×rg crossing is neither red-blue nor blue.

3.2 Configurations

Given a nested triangle drawing of K_6, a **kite** is a set of three edges radiating from a single vertex of the outer triangle to each of the vertices of the inner triangle. A kite comprises four vertices: the origin vertex, labeled o, from which the kite originates, and three internal vertices. The internal vertices are labeled in a clockwise order, with respect to the origin vertex, by the labels left (l), middle (m), and right (r); the angle $< lor$ must be acute. The kite also has three edges, two outer edges, (o, l) and

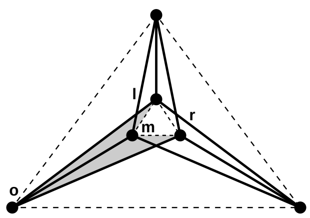


Figure 2: CCC

(o, r), and the inner edge (o, m). The origin vertex corresponds to the vertex on the outer triangle and the middle vertex is located within the sector defined by $< lor$; see Figure 2. A kite is called **concave** if m is contained within the triangle Δlor, see Figure 2, and is called **convex** if m is not contained in the triangle Δlor, see Figure 3. We shall denote a convex kite by V and a concave kite by C. A vertex is said to be inside a kite if it is within the convex hull of that kite, otherwise the vertex is said to be outside the kite.

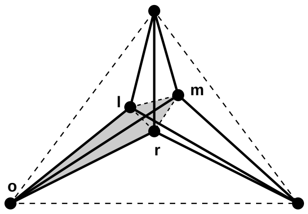


Figure 3: **VVV**

A **configuration** of kites is a set of three kites in a nested triangle drawing of K_6. Each kite originates from a different vertex of the outer triangle and is incident on the same inner triangle. There are four different configurations: CCC, CCV, CVV, and VVV, corresponding to the number of concave and convex kites in the drawing.

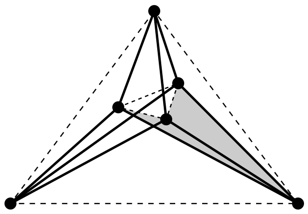


Figure 4: **CVV**

A configuration determines how many non-concentric crossings there are, i.e., the number of edges intersecting the inner triangle; CCC has zero, CCV has one, CVV has two, and VVV has three non-concentric edge crossings. A **sub-configuration** corresponds to the number of distinct middle vertices of concave kites; this can vary depending on whether the concave kites share the middle vertex.

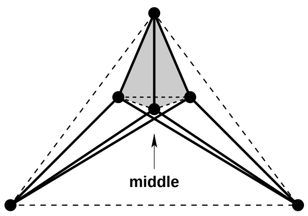


Figure 5: **Unary CCV**

Remark 3.1 *A CCV configuration is the only one that has more than one sub-configuration. A VVV configuration has no concave kites, a CVV configuration has only one concave kite, and in a CCC configuration no two kites share a middle vertex.*

In configuration CCC, Figure 2, there are three distinct middle vertices of concave kites, and in configuration VVV, Figure 3, there are zero because there are no concave kites. Configuration CVV, Figure 4, has only one middle vertex that belongs to a concave kite because it has only one concave kite.

The configuration CCV has two sub-configurations; the first, termed **unary**, has one middle vertex that is shared by both concave kites; see Figure 5. The second, termed **binary**, has two distinct middle vertices belonging to each of the concave kites; see Figure 6.

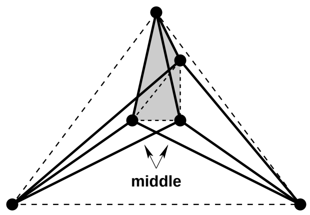


Figure 6: **Binary CCV**

Theorem 3.2 *A nested triangle drawing of K_6 belongs to one of the four configurations: CCC, CCV, CVV or VVV.*

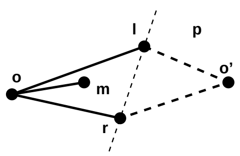
Proof: According to [Ros00] there are exactly two different rectilinear drawings of K_4, of which the convex hull is either a triangle or a quadrilateral. The former has no crossings and corresponds to the concave kite. The latter has one crossing and corresponds to the convex kite.

Since the drawing is comprised of nested triangles, a kite originates at each of the three outer vertices. Since the vertices are non-collinear, each of the kites is either convex or concave. The drawing can have, zero (CCC), one (CCV), two (CVV), or three (VVV) convex kites, with the rest being concave. ■

Lemma 3.3 *If m is a middle vertex of a concave kite in a nested triangle drawing of K_6, then m is contained within a quadrilateral composed of kite edges.*

Proof:

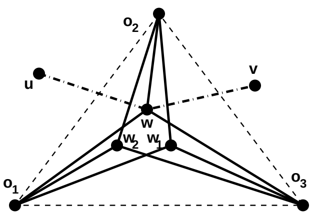
Let κ be a concave kite in a nested triangle drawing with the standard vertex labels o, l, m, and r. Since κ is concave, the middle vertex m is within the triangle Δlor. The vertices l and r determine a line that defines a half-plane p that does not contain κ. Since the vertices l, m, and r comprise the inner triangle of the drawing and must be contained within the outer triangle, there must be an outer triangle vertex located in the half-plane p. Denote this vertex by o' and note that a kite originates from it; hence, there are kite edges (o', l) and (o', r). Thus, m is contained within the quadrilateral (o, l, o', r). ■



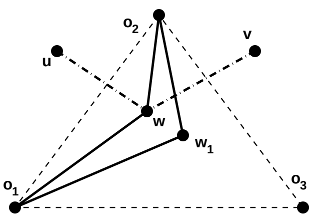
Corollary 3.4 *If m is a middle vertex of a concave kite in a nested triangle drawing of K_6 and an edge (v, m), originating outside the drawing, is incident on m, then the edge (v, m) must cross one of the kite edges.*

Remark 3.5 (Containment Argument) *Lemma 3.3 uses what will henceforth be referred to as the containment argument. Consider two vertices contained in a polygon. These vertices define a line that bisects the plane. In order for these vertices to be contained within the polygon, the two half-planes must each contain at least one vertex of the polygon. Similarly, if a vertex is contained inside two nested polygons and has edges incident on all vertices of the outer polygon, then at least two distinct edges of the inner polygon must be crossed by edges incident on the contained vertex.*

Lemma 3.6 (Barrier Lemma) *Let o_1, o_2, and o_3 be the outer vertices of a nested triangle drawing of K_6, let w be an inner vertex of the drawing, and let u and v be two additional vertices located outside the outer triangle of the drawing. If the edge (u, w) crosses (o_1, o_2) and the edge (v, w) crosses (o_2, o_3), then the total number of kite edge crossings contributed by (u, w) and (v, w) is at least two.*



Proof: If both edges (u, w) and (v, w) each cross at least one kite edge, then we are done. Without loss of generality, assume that (u, w) does not cross any kite edges. Let w_1 and w_2 be the other two inner vertices, and consider the path (o_1, w_1, o_2). Since edge (u, w) does not intersect the path, (o_1, w_1, o_2) creates a barrier on the other side of path (o_1, w, o_2). The same argument with edge (u, w) applies to path $(o_1 w_2 o_2)$, hence two barriers are present, forcing two crossings. ■



To deal with the unary CCV configuration, see Figures 5 and 7, we need to say something about the orientation of the kites. In a unary CCV configuration, the labels of the internal vertices of the two concave kites must match; given a label, left, middle, or right, and a vertex, it is impossible to distinguish one concave kite from the other. For example, the left vertex of one concave kite is also the left vertex of the other concave kite.

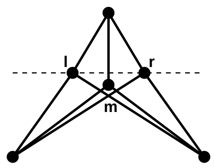


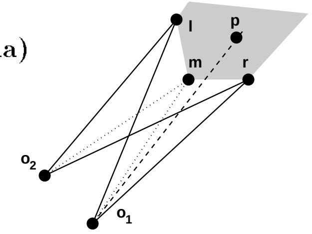
Figure 7: **Inside the unary CCV**

Lemma 3.7 *If a nested triangle drawing of K_6 is in a unary CCV configuration, then all three internal vertices of the two concave kites share the same labels.*

Proof: Since the two concave kites share the same middle vertex, there are two possible cases. Either the labels of the internal vertices match, in which case we are done. Otherwise, the left and right labels are interchanged. By way of contradiction, assume that they are interchanged; this implies that the kites are disjoint, i.e. do not overlap. Consequently, they cannot share the middle vertex that is inside both of the kites; this is contradiction. ■

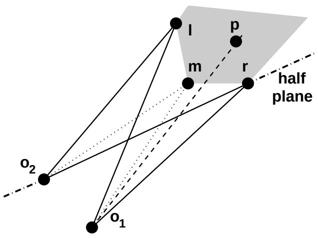
Lemma 3.7 implies that both concave kites are in the half-plane defined by their left and right vertices, which contains the shared middle vertex. Moreover, by the containment argument (Remark 3.5), the convex kite must be in the other half-plane. Furthermore, no two kites in a CCC configuration share a middle vertex.

Just like the Barrier Lemma, the Kite Lemma, CCC Lemma, and K_5 Principle Lemma, are general lemmas that are used to derive properties of specific drawings.



Lemma 3.8 (Kite Lemma) *Let $\kappa_1 = (o_1, l, m, r)$ and $\kappa_2 = (o_2, l, m, r)$ be two concave kites such that they share the same internal vertices, the internal vertices are labeled identically, and kite κ_2 does not contain vertex o_1 within it. Let A be the intersection of the sectors give by $<lo_1r$ and $<lmr$. If p is a vertex located in region A and is noncollinear with any other pair vertices, then the edge (o_1, p) must cross edge (o_2, l) or edge (o_2, r).*

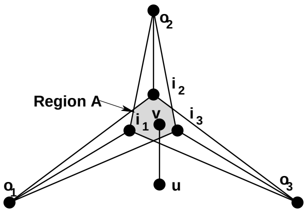
Proof: Either vertex o_2 is contained in kite κ_1 or not. If o_2 is inside κ_1, then, because kite κ_2 is concave, a barrier path (l, o_2, r) is created between vertex o_1 and vertex p. Hence, edge (o_1, p) must cross the path (l, o_2, r), intersecting one of the path's two edges.



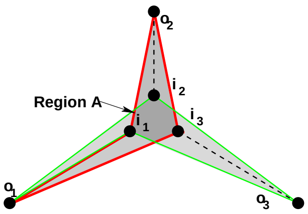
If vertex o_2 is not contained in kite κ_1, then assume, that vertex o_2 is on the left side of kite κ_1 (clockwise with respect to o_1). The edge (o_2, r) defines a half-plane that separates vertex p from vertex o_1. Furthermore, the segment defining the half-plane located within the sector $<lo_1r$ corresponds to part of the edge (o_2, r). Since the edge (o_1, p) must be within the sector $<lo_1r$, it must cross edge (o_2, r).

If vertex o_2 is on the right, by a similar argument, the edge (o_1, p) will cross edge (o_2, l). ■

Lemma 3.9 (CCC Lemma) *Given three kites in a CCC configuration, denote the internal vertices i_1, i_2, i_3, and outer vertices o_1, o_2, o_3 such that the middle vertex of a kite originating at o_j is i_j. Let A be the region defined by the intersection of sectors $< i_1o_2i_3$, $< i_2o_3i_1$, and $< i_3o_1i_2$. Let vertex u not be contained in any kite, let vertex v be located in region A, and assume that no three vertices are collinear. The edge (u, v) must cross at least two kite edges.*



Proof:



Using the kite edges we construct two polygons $(o_3, i_2, o_1, i_1, o_3)$ and $(o_2, i_3, o_1, i_1, o_2)$. Since both polygons contain region A and since the only shared edge, is a middle edge, edge (u, v) must cross into both polygons, contributing at least one kite edge crossing from each. ■

Lemma 3.10 (K_5 Principle) *Let a drawing of K_n have a triangular convex hull with the hull coloured red and $n - 3$ vertices contained within it coloured green. The drawing has exactly $\binom{n-3}{2}$ rg×rg edge crossings.*

Proof:
Select a pair of green vertices and remove all other green vertices from the drawing. This forms a K_5 with exactly one rg×rg edge crossing that is uniquely identified by the two green vertices. Since there are $\binom{n-3}{2}$ pairs of green vertices, there must be $\binom{n-3}{2}$ rg×rg edge crossings. ■

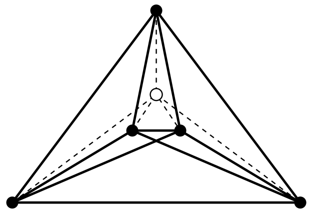


Figure 8: K_5 **principle**

4 The Proof

Using configurations to abstract the vertex positions in drawings we are now ready to combinatorially compute $\overline{\text{cr}}(K_9)$ and $\overline{\text{cr}}(K_{10})$. We first reproduce the results from [Sin71] and [Guy72] proving that $\overline{\text{cr}}(K_9) = 36$ and use these results to show that $\overline{\text{cr}}(K_{10}) = 62$.

The argument is as follows:

1. Since $\overline{\text{cr}}(K_{10}) \geq 61$, assume $\overline{\text{cr}}(K_{10}) = 61$.
2. If $\overline{\text{cr}}(K_{10}) = 61$ then the convex hull of an optimal of K_{10} must be a triangle.
3. If the convex hull of a drawing of K_{10} is triangular then that drawing has 62 or more crossings, contradiction.
4. Therefore, $\overline{\text{cr}}(K_{10}) \geq 62$

4.1 The Rectilinear Crossing Number of K_9

We know from [Sin71] and [Guy72] that the convex hull of an optimal rectilinear drawing of K_9 must be a triangle. By a counting argument in [Sin71], the drawing must be composed of three nested triangles, which we colour red, green, and blue. Furthermore, the same paper argues that the red and green triangles are pairwise concentric. We derive these results for completeness.

As mentioned in the introduction, the rectilinear crossing numbers of K_6 and K_9 are 3 and 36 respectively (see Table 1); we make use of these facts throughout the following proofs. We first reproduce a result from [Sin71] that states that an optimal rectilinear drawing of K_9 must comprise of three nested triangles.

Lemma 4.1 (Singer, [Sin71]) *An optimal rectilinear drawing of K_9 consists of three nested triangles.*

Proof: That the convex hull of an optimal rectilinear drawing of K_9 is a triangle has been shown in [Guy72] and [Sin71]. Using a counting technique similar to [Sin71], consider a drawing composed of a red triangle that contains a green convex quadrilateral that contains two blue vertices. By the K_5 principle there are $\binom{4}{2} = 6$ rg×rg crossings. At least two rg×gg crossings are present because a convex quadrilateral cannot be concentric with a triangle. Selecting one green and one blue vertex at a time and applying the K_5 principle yields, $4 \cdot 2 = 8$ rb×rg crossings. Six rb×gg crossings are due to the red-blue edges entering the green quadrilateral. Applying the K_5 principle to the blue vertices yields one rb×rb crossing. There are $2 + 4 = 6$ gb×gb and gb×gg crossings; the green quadrilateral is initially partitioned into four parts by one gg×gg crossing, adding the first blue vertex creates two gb×gg and adding the second vertex creates two more gb×gg crossings and two gb×gb crossings. This totals 30 crossings. An additional eight rb×gb and rb×gg crossings occur inside the green quadrilateral, four per blue vertex, totaling 38 crossings, which is greater than the optimal 36. By a similar argument any drawing whose second hull is not a triangle will also be non-optimal; see Appendix A. ■

Crossing	Count
rg×rg	6
rg×gg	2
rb×rg	8
rb×gg	6
rb×rb	1
gg×gg	1
gb×gg	4
gb×gb	2
rb×gb/gg	8
Total	38

Table 2: **Crossing contributions**

Lemmas 4.2, 4.3, 4.4, 4.5, and 4.6, count the number of different crossings in an optimal drawing of K_9, making use of the nested triangle property of Lemma 4.1.

Lemma 4.2 *A rectilinear drawing of K_9 comprising of nested triangles has a minimum of three 2-coloured crossings of red-green, red-blue, and green-blue.*

Proof: Select two of the three red, green, and blue triangles. These two triangles form a nested triangle drawing of K_6 with three 2-colour crossings. Hence, there are three 2-colour edges of each type. ■

Lemma 4.3 *A rectilinear drawing of K_6 comprising of nested non-concentric triangles has more than three crossings.*

Proof: Let the outer triangle be red and the inner green. By the K_5 Principle (Lemma 3.10) there are three rg×rg edge crossings. If the two triangles are non-concentric then there is at least one rg×gg crossing. ■

Lemma 4.4 *A rectilinear drawing of K_9 comprising of nested triangles has exactly nine rb×gg crossings.*

Proof: The red triangle contains the green triangle and the green triangle contains the blue triangle. Therefore, every red-blue edge must cross into the green triangle. Since there are nine red-blue edges, there are nine rb×gg crossings. ■

Lemma 4.5 *A rectilinear drawing of K_9 comprising of nested triangles has at least nine rb×rg crossings.*

Proof: The are three green and three blue vertices, thus there are nine unique green-blue pairs of vertices. By the K_5 principle, each pair contributes exactly one rg×rb crossing. Hence, a nested triangle drawing of K_9 has exactly nine rg×rb crossings. ■

We call a crossing **internal** if it is coloured either rb×gb or gb×bb. The set of internal crossings consists of all internal crossings in a drawing. Intuitively, all internal crossings take place within the green triangle. We call a red-blue kite **full** if it contains a green vertex; otherwise we call it **empty**. Intuitively a full red-blue kite contains a green-blue kite.

Lemma 4.6 *The number of internal crossings in a nested triangle drawing of K_9 is at least nine.*

Proof: We make use of the fact that the green and blue triangles form a K_6 and that any rectilinear drawing of K_6 falls into one of the five configurations: CCC, VVV, CVV, binary CCV, and unary CCV. The proof is by case analysis on the green-blue K_6 subdrawing. The green-blue K_6 is drawn in one of the five configurations:

CCC configuration: Since each of the blue vertices is a middle vertex of a concave kite, and all middle labels are distinct, by Corollary 3.4 each of the nine red-blue edge crosses one green-blue edge, hence there are nine rb×gb crossings.

VVV or CVV configuration: If the drawing is in a VVV configuration, by the Barrier Lemma

(Lemma 3.6) there are two rb×gb crossings per blue vertex. Adding the three gb×bb crossings yields nine. In the CVV configuration one of the blue vertices is responsible for at least three rb×gb crossings rather than two; adding the two gb×bb crossings yields the required result.

Binary CCV configuration: Note that two of the blue vertices are responsible for three rb×gb crossings, and the third vertex is responsible for two. Adding the single gb×bb crossing yields nine.

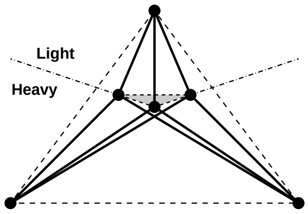
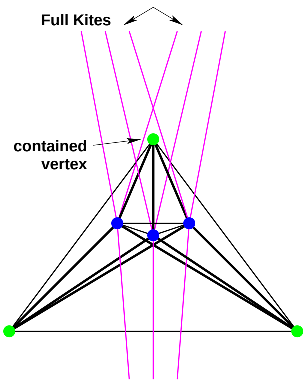


Figure 9: **Partition of Drawing**

Unary CCV configuration: In the case of the unary CCV configuration, the drawing is partitioned into a heavy and light part by extending the blue edges incident on the middle vertex of the convex kite; see Figure 9. A red-blue kite whose origin vertex is in the heavy side of the drawing is responsible for four or six rb×gb crossings while a red-blue kite originating in the light side of the partition is responsible for three crossings if it is empty, and one crossing if it is full; the six edge crossings occur if there is an empty red-blue kite between the two concave kites. In order for the green triangle to be nested within the red, by the containment argument, at least one of the red-blue kites must originate in the heavy partition. This implies that in order to get fewer than eight rb×gb crossings, two of the red-blue kites must be full and contain the green-blue kite in the light partition. This implies that the third red-blue kite must be an empty kite between the two concave red-green kites. Since this kite is responsible for six crossings, it follows that there are at least eight rb×gb crossings and therefore at least nine internal crossings. ■



Singer's Theorem [Sin71] follows from the previous lemmas. A stronger version of the theorem is given next.

Theorem 4.7 *An optimal rectilinear drawing of K_9 consists of three nested triangles. Furthermore, the red and green triangles, and the red and blue triangles are concentric.*

Proof: The first part of the statement is proven in [Guy72] and the counting argument in Lemma 4.1. Putting Corollary 4.2, Lemma 4.4, and Lemma 4.5 together accounts for 27 of the 36 crossings in an optimal drawing. Lemma 4.6 states that there are at least nine internal crossings. Since $\overline{\text{cr}}(K_9) = 36$, the number of rg×gg and rb×bb crossings must be zero; this implies concentricity. ■

Contribution	Count
Lemma 4.2	≥ 9
Lemma 4.4	9
Lemma 4.5	9
Lemma 4.6	≥ 9
Total	≥ 36

Table 3: **Lower bound**

Corollary 4.8 *An optimal rectilinear drawing of K_9 has at most nine rb×gb crossings, at most two gb×bb crossings and the total number of internal crossings is exactly nine.*

Proof: By Theorem 4.7, an optimal drawing of a K_9 has 36 crossings. Referring to Table 3, an optimal drawing has at least 27 non-internal edge crossings (Lemmas 4.2, 4.4, and 4.5). By Lemma 4.6, there are at least nine internal edge crossings and hence, an optimal drawing has exactly nine internal edge crossings.

Three gb×bb crossing occur if the green-blue K_6 part of the drawing has configuration VVV. However by a Barrier argument similar to Lemma 3.6 the configuration VVV creates nine rb×gb crossings plus three gb×bb crossings, which totals 12 internal crossings and cannot occur in an optimal drawing of K_9. Consequently at most two gb×bb crossings may occur. ■

4.1.1 Optimal K_9 Drawings

One is tempted to believe that an optimal drawing of K_9 is necessarily comprised of three nested triangles that are pairwise concentric. However, this belief is fallacious, as is shown in Figures 11 and 12.

4.2 The Rectilinear Crossing Number of K_{10}

We begin by reproducing a proof from [Sin71] that $\overline{\text{cr}}(K_{10}) > 60$. Since Singer [Sin71, Gar86] exhibited a 62 crossing rectilinear drawing of K_{10}, it follows that $61 \leq \overline{\text{cr}}(K_{10}) \leq 62$.

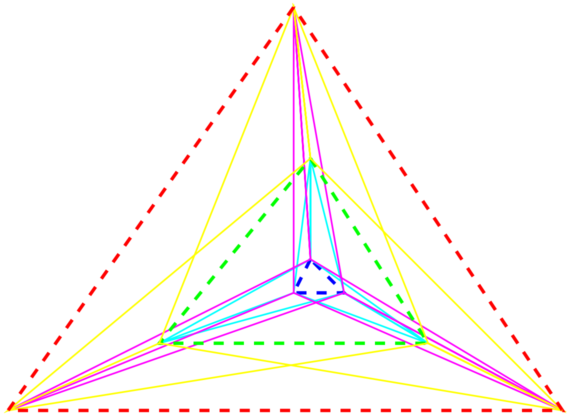


Figure 10: **Blue-Green CCC Drawing**

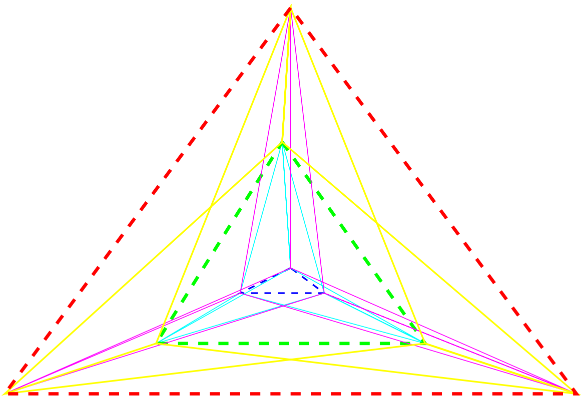


Figure 11: **Blue-Green CCV Drawing**

Theorem 4.9 (Singer, [Sin71]) $\overline{\text{cr}}(K_{10}) > 60$.

Proof: By way of contradiction, assume that there exists a rectilinear drawing of K_{10} with 60 crossings. Since each edge crossing comprises of four vertices, the sum of responsibilities of each vertex totals $4 \cdot 60$. Therefore, the average responsibility of each vertex is $\frac{4 \cdot 60}{10} = 24$. Furthermore, each vertex in the drawing is responsible for exactly 24 edge crossings. For if a vertex is responsible for more than 24 edge crossings, then removing the vertex from the drawing yields a drawing of K_9 with fewer than 36 edge crossings, which contradicts $\overline{\text{cr}}(K_9) = 36$. Similarly, if the drawing has a vertex that is responsible for fewer than 24 crossings, then by the averaging argument, there must be a vertex that is responsible for more than 24 crossings, leading to the same contradiction. Therefore, each vertex is responsible for 24 crossings. Thus, any drawing of K_{10} with 60 crossings contains an optimal drawing of K_9.

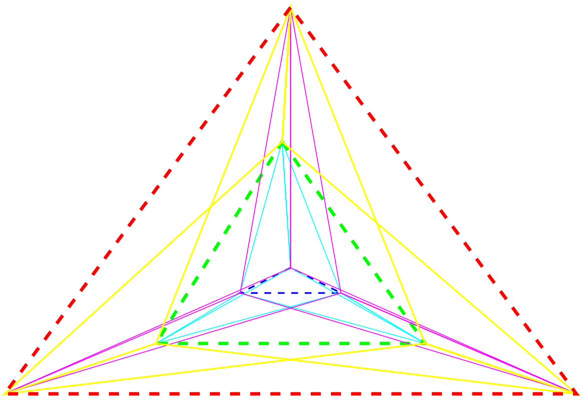


Figure 12: **Blue-Green CVV Drawing**

Starting with an optimal drawing of K_9 we try to place the tenth vertex. We have two choices; either place it such that one of the hulls of the K_{10} drawing is a convex quadrilateral or the drawing comprises of nested triangles with a vertex in the inner triangle. In the latter case, the edge connecting the tenth vertex to one of the outer triangle vertices must intersect an inner triangle edge. Removing the inner triangle vertex that is opposite the intersected edge creates a drawing of K_9 that fails the concentricity condition. Hence, the latter drawing will not be optimal. If the former situation arises there are two subcases. If the quadrilateral is the outer or the second hull, then removing an inner vertex creates a non-optimal K_9 drawing, which is a contradiction. If the innermost hull is a convex quadrilateral, then a priori it is not concentric with the outer triangle. Let b be the vertex such that there is an edge from it to a vertex in the outer triangle that intersects the quadrilateral. Remove a vertex from the quadrilateral that is antipodal to b. This creates a non-optimal K_9 drawing. The result follows. By an identical argument any rectilinear drawing of K_{10} cannot have fewer than 60 crossings. ∎

Next, we study drawings of K_{10} that have a nested triangle sub-drawing of K_9 coloured in the standard way. Let the tenth vertex be coloured white; the responsibility of the tenth vertex is the number of white crossings in the corresponding drawing of K_{10}. The following technical Lemma is needed in the proof of Theorem 4.14. This Lemma gives a lower bound on some of the white crossings that occur within the green triangle.

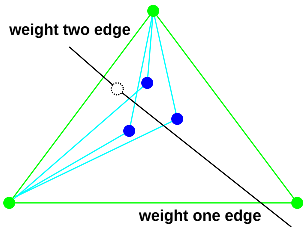
Lemma 4.10 *If a white vertex is added to a nested triangle drawing of K_9 such that it is contained in the green triangle, then at least six crossings must exist of the types rw×gb, rb×bw, gb×bw, and rg×gg.*

Proof: At least two of the red-white edges must cross into the green triangle on distinct green-green edges as a consequence of the nested triangle requirement and the containment argument. Select two of the three red-white edges such that they cross into the green triangle on distinct green-green edges and such that the total number of rw×gb crossings is minimized. Let c_1 and c_2 be the number of rw×gb crossings for which each of the two red-white edges is responsible and assume, without loss of generality, that $c_1 \leq c_2$. The lower bound on the total number of rw×gb crossings is $2c_1 + c_2$. We say that the red-white edge of lesser responsibility (c_1) has **weight two**, and we say that the other red-white edge, of responsibility c_2, has **weight one**.

Upon examining rw×gb crossings the proof falls into three main cases corresponding to the numbers of rw×gb crossings; if there are six or more rw×gb crossings then we are done. We consider the cases when the number of rw×gb crossings is $\{0, 1, 2\}$, $\{3\}$, and $\{4, 5\}$, the latter being the most challenging.

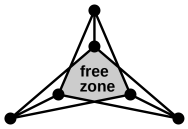
Case 1: 0, 1, or 2 rw×gb crossings
By the Barrier Lemma, every blue vertex forces at least one rw×gb crossing. Hence, there must be at least three rw×gb crossings.

Case 2: 3 rw×gb crossings
Considering only the rw×gb crossings, the configuration that minimizes the number of rw×gb crossings occurs when the red-white edge of weight two crosses zero green-blue edges and the red-white edge of weight one crosses three. However, we must consider blue-white edges also; by the Barrier principle one of the blue-white edges must cross at least two green-blue edges, and the other must cross at least one. This brings the total up to at least six.



Case 3: 4 or 5 rw×gb crossings
Assume there are at least four rw×gb crossings. If there are two or more gb×bw crossings then we are done. It remains to consider two subcases: that of zero or one gb×bw crossings.

Subcase 3.1: 0 gb×bw crossings
Assume there are zero gb×bw crossings. This case can only occur when no green-blue edge intersects the blue triangle, i.e., the green-blue kites are in a CCC configuration because there are no gb×bb crossings. The white vertex is in the green-blue free zone; a **free zone** consists of all regions of a nested triangle drawing of K_6 where a seventh vertex can be placed such that no kite edge blocks visibility of any inner vertices. Note that removal of the inner edges of all convex kites in a configuration creates a free zone. A free zone occurs naturally in a CCC configuration.



If there is a green-blue edge intersecting the blue triangle, then there exists a green-blue-green path between two of the blue vertices that forces at least one gb×bw crossing. Since the white vertex must be in the naturally occurring green-blue free zone, i.e., a green-blue CCC configuration, by the CCC Lemma (Lemma 3.9, this forces every red-white edge to generate at least two rw×gb crossings. This yields a total of at least six crossings.

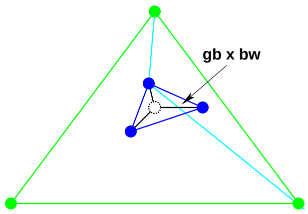


Figure 13: **The path**

Remark 4.11 *We reach a count of five crossings of the required type. The remainder of the proof is devoted to producing one more edge crossing of one of the required types.*

Subcase 3.2: 1 gb×bw crossing
We now consider the rb×bw crossings. Consider the red-blue kite configuration. Either the configuration is a CCC or not.

Subcase 3.2.1: Non-CCC red-blue configuration
Assume that the red-blue kite configuration is not in a CCC configuration. By the converse of the argument used in Subcase 3.1 there is at least one rb×bw crossing. Adding to the existing five yields at least six distinct crossings of the required type. This leaves only one case: the CCC red-blue configuration.

Subcase 3.2.2: CCC red-blue configuration
We now consider the five subcases corresponding to the distinct green-blue configurations within the red-blue CCC configuration.

Subcase 3.2.2.1: CCC green-blue configuration
If the green-blue kites are in a CCC configuration, then this case is covered by subcase 3.1.

Subcase 3.2.2.2: CVV and VVV green-blue configurations
For every green-blue edge that intersects the blue triangle, there is at least one gb×bw edge crossing; see Figure 13. Hence, if the green-blue kites are in a CVV or a VVV configuration then we have at least two gb×bw crossings. This sums to at least six crossings.

Subcase 3.2.2.3: Unary CCV green-blue configuration
If the green-blue configuration is a unary CCV configuration then the red and green triangles are not concentric; therefore, there is at least one rg×gg crossing. Adding at least four rb×bw crossings, and at least one gb×bw crossing, by the same argument as in subcase 3.2.2.2, yields at least six crossings.

Subcase 3.2.2.4: Binary CCV green-blue configurations
We are now left with the case of a CCC red-blue kite configuration and a binary CCV green-blue kite configuration with the white vertex either inside the red-blue free zone or not.

If the white vertex is not inside the red-blue free zone then there is at least one rb×bw crossing, by the same argument used in subcase 3.1, plus at least one gb×bw crossing, by the same argument as in subcase 3.2.2.2, plus at least four rw×gb crossings. The sum of these crossings is at least six.

Thus, assume that the white vertex is in the red-blue free zone. We will argue that there must always be either at least five rw×gb crossings plus at least one gb×bw crossing, or at least four rw×gb crossings plus at least two gb×bw crossings.

Consider the drawing minus the single green-blue edge in the only convex green-blue kite, i.e., the inner edge of the convex kite. This creates a green-blue free zone, inside of which there are no gb×bw edge crossings.

Remark 4.12 *In order to cross into the green-blue free zone, a red-white edge must cross a green-blue edge. Furthermore, if a green-blue kite and a red-blue kite are both concave, and have their internal (blue) vertices labeled identically, then we may invoke Lemma 3.8 (Kite Lemma). That is, the red-white edge, incident on the origin vertex (red) of the red-blue kite, must cross into the concave green-blue kite before crossing into the free zone. This produces an additional rw×gb crossing.*

The white vertex is either inside the green-blue free zone or not.

If the white vertex is inside the green-blue free zone, then the red-blue CCC configuration together with the pigeon-hole principle implies that we can match up a concave red-blue kite with each of the two concave blue-green kites. By remark 4.12, each of these match-ups contribute at least two rw×gb crossings, and the third red-white edge contributes at least one rw×gb crossing. Thus, if the white vertex is in the green-blue free zone there are five rw×gb crossings. By the argument used in subcase 3.2.2.2, the single convex green-blue kite contributes to at least one gb×bw crossing. Thus we get at least six crossings.

If the white vertex is outside the green-blue free zone, then we get at least one gb×bw crossing by the same argument used in subcase 3.1 and at least one gb×bw crossing by the same argument used in subcase 3.2.2.2. Since we have at least four rw×gb crossings (case 3), we get a grand total of at least six crossings.

In all possible cases that can occur we have shown that the number of crossings of the required type is at least six. ■

Lemma 4.1 imposed a nested triangle requirement on any optimal rectilinear drawing of K_9. The following lemma imposes a similar constraint on optimal rectilinear drawings of K_{10}.

Lemma 4.13 *If $\overline{\mathrm{cr}}(K_{10}) = 61$ then the first two hulls of an optimal rectilinear drawing of K_{10} must be triangles.*

Proof: By way of contradiction, assume that there exits an optimal rectilinear drawing of K_{10} whose convex hull is not a triangle and 61 edge crossings. By the same averaging argument used in Theorem 4.9, at least four of the vertices are responsible for 25 edge crossings; removing any of them yields an optimal drawing of K_9 with 36 crossings. If any of the vertices with responsibility 25 are not on the convex hull, then removing such a vertex yields a drawing of K_9 with a non-triangular convex hull, which is a contradiction. Therefore, all the vertices of responsibility 25 must be on the convex hull of the original drawing. Since we can always remove one of the four vertices such that the outer hull of the new drawing is not a triangle, this contradicts the original assumption. Hence, the first convex hull must be a triangle.

Assume that the second hull is not a triangle. Either the second hull is a convex quadrilateral or the second hull has more than four vertices; assume the latter. Since at least four of the vertices must have responsibility 25 and the outer hull is a triangle, at least one vertex of responsibility 25 must either belong to the second hull, or be contained within it. In either case, removing said vertex creates a drawing of K_9 that has 36 crossings and whose second hull is not a triangle. This is a contradiction.

Finally, assume that the second hull is a convex quadrilateral. If within the second hull there is a vertex of responsibility 24 or higher, removing said vertex creates a drawing of K_9 with 37 or fewer vertices. By Lemma 4.1 such a drawing should have at least 38 crossings, contradiction. Hence, assume that all three vertices inside the second hull have responsibility 23. Consequently, the remaining 7 vertices, must have responsibility 25. Since, the second hull is non-concentric with the first, by the same argument used in Theorem 4.9, we can always remove one of the vertices from the second hull such that the outer two hulls are non-concentric. This implies that we can create an optimal drawing of K_9 whose outer two hulls are non-concentric, a contradiction of Theorem 4.7.

Hence, the outer two hulls must be triangular. ■

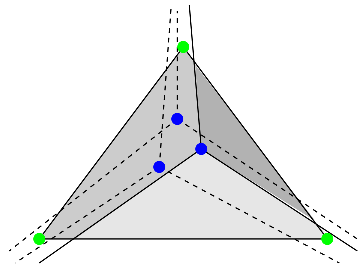
Theorem 4.14 *If $\overline{\text{cr}}(K_{10}) = 61$ then an optimal drawing of K_{10} will consist of two nested triangles containing a convex quadrilateral.*

Proof:

Crossing	Count
gw×bb	3
gw×gb	3
gw×rb	3
rw×gg	3
rw×bb	3
rw×rg	3
rw×rb	3
rw×gb	6
rb×bw	
gb×bw	
rg×gg	
Total	27

By Lemma 4.13 the outer two hulls of the optimal drawing K_{10} must be triangles. We must still account for the four internal vertices. If the four vertices form a convex quadrilateral then we are done; otherwise, assume the tenth vertex is inside the third nested triangle.

Colour the tenth vertex white. Now count the number of red-white and green-white edge crossings, starting with the green-white edge crossings. Each green-white edge must cross into the blue triangle; multiplying by three yields a total of three gw×bb crossings. By the K_5 principle there are three gw×gb crossings. Each blue vertex has three incident red-blue edges that partition the green triangle into three regions. The white vertex must be in one of the regions; by the Barrier argument there is at least one gw×rb crossing per blue vertex. The total of the green-white edge crossings sums to nine.



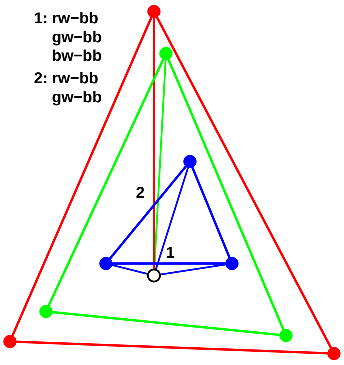
Each red-white edge must cross into both the green and blue triangles, totaling six edge crossings. By the K_5 principle, there are three rw×rg crossings and three rw-rb crossings. This gives an additional 12 crossings.

By Lemma 4.10 there are at least six additional crossings of the rw×gb, rb×bw, gb×bw and rg×gg type, of which at least three are rw×gb crossings.

Altogether, the number of white and rg×gg crossings is 27. Since $\overline{\text{cr}}(K_9) = 36$, the number of edge crossings in the drawing of K_{10} with the white vertex in the blue triangle is, $36 + 27 = 63 > 61$. ■

Theorem 4.15 $\overline{\text{cr}}(K_{10}) > 61$.

Proof: By way of contradiction assume that $\overline{\text{cr}}(K_{10}) = 61$. By Theorem 4.14 the inner hull must be a convex quadrilateral. Repeat the argument from Theorem 4.14 disregarding the rw×bb and gw×bb edge crossings (because there is no blue triangle). This gives us an initial count of $63 - 6 = 57$ edge crossings. Let the entire inner convex quadrilateral be coloured blue. Inside the quadrilateral there will be one bb×bb crossing (the diagonals). Furthermore, since the quadrilateral is neither concentric with the red triangle nor the green triangle, there will be a minimum of two rb×bb edge crossings and two gb×bb edge crossings. Summing the edge crossings yields $57 + 5 = 62 > 61$. ■



Theorem 4.16 $\overline{\text{cr}}(K_{10}) = 62$.

Proof: Singer's rectilinear drawing of K_{10} with 62 edge crossings [Sin71] is exhibited in [Gar86, p. 142], and hence $\overline{\text{cr}}(K_{10}) \leq 62$. By Theorem 4.15 $\overline{\text{cr}}(K_{10}) \geq 62$. The result follows. ■

An even stronger statement can be made. Just as in the case of K_9, the outer two hulls of an optimal rectilinear drawing of K_{10} must be triangles; see Appendix B. These properties could be useful, just as in the case of K_{10}, for determing the rectilinear crossing number of K_{11}.

Theorem 4.16 enables us to improve the lower bound in equations (1) and (3).

5 Asymptotic Lower Bounds

Given $\overline{\text{cr}}(K_a)$ for a fixed a, one can derive lower bounds for all $\overline{\text{cr}}(K_n)$, $n > a$. Any complete subgraph of a vertices drawn from a rectilinear drawing of K_n will include at least $\overline{\text{cr}}(K_a)$ crossings. There are $\binom{n}{a}$ complete subgraphs of size a. Each crossing consists of four vertices and each will be included in all other subgraphs containing the same four vertices. The number of such subgraphs that share four given vertices is $\binom{n-4}{a-4}$. Guy [Guy60], Richter and Thomassen [RT97], and Scheinerman and Wilf [SW94] each use this argument to show that

$$\overline{\text{cr}}(K_n) \geq \overline{\text{cr}}(K_a)\binom{n}{a} / \binom{n-4}{a-4} . \tag{4}$$

Scheinerman and Wilf [SW94] show that this can be rearranged to get

$$\frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} \geq \frac{\overline{\text{cr}}(K_a)}{\binom{a}{4}} . \tag{5}$$

Thus, one obtains a general lower bound for $\overline{\text{cr}}(K_n)$ from any known $\overline{\text{cr}}(K_a)$. Since $\overline{\text{cr}}(K_{10}) = 62$ and $\binom{10}{4} = 210$, one gets

$$\forall n \geq 10,\ \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} \geq \frac{62}{210} \approx 0.2952 . \tag{6}$$

This raises the lower bound for $\overline{\text{cr}}(K_{11})$ to 98. We conjecture $\overline{\text{cr}}(K_{11}) = 102$. Since crossing numbers are integers, each lower bound can be slightly increased by taking its ceiling. Thus,

$$\overline{\text{cr}}(K_n) \geq \left\lceil \overline{\text{cr}}(K_a)\binom{n}{a} / \binom{n-4}{a-4} \right\rceil . \tag{7}$$

If one sets $a = n - 1$, equation (7) gives a recursive definition whose recursive ceilings provide an improved lower bound for $\overline{\text{cr}}(K_n)$. For example, one finds that, $\overline{\text{cr}}(K_{400}) \geq 315356975$. This leads to a general lower bound of

$$\lim_{n\to\infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} \geq \frac{315356975}{\binom{400}{4}} = \frac{315356975}{1050739900} \approx 0.3001 . \tag{8}$$

As n increases, the limit converges. Whenever $\overline{\text{cr}}(K_{a'})$ is discovered for a new a', one can find an improved lower bound for a general $\overline{\text{cr}}(K_n)$, $n > a'$.

Consequently, $\overline{\text{cr}}(K_n)$ can be bound from below by using the technique describe here and from above by the drawing described by Brodsky, Durocher, and Gethner in [BDG00] to achieve the following lower and upper bounds:

$$0.3001 \approx \frac{315356975}{1050739900} \leq \lim_{n\to\infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} \leq \frac{6467}{16848} \approx 0.3838 . \tag{9}$$

6 Conclusion

6.1 Current and Future Work

The flavour of finding the crossing number of a graph, particularly in a rectilinear drawing, is similar to that of determining properties of line arrangements in the plane; this area is well known to be delicate and difficult. Therefore, one expects improvements to occur at a slow rate and specific instances of the problem for small graphs to be hard, though interesting.

An approach that has proved quite useful is to catalogue all inequivalent drawings of a given graph. With such a catalogue one can determine many specific properties of small graphs; see, for example, [GH90, HT96]. In particular, to find the crossing number or rectilinear crossing number of K_n and $K_{m,n}$, one can adopt a brute-force computational approach to find exact values of the crossing number for small graphs. Such an approach is currently underway for determining $\overline{\text{cr}}(K_n)$ by Applegate, Cook, Dash, and Dean [Dea00], where not only will they independently confirm that $\overline{\mathrm{cr}}(K_{10}) = 62$, but they will determine exact values of $\overline{\mathrm{cr}}(K_n)$ for other values of $n \geq 11$ as well.

In fact, when each new value of $\overline{\mathrm{cr}}(K_n)$ is found, the lower bounds in equation (3) and equation (7) will improve by way of the technique given in Section 5. For example, we have seen that $\overline{\mathrm{cr}}(K_{11}) \geq 98$. There exists a rectilinear drawing of K_{11} with 102 edge crossings [Jen71, SW94]; by [AR88], $\overline{\mathrm{cr}}(K_{11})$ is even. Therefore, $\overline{\mathrm{cr}}(K_{11}) \in \{98, 100, 102\}$. If $\overline{\mathrm{cr}}(K_{11}) = 100$ or 102 then the lower bound in equation (7) becomes .30544 or .31085 respectively. Similarly, the best drawing of K_{12} known to date has 156 edge crossings [Jen71]; if $\overline{\mathrm{cr}}(K_{12}) = 156$, then the lower bound reaches .31839.

Clearly, finding exact values for $\overline{\mathrm{cr}}(K_n)$ for any value of n will make relatively large improvements on the asymptotic lower bounds for the determination of $\overline{\nu}^*$.

6.2 Open Problems

We mention a small subset of open problems that arose from our investigations.

1. We know from [Guy72] that if $\mathrm{cr}(K_n) = \overline{\mathrm{cr}}(K_n)$ then the convex hull of any optimal rectilinear drawing of K_n is a triangle. Prove that the convex hull of any optimal rectilinear drawing of K_n is a triangle.

2. Given a rectilinear drawing of G, the **planar subdivision of** G is the graph obtained by adding vertices (and corresponding adjacencies) at each of the edge crossings of the particular drawing of G. Is the planar subdivision of any rectilinear drawing of K_n necessarily 3-connected? This question was also posed by Nate Dean.

3. Does there exist an optimal rectilinear drawing of K_n, for some n, such that it does not contain a sub-drawing that is an optimal rectilinear drawing of K_{n-1}? Furthermore, does there exist some n for which none of the optimal rectilinear drawings of K_n contain a sub-drawing that is an optimal rectilinear drawing of K_{n-1}?

4. Often an optimal rectilinear drawing of K_n is not unique. For a given n, how many optimal drawings of K_n are there? For what values of n is the optimal drawing unique?

5. Let G be an arbitrary graph. What is the complexity of determining $\overline{\mathrm{cr}}(G)$? Similarly, where in the complexity hierarchy does the determination of $\overline{\mathrm{cr}}(K_n)$ live? Recall that for a not-necessarily-rectilinear drawing of G, the general problem is known to be NP-complete [GJ83]. For some thoughts on such problems, see [Bie91].

6. We have seen that $\overline{\mathrm{cr}}(K_{11}) \in \{98, 100, 102\}$ and believe $\overline{\mathrm{cr}}(K_{11})$ to be 102. Give a combinatorial proof.

7. Finally, in the spirit of the present paper we feel compelled to mention the following problem, for which we sincerely apologize. Since 1970 it has been known that $\mathrm{cr}(K_{7,7}) \in \{77, 79, 81\}$ [Kle70]. What is the final answer?

7 Acknowledgments

Crossing number problems are easy to state but notoriously and profoundly difficult to solve. Throughout our investigations we discovered the inadequacy of simply searching the existing literature databases and subsequently reading papers. In particular, much of what is known or claimed to be known can only be discovered by communicating directly with those who are acquainted with the crossing number realm. For this reason, we thank Mike Albertson, Nate Dean, Richard Guy, Heiko Harborth, Joan Hutchinson, David Kirkpatrick, Jiří Matoušek, Nick Pippenger, David Singer, and Herb Wilf, all of whom shared their insights with us.

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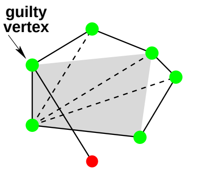
A Other K_9 Drawings

As before, colour the outer hull red, the second hull green, and the vertices inside the second hull blue.

Lemma A.1 *If the first hull of a rectilinear drawing of K_9 is a triangle, and the second hull has six vertices, then the drawing has more than 36 crossings.*

Proof: This drawing is coloured by only two colours: red and green. By the K_5 principle there are $\binom{6}{2} = 15$ rg×rg crossings. Since the six green vertices comprise the second hull, there are $\binom{6}{4} = 15$ gg×gg crossings. The 30 crossings counted so far include all except the rg×gg crossings.



We now consider the rg×gg crossings. Select four of the green vertices; these form a convex quadrilateral and at least one green vertex, the guilty vertex, has a red-green edge that intersects the quadrilateral. This edge partitions the green hull into two parts with one green vertex on one side of the green hull and three on the other, or two on each side. In the former case the red-green edge crosses three green-green edges that are incident on the single vertex. In the latter case, the red-green edge intersects four green-green edges that are incident on the two green vertices in one of the partitions. In both cases, there is an additional rg×gg crossing due to the red-green edge crossing an edge of the quadrilateral. Hence, at minimum four rg×gg crossings are due to the single red-green edge. Since there are at least three guilty vertices in a hull on six vertices. There must be at least 12 rg×gg crossings.

Therefore, the total number of crossings is at least $42 > 36$. ■

Lemma A.2 *If the first hull of a rectilinear drawing of K_9 is a triangle, and the second convex hull has five vertices, then the drawing has more than 36 crossings.*

Proof: As before, the single vertex inside the second hull is coloured blue. By the K_5 principle there are $\binom{5}{2} = 10$ rg×rg crossings and $\binom{5}{1} = 5$ rg×rb crossings. By the same argument used in the previous lemma there are $\binom{5}{4} = 5$ gg×gg crossings. There are at least five gb×gg crossings. Thus, we reach a count of 25 crossings without having considered the rb×gg, rg×gg, and rg×gb crossings.

We count the rg×gg, and rg×gb crossings by the guilty vertex argument used in the previous lemma. A hull on five vertices will have at least two guilty vertices. Each guilty vertex is responsible for at least three rg×gg crossings and, by the Barrier argument, at least one rg×gb crossing. This yields an additional eight crossings, bringing the total up to 33.

Finally, consider the rb×gg crossings. At least three occur from the red-blue edges having to cross into the green hull. By the containment argument, at least one of these three edges has to cross two of the green-green diagonals within the green hull. This brings up the total to at least five rb×gg crossings. Adding this to the running total yields $38 > 36$. ■

Crossing	Min
rg×rg	10
rg×rb	5
gg×gg	5
gb×gg	5
rg×gg	6
rg×gb	2
rb×gg	5
Total	38

Table 4: **Crossings**

B Outer Hulls of Optimal K_{10} Drawings

In this section we prove that the outer two hulls of optimal rectilinear drawings of K_{10} must be triangles. We first need three lemmas.

Lemma B.1 (Singer, [Sin71]) *If a rectilinear drawing of K_9 has a quadrilateral convex hull, then it has at least 40 crossings.*

Proof: We first count the number of rr×rg, rr×rb, rg×rg, rg×rb, rb×rb crossings. Placing green and blue vertices inside the red hull one at a time and using the same argument as in Lemma 4.1 reveals that the number of crossings is $2+4+6+8+10 = 30$. Additionally there is one rr×rr crossing. This gives us a sum of 31. We now consider the vertices inside the red hull. There are three cases: five green vertices and zero blue vertices, four green vertices and one blue vertex, or three green vertices and two blue vertices.

In the first case, there are $\binom{5}{4} = 5$ gg×gg crossings and, by the guilty argument of Lemma A.1 there are $2 \cdot 3 = 6$ rg×gg crossings. Adding these to our initial sum yields 42.

In the second case there is one gg×gg crossing, four rb×gg crossings, two gg×gb crossings and, by the guilty argument, two rg×gg crossings. The final sum is $31 + 9 = 40$.

In the last, there are $2 \cdot 4 = 8$ rb×gg crossings and, by the K_5 principle, one gb×gb crossing; this yields 40 crossings. ∎

Lemma B.2 *Let D be a drawing of K_{n+m} consisting of a red convex hull comprising of n vertices that contains m green vertices. The number of rr×rg crossings is at least*

$$\eta(n, m) = m \cdot \frac{n^3 - 4n^2 + (3 + (n + 1 \bmod 2))n}{8}$$

Proof: We first consider the lowest possible cost of a single green vertex inside the red convex hull and then multiply by m to get the bound for m vertices.

Select any red vertex v, and assign a clockwise order, $[1, n-1]$, on the red edges incident on it such that the first edge and the last edge are on the convex hull of D. With respect v, the ith edge in the order has $i-1$ red vertices to the left of it (counter-clockwise with respect to v) and $n-1-i$ red vertices on the right. A green vertex inside the red convex hull, will either be on the right side or the left side of the ith edge. If the green vertex is on the left side of the ith edge, the ith edge will be responsible for $n-1-i$ rr×rg crossings; if the green vertex is on the right side of the edge, then there are $i-1$ such crossings.

We assume that the green vertex is on the good side of each red edge, the one that yields the fewest crossings. Furthermore, the first half of the red edges incident on v always have their good side on the right and the second half of the red edges always have their good side on the left. Hence, for each red vertex v we need only consider the first half of the edges in the order because the rest will accounted for by other red vertices. Since every red edge is responsible for it's own crossings, we need only sum the edges and their optimal responsibility to get the lower bound. Furthermore, because we are assuming optimality, we can compute compute the optimal sum for only one of the red vertices and then multiply by the number of red vertices.

Since we are taking the first half of the edges in the order, $[1, \lfloor \frac{n}{2} \rfloor]$. The right side is always the good side and the responsibility is $i-1$. Hence, the sum for n odd is

$$n \cdot \sum_{i=1}^{\frac{n-1}{2}} (i-1) = \frac{n^3 - 4n^2 + 3n}{8}.$$

For n even we must be careful not to double-count the diagonals that bisect the convex hull, i.e., $i = \frac{n}{2}$. These diagonals have good sides on both left and right simultaneously. Hence, they are good on their right side with respect to two red vertices, but must only be counted once. Thus, for n even the sum is

$$\frac{n-2}{2} \cdot \frac{n}{2} + n \cdot \sum_{i=1}^{\frac{n}{2}-1} (i-1) = \frac{n^3 - 4n^2 + 4n}{8}.$$

Combining the two sums and multiplying by m yields the result. ∎

Lemma B.3 *Let D be a drawing of a K_{n+m} consisting of a red convex hull comprising of n vertices that contains m green vertices. The number of rg×rg crossings is at least*

$$\zeta(n, m) = (n-2)\binom{m}{2}.$$

Proof: For every pair of green vertices perform the following procedure. Select two red vertices.

Select a third red vertex from the set of unselected red vertices. By the K_5 principle these five vertices (two green and three red) are responsible for at least one unique rg×rg crossing. Select one of the red vertices that is responsible for a rg×rg crossing and discard it, i.e., the vertex may not be selected in this round. Thus, this crossing will never be double counted. We are left with two red and two blue vertices. Repeat until we run out red vertices.

This procedure will take exactly $n-2$ iterations and the procedure will be executed for $\binom{m}{2}$ pairs of green vertices. Hence, the total number of rg×rg crossings is at least $(n-2)\binom{m}{2}$. ∎

For convenience we define $\xi(n)$ to be the the crossing number of a drawing of K_n where all n vertices are on the convex hull of the drawing, i.e.,

$$\xi(n) = \binom{n}{4}.$$

Theorem B.4 *The convex hull of an optimal rectilinear drawing of K_{10} must be a triangle.*

Proof: Colour the convex hull red and the vertices contained inside green. If the convex hull has $n-m$ vertices on it, and m vertices inside, then the number of rr×rr crossings is $\xi(n-m)$. By Lemma B.2 the number of rr×rg crossings is at least $\eta(n-m, m)$ and by Lemma B.3, the number of rg×rg crossings is at least $\zeta(n-m, m)$. Finally the number of gg×gg crossings is at least $\overline{\text{cr}}(K_m)$. Summing these counts yields a lower bound of

$$\begin{aligned}\chi(n,m) &= \xi(n-m)+\eta(n-m,m)+ \\ &\quad \zeta(n-m,m)+\overline{\text{cr}}(K_m).\end{aligned}$$

Fixing n at 10, and letting m range from 0 to 4 yields crossing numbers that exceed 62. Hence, this leaves the case when the convex hull has five or four vertices.

In the case of $m = 5$, $\chi(10,5) = 61$. This does not account for the rg×gg crossings. If the K_5 sub-drawing inside the convex hull does not have convex hull of five vertices, then at least five rg×gg crossings occur. Otherwise, the green K_5 sub-drawing is responsible for an additional four gg×gg crossings. In either case, the total sum exceeds 62.

In the case of a quadrilateral convex hull, we use a different argument. By Lemma B.1 a rectilinear drawing of K_9 with a convex quadrilateral hull has at least 40 crossings. By the same argument as in Theorem 4.9, an optimal rectilinear drawing of K_{10} must have a vertex of responsibility 24 or greater inside the hull. Removing the vertex yields a drawing of K_9 with a quadrilateral convex hull and a crossing number of 38, contradicting Lemma B.1. ■

Theorem B.5 *The second hull of an optimal rectilinear drawing of K_{10} must be a triangle.*

Proof: There are two cases, either the second hull has four vertices, or more than four vertices. In the latter case, by the same argument as in Theorem 4.9 there must be a vertex of responsibility 25 or greater either inside or on the green hull. Removing said vertex yields a drawing of K_9 with fewer than 38 crossings but with a non-triangular second hull, contradicting one of Lemmas 4.1, A.1, or A.2.

Assume that the second hull is a quadrilateral. If there is a blue vertex with responsibility greater than 24 inside the green hull, then removing it yields a contradiction as above. Otherwise, all blue vertices have responsibility 24 or less. The rectilinear crossing number of any drawing of K_9 must be even [Guy72]. If all the vertices on the second hull have responsibility 26, then we can remove one, creating an optimal drawing of K_9 that does not satisfy the concentricity constraints. If not, one of the vertices on the second hull has responsibility fewer than 26. Therefore, all vertices on the outer hull have responsibility 24 or more. We can always remove a vertex from the outer hull such that the new drawing has a quadrilateral convex hull. This yields a drawing of K_9 with a quadrilateral convex hull and a crossing number fewer than 40, which contradicts Lemma B.1. ■