Random Interval Graphs

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Abstract: We consider models for random interval graphs that are based on stochastic service systems, with vertices corresponding to customers and edges corresponding to pairs of customers that are in the system simultaneously. The number N of vertices in a connected component thus corresponds to the number of customers arriving during a busy period, while the size K of the largest clique (which for interval graphs is equal to the chromatic number) corresponds to the maximum number of customers in the system during a busy period. We obtain the following results for both the $M/D/\infty$ and the $M/M/\infty$ models, with arrival rate λ per mean service time. The expected number of vertices is e^λ, and the distribution of the N/e^λ tends to an exponential distribution with mean 1 as λ tends to infinity. This implies that $\log N$ is very strongly concentrated about $\lambda - \gamma$ (where γ is Euler's constant), with variance just $\pi^2/6$. The size K of the largest clique is very strongly concentrated about $e\lambda$. Thus the ratio $K/\log N$ is strongly concentrated about e, in contrast with the situation for random graphs generated by unbiased coin flips, where $K/\log N$ is very strongly concentrated about $2/\log 2$.

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1. Introduction

Our goal in this paper is to study some models for random interval graphs. We are by no means the first to do this, but our approach is somewhat different from those taken in previous attempts. Thus we shall begin by describing our approach and some of our results, and afterward compare it with others.

Consider a stochastic service system, to which customers arrive according to some random process, are served for randomly distributed intervals of time, and then depart. We shall confine our attention to systems in which any number of customers may be served simultaneously (so that there is no queueing for service), but other service disciplines could be considered as well. Construct a graph by associating with each customer a vertex and joining with an edge each pair of vertices associated with customers that are in the system simultaneously. The resulting graph is infinite, but under the mild assumption that there is a recurrent state in which the system is empty, and from which the system begins its operation, this graph breaks into infinitely many independent and identically distributed connected components. We shall be interested in the probability distribution of such a component. Since each customer is in the system for a contiguous interval of time, these components are interval graphs, and we have thus defined a probability measure on the family of interval graphs.

We first consider the $M/D/\infty$ model, in which customers arrive according to a Poisson distribution with rate λ per unit time, and for which the service times are equal and taken as the unit of time. For this model, the number N of vertices is geometrically distributed with mean e^{λ}. The random variable N is not concentrated about its mean (the variance is $e^{2\lambda} - e^{\lambda}$, which grows as the square of the mean), but the distribution of N/e^{λ} tends to an exponential distribution with mean 1 as λ tends to infinity. From this it follows that $\log N$ is very strongly concentrated about its mean $\lambda - \gamma$ (where $\gamma = 0.557\ldots$ is Euler's constant), with variance just $\pi^2/6 = 1.6449\ldots$. We also show that the size K of the largest clique (which for an interval graph is equal to the chromatic number) is similarly strongly concentrated about $e\lambda$. This implies that the ratio $K/\log N$ tends with high probability to $e = 2.718\ldots$ as λ tends to infinity. This may be compared with the situation for the model $G_{n,1/2}$, where the ratio $K/\log n$ tends with high probability to $2/\log 2 = 2.885\ldots$ (see Grimmett and McDiarmid [G2]), but the chromatic number grows with high probability as n/K rather than K (see Bollobás [B2]).

We then turn our attention to the $M/M/\infty$ model, where customers again arrive according to a Poisson distribution with rate λ per unit time, but now service times are independent and exponentially distributed with mean taken as the unit of time. In this case the distribution of N is much more complicated (its generating function involves the ratio of two confluent hypergeometric functions), but the mean is again e^{λ} and we obtain exactly the same results concerning the limiting distributions of N/e^{λ} and $\log N$. We also obtain similar results for the distribution of K, which is again very strongly concentrated about $e\lambda$.

Interval graphs arise in a variety of situations and have an extensive literature (see, for example, Fishburn [F]). In many situation these graphs arise through some random process, so that it is desirable to have models for random interval graphs. The explicit study of models for random graphs begins with the work of Erdős and Rényi [E1, E2], who proposed several models (see also Bollobás [B1] and Palmer [P]). These models assign very little probability to interval graphs, however, and even when conditioned on the event that the outcome is an interval graph they give highly skewed distributions. The random graph $\mathbf{G}_{n,p}$, for example, for which the graph has n vertices and each edge is present independently with probability p, yields interval graphs with significant probability only when p is so small that with high probability all components are trees with at most six vertices, or so large that with high probability there is at most one edge missing. This has led to a search for natural models of random graphs that assign positive probabilities exclusively to interval graphs.

The first such model was proposed by Scheinerman [S1]. This model constructs a graph $\mathbf{G}_n^{(1)}$ by independently choosing $2n$ points $X_1, \ldots, X_{2n}$ uniformly distributed in $[0, 1]$, then taking n vertices corresponding to the intervals $I_k = [\min\{X_{2k-1}, X_{2k}\}, \max\{X_{2k-1}, X_{2k}\}]$. Since all $(2n)!$ orders of the points $X_1, \ldots, X_{2n}$ are equally likely, and since an interval graph is determined by the order of the endpoints, this is equivalent to choosing the points $X_1, \ldots, X_{2n}$ uniformly without replacement from $\{1, \ldots, 2n\}$. The result is a dense graph (the expected number of edges is $n(n-1)/3$) that is connected with high probability. These graphs have many interesting properties, but (as for the random graphs $G_{n,p}$ with p fixed), the fact that the number of edges grows as the square of the number of vertices makes them inappropriate as models for certain phenomena.

This circumstance led Scheinerman [S2] to propose a second model, in which a second parameter can be varied to produce a variety of interval graphs, ranging from very sparse to very dense. This model constructs a graph $\mathbf{G}_{n,r}^{(2)}$ by independently choosing n points $X_1, \ldots, X_n$ uniformly distributed in $[0, 1]$ and independently choosing n radii $R_1, \ldots, R_n$ uniformly distributed in $[0, r]$, then taking n vertices corresponding to the intervals $I_k = [X_k - R_k, X_k + R_k]$. Here the intervals have average length $2\mathrm{Ex}(R_k) = r$. Taking this length as the unit of time, we see that the intervals "arrive at the rate" nr per unit time.

Thus if $n \to \infty$ and $r \to 0$ in such a way that $nr \to \lambda$, we might expect $\mathbf{G}_{n,r}^{(2)}$ to behave like a finite portion of the graph corresponding to the stochastic service system $M/U/\infty$, where "U" indicated that service times are uniformly distributed in $[0, 2]$ (so that the mean service time is 1). Scheinerman shows (Theorem 4.3) that the number of components tends with high probability to n/e^{λ} in this limit, so that the average component size is e^{λ}. (The components in $\mathbf{G}_{n,r}^{(2)}$ are not identically distributed, since edge effects at the endpoints 0 and 1 affect the early and late components more than those in the middle.) While we do not have results for this system, the expected component size e^{λ} is the same for the models that we do study. Since $\mathbf{G}_{n,r}^{(2)}$ consists of a large number of small components in this limit, its maximum clique size with be the maximum of many random variable with small expectations. Thus while we find maximum clique sizes concentrated about the constant $e\lambda$ in our models, the maximum clique size for $\mathbf{G}_{n,r}^{(2)}$ grow logarithmically with n.

A somewhat different model was proposed by Godehardt and Jaworski [G1]. Their model constructs a graph $\mathbf{G}_{n,d}^{(3)}$ like $\mathbf{G}_{n,r}^{(2)}$, but with the radii $R_1, \ldots, R_n$ being deterministically set equal to $d/2$, rather than being uniformly distributed in $[0, r]$. Here the interval length is d. Thus if $n \to \infty$ and $d \to 0$ in such a way that $nd \to \lambda$, we might expect $\mathbf{G}_{n,d}^{(3)}$ to behave like a finite portion of the graph corresponding to the system $M/D/\infty$. (Again, edge effects will prevent the correspondence from being exact.) Godehardt and Jaworski also obtain an expected number of components n/e^{λ}, corresponding to the expected component size e^{λ} in our analysis of the system $M/D/\infty$.

The use of the $M/D/\infty$ system as a basis for generating random interval graphs may be criticized on the grounds that all of the intervals in the resulting representation are of equal length, so that the generated graph is in fact an indifference graph (see Roberts [R]), also known as a unit interval graph. This criticism does not apply to the use of the $M/M/\infty$ system, and this is one of our motives for studying this case. One may ask, of course, whether there is a statistically significant difference difference between these two models. We shall show in the Appendix that there is: the probability that an interval graph arising from the $M/M/\infty$ system is an indifference graph tends to zero as λ tends to infinity.

All of our results can be interpreted in terms of the busy periods of stochastic service systems, and these busy periods have of course also been studied before. The first such study was that of Takács [T], who was interested primarily in the length of the busy period in time. The number N of customers arriving in a busy period was first studied by Kingman [K], who gives formulas that allow the distribution of N to be calculated for the $M/G/\infty$ system (with a general service time distribution). We have not succeed, however, in extracting the information we seek from these formulas in the case of the $M/M/\infty$ system, and in the case of the $M/D/\infty$ system the results are more easily obtained by more direct methods. The maximum number K of customers during a busy period has not to our knowledge been studied for $M/D/\infty$ or $M/M/\infty$ systems, but the analogous question for the classical $M/M/1$ queueing system has been studied by Neuts [N], and our approach to the problem for the $M/M/\infty$ system is similar in spirit to his.

2. The $M/D/\infty$ Model

We shall begin by examining the interval graph based on the $M/D/\infty$ system, which is easier in many respects to analyze than that based on the $M/M/\infty$ system. In the $M/D/\infty$ system, customers arrive according to a Poisson process with rate λ per unit time, and they each remain in the system for one unit of time.

Let t_0 be the time of an arrival that occurs when the system is empty, so that t_0 marks the beginning of a busy period. Let $t_1, t_2, \ldots$ be the times of subsequent arrivals, and let $\Delta_1 = t_1 - t_0, \Delta_2 = t_2 - t_1, \ldots$ be the corresponding interarrival times. Since the arrivals form a Poisson process with rate λ, the interarrival times are independent and identically distributed according to an exponential distribution with mean $1/\lambda$:

$$\Pr(\Delta_j > \tau) = e^{-\lambda\tau}.$$

The busy period beginning at t_0 comprises n arrivals (counting the arrival at t_0), where $n \geq 1$ is the smallest integer such that $\Delta_n > 1$. The number N of arrivals during a busy period is thus distributed as the number of independent trials, each of which succeeds with probability $e^{-\lambda}$, required to produce the first success. The distribution of N is therefore geometric,

$$\Pr(N = n) = e^{-\lambda}(1 - e^{-\lambda})^{n-1},$$

with mean

$$\mathrm{Ex}(N) = e^{\lambda}$$

and variance

$$\mathrm{Var}(N) = e^{2\lambda} - e^{\lambda}.$$

Since the variance grows as the square of the mean, the distribution does not tend to concentrate about the mean as λ increases. But if we rescale N by its mean, we get a random variable N/e^λ that whose distribution tends to an exponential with mean 1 as λ tends to infinity. We have

$$\begin{aligned}\Pr(N/e^\lambda > u) &= e^{-\lambda} \sum_{n > u e^\lambda} (1 - e^{-\lambda})^{n-1} \\ &= (1 - e^{-\lambda})^{\lfloor u e^\lambda \rfloor}.\end{aligned}$$

This gives

Theorem 2.1:

$$\Pr(N/e^\lambda > u) \to e^{-u}$$

as $\lambda \to \infty$ with $u \geq 0$ fixed.

The distribution of $\log N - \lambda$ thus tends to that of $\log M$, where M is exponentially distributed with mean 1. The random variable $\log M$ has mean

$$\mathrm{Ex}(\log M) = \int_0^\infty e^{-x} \log x \, dx = \Gamma'(1) = -\gamma,$$

where $\Gamma(s)$ denotes the gamma function and $\gamma = 0.557\ldots$ denotes Euler's constant (see Whittaker and Watson [W], Chapter XII). Its second moment is

$$\mathrm{Ex}\big((\log M)^2\big) = \int_0^\infty e^{-x} (\log x)^2 \, dx = \Gamma''(1) = \frac{\pi^2}{6} + \gamma^2,$$

and thus its variance is

$$\mathrm{Var}(\log M) = \mathrm{Ex}\big((\log M)^2\big) - \mathrm{Ex}(\log M)^2 = \frac{\pi^2}{6}.$$

Here integrals representing the moments are expressed in terms of the derivatives of the gamma function at 1 by differentiating the integral representation

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx$$

once or twice before setting $s = 1$. The values of the resulting derivatives are obtained by differentiating the logarithm of the Weierstrass product

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{m \geq 1} \left(\frac{m}{m+s}\right) e^{s/m}$$

once or twice before setting $s = 1$. The evaluation of the second derivative is completed using the formula $\sum_{m \geq 1} \frac{1}{m^2} = \zeta(2) = \frac{\pi^2}{6}$, where $\zeta(s)$ is the zeta function of Riemann (see Whittaker and Watson [W], Chapter XIII). This gives

Corollary 2.2:

$$\Pr\big((\log N)/\lambda > u\big) \to \begin{cases} 1, & 0 \le u < 1, \\ 0, & 1 \le u, \end{cases}$$

as $\lambda \to \infty$ with $u \ge 0$ fixed.

Next we shall turn our attention to the maximum number K of customers present in the system simultaneously during a busy period. We shall show that with high probability K lies in the interval $[e\lambda - 2\log\lambda, e\lambda]$. To do this, we shall obtain estimates for the mean and variance of the number X_k of customers that depart from the system only after $k-1$ additional customers have arrived. We shall obtain all our estimates by first conditioning on the event $N = n$ that there are exactly n customers, then averaging over the distribution of N. We then have

$$\Pr(K \ge k \mid N = n) = \Pr(X_k \ge 1 \mid N = n).$$

We rely here on the fact that, since all service times are equal, the customers arrive and depart according to a first-in first-out discipline, so if there are k customers in the system simultaneously, then there are k *consecutive* customers in the system simultaneously.

Let $X_{k,i}$ (where $0 \le i \le n-k$) denote the event that the the customer arriving at time t_i departs (at time $t_i + 1$) only after $k-1$ additional customers have arrived (at times $t_{i+1}, \ldots, t_{i+k-1}$). Thus $X_{k,i}$ occurs if and only if $\Delta_{i+1} + \cdots + \Delta_{i+k-1} \le 1$, and we have

$$X_k = \sum_{0 \le i \le n-k} X_{k,i}$$

(where we have identified the event $X_{k,i}$ with its $\{0,1\}$-valued indicator variable).

When we condition on the event $N = n$, the interarrival times $\Delta_1, \ldots, \Delta_{n-1}$ remain independent, but they now have the distribution obtained by conditioning on the event $\Delta_h \le 1$, namely

$$\Pr(\Delta_h \le \tau) = \frac{\int_0^\tau e^{-\lambda\sigma}\, d\sigma}{\int_0^1 e^{-\lambda\sigma}\, d\sigma} = \frac{1 - e^{-\lambda\tau}}{1 - e^{-\lambda}}.$$

Note that this distribution differs from the unconditional distribution by the factor $1 - e^{-\lambda}$. Thus we have

$$\begin{aligned} \Pr(X_{k,i} \mid N = n) &= \Pr(\Delta_{i+1} + \cdots + \Delta_{i+k-1} \le 1 \mid N = n) \\ &= \Pr(\Delta_{i+1} + \cdots + \Delta_{i+k-1} \le 1)/(1 - e^{-\lambda})^{k-1}. \end{aligned}$$

Now $\Pr(\Delta_{i+1} + \cdots + \Delta_{i+k-1} \le 1)$ is just the probability that $k-1$ or more arrivals occur during an interval of unit length in a Poisson process of arrival rate λ:

$$\Pr(\Delta_{i+1} + \cdots + \Delta_{i+k-1} \le 1) = e^{-\lambda} \sum_{k-1 \le h < \infty} \frac{\lambda^h}{h!}.$$

(Note that this formula is exact, with the sum over h extending to infinity, even though we are considering a process with just n arrivals.) Thus

$$\Pr(X_{k,i} \mid N = n) = \frac{e^{-\lambda}}{(1-e^{-\lambda})^{k-1}} \sum_{k-1 \le h < \infty} \frac{\lambda^h}{h!}.$$

Summing over i we obtain

$$\mathrm{Ex}(X_k \mid N = n) = \frac{(n-k+1)e^{-\lambda}}{(1-e^{-\lambda})^{k-1}} \sum_{k-1 \le h < \infty} \frac{\lambda^h}{h!}.$$

This formula holds when $n \ge k-1$; when $n < k$, the conditional expectation is of course 0. We thus have

$$\mathrm{Ex}\big(\max\{n-k+1, 0\}\big) = (1-e^{-\lambda})^{k-1} e^{\lambda},$$

so that

$$\mathrm{Ex}(X_k) = \sum_{k-1 \le h < \infty} \frac{\lambda^h}{h!}. \tag{2.1}$$

By obtaining upper bounds for this expression, we shall be able to show that $\Pr(X_k \ge 1)$ is small when k is large, which yields bounds on the upper tail of the distribution of K. To obtain corresponding bounds on the lower tail, we shall need to show that $\Pr(X_k = 0)$ is small when k is small, and for this we shall need upper bounds for the variance of X_k as well as lower bounds for the expectation.

To this end, we start with the formula

$$\begin{aligned}\mathrm{Var}(X_k) &= \sum_{0 \le i,j \le n-k} \Pr(X_{k,i}, X_{k,j}) - \Pr(X_{k,i}) \Pr(X_{k,j}) \\ &\le \mathrm{Ex}(X_k) + 2 \sum_{0 \le i < j \le n-k} \Pr(X_{k,i}, X_{k,j}) - \Pr(X_{k,i}) \Pr(X_{k,j}).\end{aligned}$$

When $j \ge i + k - 1$, the events $X_{k,i}$ (which is equivalent to $\Delta_{i+1} + \cdots + \Delta_{i+k-1}$) and $X_{k,j}$ (which is equivalent to $\Delta_{j+1} + \cdots + \Delta_{j+k-1}$) are determined by disjoint sets of independent interarrival times, and thus are themselves independent, so that

$\Pr(X_{k,i}, X_{k,j}) = \Pr(X_{k,i}) \Pr(X_{k,j})$, and these terms make no contribution to the sum. Thus we have

$$\begin{aligned}
\operatorname{Var}(X_k) &\le \operatorname{Ex}(X_k) + 2 \sum_{\substack{0 \le i < j \le n-k \\ j < i+k-1}} \Pr(X_{k,i}, X_{k,j}) - \Pr(X_{k,i}) \Pr(X_{k,j}) \\
&\le \operatorname{Ex}(X_k) + 2 \sum_{\substack{0 \le i < j \le n-k \\ j < i+k-1}} \Pr(X_{k,i}, X_{k,j}).
\end{aligned}$$

Define $Y_{k,i,j}$ to be the event $\Delta_{i+k} + \cdots + \Delta_{j+k-1} \le 1$. If $j < i+k-1$, then $Y_{k,i,j}$ is implied by $X_{k,j}$, so that $\Pr(X_{k,i}, X_{k,j}) \le \Pr(X_{k,i}, Y_{k,i,j})$. Furthermore, $X_{k,i}$ and $Y_{k,i,j}$ depend on disjoint sets of independent interarrival times, and thus are themselves independent, so that $\Pr(X_{k,i}, Y_{k,i,j}) = \Pr(X_{k,i}) \Pr(Y_{k,i,j})$. Thus we have

$$\operatorname{Var}(X_k) \le \operatorname{Ex}(X_k) + 2 \sum_{0 \le i \le n-k} \Pr(X_{k,i}) \sum_{\substack{i < j \le n-k \\ j < i+k-1}} \Pr(Y_{k,i,j}).$$

Now $\Pr(Y_{k,i,j}) = \Pr(\Delta_{i+k} + \cdots + \Delta_{j+k-1} \le 1)$ is just the probability that $j - i$ or more arrivals occur during an interval of unit length in a Poisson process of arrival rate λ:

$$\Pr(Y_{k,i,j}) = e^{-\lambda} \sum_{j-i \le h < \infty} \frac{\lambda^h}{h!}.$$

If we sum the probability that d or more arrivals occur over all $d \ge 1$, we obtain the expected number of arrivals, which for the Poisson process in question is λ. Thus we have

$$\sum_{\substack{i < j \le n-k \\ j < i+k-1}} \Pr(Y_{k,i,j}) \le \lambda,$$

so that

$$\begin{aligned}
\operatorname{Var}(X_k) &\le \operatorname{Ex}(X_k) + 2\lambda \sum_{0 \le i \le n-k} \Pr(X_{k,i}) \\
&\le (1 + 2\lambda) \operatorname{Ex}(X_k).
\end{aligned}$$

By Markov's inequality we have

$$\Pr(X_k \ge 1) \le \operatorname{Ex}(X_k),$$

and by Chebyshev's inequality we have

$$\Pr(X_k = 0) \le \frac{\operatorname{Var}(X_k)}{\operatorname{Ex}(X_k)^2} \le \frac{1 + 2\lambda}{\operatorname{Ex}(X_k)}.$$

Thus, since K is the largest k such that $X_k \geq 1$, we can obtain bounds on the tails of the distribution of K from estimates for $\mathrm{Ex}(X_k)$. For $k-1 \geq e\lambda$, successive terms in the sum (2.1) decrease at least geometrically with ratio $1/e$, so the sum is bounded above by $e/(e-1)$ times its largest term:

$$\mathrm{Ex}(X_k) \leq \frac{e}{e-1} \frac{\lambda^{k-1}}{(k-1)!}.$$

For $k-1 = e\lambda$, Stirling's formula yields

$$\mathrm{Ex}(X_k) \leq \frac{e}{e-1} \frac{1}{(2\pi e\lambda)^{1/2}} + O\left(\frac{1}{\lambda^{3/2}}\right).$$

This implies that $\Pr(X_k \geq 1) \to 0$ as $\lambda \to \infty$ with $k-1 \geq e\lambda$. In any case, the sum in (2.1) is bounded below by its largest term. For $k-1 = e\lambda - 2\log\lambda$, Stirling's formula yields

$$\mathrm{Ex}(X_k) = \frac{\lambda^{3/2}}{(2\pi e)^{1/2}} + O\left(\lambda^{1/2}\right).$$

This implies that $\Pr(X_k = 0) \to 0$ as $\lambda \to \infty$ with $k-1 \leq e\lambda - 2\log\lambda$. Combining these results we have

Theorem 2.3:

$$\Pr\bigl(K/(e\lambda) > u\bigr) \to \begin{cases} 1, & 0 \leq u < 1, \\ 0, & 1 \leq u, \end{cases}$$

as $\lambda \to \infty$ with $u \geq 0$ fixed.

3. The $M/M/\infty$ Model

We now turn our attention to the $M/M/\infty$ model, for which we shall obtain analogous results by different methods. We shall reuse the notation of the preceding section, so N will denote the number of customers arriving during a busy period of the $M/M/\infty$ system, and K will denote the maximum number of customers served simultaneously during such a period. The most striking feature of the $M/M/\infty$ system is that it has a countable state space. Indeed, the number of customers being served constitutes the state of the system, with the system making transitions among states at the rates indicated in the following diagram.

$$0 \underset{1}{\overset{\lambda}{\rightleftharpoons}} 1 \underset{2}{\overset{\lambda}{\rightleftharpoons}} 2 \underset{3}{\overset{\lambda}{\rightleftharpoons}} \cdots \underset{j}{\overset{\lambda}{\rightleftharpoons}} j \underset{j+1}{\overset{\lambda}{\rightleftharpoons}} \cdots$$

It is easy to see that the number J of customers in the system has an equilibrium distribution

$$\Pr(J = j) = \frac{e^{-\lambda}\lambda^j}{j!}.$$

Let $T(z)$ denote the generating function of $N - 1$:

$$T(z) = \sum_{j\geq 0} z^j \Pr(N - 1 = j).$$

(The discounting of the first customer, focusing attention on $N - 1$ rather than N, is for technical convenience.) Our first order of business is to derive an expression for $T(z)$. We shall use a method given by Guillemin and Simonian [G3].

For $j \geq 1$, let $T_j(z)$ be the generating function for the number of customers arriving before the first visit to state $j - 1$, when the system starts in state j. Then $T_j(1) = 1$ (since the state 0 is recurrent, and can only be reached from state j by passing through state $j - 1$), and we seek $T(z) = T_1(z)$. We have

$$T_j(z) = \frac{j}{\lambda + j} + \frac{\lambda}{\lambda + j}\, z\, T_{j+1}(z)\, T_j(z), \tag{3.1}$$

since from state j the system can either go immediately to state $j - 1$ (which occurs with probability $\frac{j}{\lambda+j}$), or it can go by an arrival to state $j + 1$ (which occurs with probability $\frac{\lambda}{\lambda+j}$), after which it must pass through state j before reaching state $j-1$. For $j \geq 1$, define

$$S_j(z) = \prod_{1\leq i\leq j} T_i(z),$$

so that $T_1(z) = S_1(z)$. Multiplying (3.1) by $(\lambda + j)S_{j-1}(z)$ yields

$$(\lambda + j)\, S_j(z) = j\, S_{j-1}(z) + \lambda z\, S_{j+1}(z). \tag{3.2}$$

Define

$$R(y, z) = \sum_{j\geq 1} \frac{y^j}{j!} S_j(z).$$

We note that since $|S_j(z)| \leq 1$ for $|z| \leq 1$, $R(y, z)$ is an entire function of y for any z such that $|z| \leq 1$. Multiplying (3.2) by $y^j/j!$ and summing over $j \geq 2$ yields

$$\lambda\big(R(y, z) - yS_1(z)\big) + y\big(\tfrac{\partial}{\partial y}R(y, z) - S_1(z)\big) = yR(y, z) + \lambda z\big(\tfrac{\partial}{\partial y}R(y, z) - S_1(z) - yS_2(z)\big).$$

Using the case $j = 1$ of (3.2), $(\lambda + 1)\, S_1(z) = 1 + \lambda z\, S_2(z)$, allows us to eliminate $S_2(z)$, yielding

$$(\lambda z - y)\, \tfrac{\partial}{\partial y} R(y,z) = (\lambda - y)\, R(y,z) + \lambda z\, S_1(z) - y.$$

To solve this equation, we put $R(y,z) = P(y,z)\, Q(y,z)$, where $P(y,z)$ is a solution to the homogeneous equation

$$(\lambda z - y)\, \tfrac{\partial}{\partial y} P(y,z) = (\lambda - y)\, P(y,z).$$

Integrating the equation

$$\tfrac{\partial}{\partial y} \log P(y,z) = \frac{\lambda - y}{\lambda z - y}$$

yields

$$P(y,z) = e^y (y - \lambda z)^{-\lambda(1-z)} C(z),$$

where $C(z)$ is a factor arising from the constant of integration. Substitution then gives the following equation for $Q(y,z)$:

$$\tfrac{\partial}{\partial y} Q(y,z) = \frac{e^{-y} (y - \lambda z)^{\lambda(1-z)} (\lambda z\, S_1(z) - y)}{C(z)}.$$

Integrating this equation yields

$$Q(y,z) = \frac{\lambda z\, S_1(z)}{C(z)} \int_0^y e^{-\eta} (\eta - \lambda z)^{\lambda(1-z)}\, d\eta - \frac{1}{C(z)} \int_0^y e^{-\eta} (\eta - \lambda z)^{\lambda(1-z)} \eta\, d\eta.$$

Combining the expressions for $P(y,z)$ and $Q(y,z)$ yields

$$R(y,z) = \\ e^y (y - \lambda z)^{-\lambda(1-z)} \left(\lambda z\, S_1(z) \int_0^y e^{-\eta} (\eta - \lambda z)^{\lambda(1-z)}\, d\eta - \int_0^y e^{-\eta} (\eta - \lambda z)^{\lambda(1-z)} \eta\, d\eta \right).$$

We have observed that the left-hand side is an entire function of y for any z such that $|z| \le 1$. On the other hand, the factor $(y - \lambda z)^{-\lambda(1-z)}$ will cause the right-hand side to diverge when $y = \lambda z$, unless the expression in large parentheses vanishes for $y = \lambda z$. This yields the following expression for $T(z) = T_1(z) = S_1(z)$ (which is what we were seeking in the first instance):

$$T(z) = \frac{\int_0^{\lambda z} e^{-\eta} (\eta - \lambda z)^{\lambda(1-z)} \eta\, d\eta}{\lambda z \int_0^{\lambda z} e^{-\eta} (\eta - \lambda z)^{\lambda(1-z)}\, d\eta},$$

or (making the change of variable $\eta = \lambda z(1-s)$)

$$T(z) = \frac{\int_0^1 e^{\lambda z s}(1-s)s^{(1-z)\lambda-1}\,ds}{\int_0^1 e^{\lambda z s}s^{(1-z)\lambda-1}\,ds}.$$

The integrals in both numerator and denominator have singularities as $z \to 1$, but these singularities cancel, since both integrands behave similarly in this limit. After some further manipulations we shall be able to carry out this cancellation explicitly. The Kummer confluent hypergeometric function

$$\Phi(a,c;w) = \sum_{j\geq 0} \frac{(a)_j w^j}{(c)_j j!} \tag{3.3}$$

(where $(a)_j = a(a-1)\cdots(a-j+1)$) has the integral representation

$$\Phi(a,c;w) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{wt}(1-t)^{c-a-1}t^{a-1}\,dt$$

(see Lebedev [L], Equations (9.9.1) and (9.11.1)). Thus, using the functional equation $\Gamma(b+1) = b\Gamma(b)$, we have

$$T(z) = \frac{1}{(1-z)\lambda+1}\,\frac{\Phi\big((1-z)\lambda,(1-z)\lambda+2;\lambda z\big)}{\Phi\big((1-z)\lambda,(1-z)\lambda+2;\lambda z\big)}.$$

The expansion (3.3) gives the numerator and denominator as Laurent series with leading term $1/(1-z)\lambda$, which gives rise to the singularities mentioned above. Thus it will be convenient to multiply both expansions by the factor $(1-z)\lambda$, to obtain

$$T(z) = \frac{\frac{1}{\big((1-z)\lambda+1\big)} + (1-z)\lambda\sum_{j\geq 1}\frac{\lambda^j z^j}{\big((1-z)\lambda+j\big)\big((1-z)\lambda+j+1\big)j!}}{1+(1-z)\lambda\sum_{j\geq 1}\frac{\lambda^j z^j}{\big((1-z)\lambda+j\big)j!}}. \tag{3.4}$$

Let

$$A(\lambda,z) = \frac{1}{\big((1-z)\lambda+1\big)} + (1-z)\lambda\sum_{j\geq 1}\frac{\lambda^j z^j}{\big((1-z)\lambda+j\big)\big((1-z)\lambda+j+1\big)j!},$$

and

$$B(\lambda,z) = 1+(1-z)\lambda\sum_{j\geq 1}\frac{\lambda^j z^j}{\big((1-z)\lambda+j\big)j!},$$

so that $T(z) = A(\lambda, z)/B(\lambda, z)$. We note that $A(\lambda, 1) = B(\lambda, 1) = 1$, so that $T(z) = 1$, as it must be.

By expanding the first few terms of $A(\lambda, z)$ and $B(\lambda, z)$, we obtain

$$
\begin{aligned}
T(z) = {} & \left(\frac{1}{\lambda+1}\right) + \left(\frac{2\lambda}{(\lambda+1)^2(\lambda+2))}\right) z \\
& + \left(\frac{6\lambda^3}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)} + \frac{4\lambda^2}{(\lambda+1)^3(\lambda+2)^2}\right) z^2 + \cdots,
\end{aligned}
$$

which agrees with the following direct calculation. We obtain $\Pr(N = n)$ by summing the probabilities of the trajectories in state space, starting at state 0, and having exactly n ascents and n descents before first returning to state 0. Each such trajectory contributes a probability that is a product of $2n$ factors, with a factor $\lambda/(\lambda + j)$ for each ascent from state j to state $j + 1$, and a factor of $j/(\lambda + j)$ for each descent from state j to state $j - 1$. The cases $N = 1$ and $N = 2$ each have one possible trajectory, as follows.

$$
\begin{array}{cc}
\nearrow & \searrow \\
\frac{\lambda}{\lambda+0} & \frac{1}{\lambda+1}
\end{array}
\qquad\qquad
\begin{array}{cccc}
 & \nearrow & \searrow & \\
\nearrow & & & \searrow \\
\frac{\lambda}{\lambda+0} & \frac{\lambda}{\lambda+1} & \frac{2}{\lambda+2} & \frac{1}{\lambda+1}
\end{array}
$$

The probabilities of these trajectories give the constant term and the coefficient of z in $T(z)$. The case $N = 3$ has two possible trajectories.

$$
\begin{array}{cccccc}
 & & \nearrow & \searrow & & \\
 & \nearrow & & & \searrow & \\
\nearrow & & & & & \searrow \\
\frac{\lambda}{\lambda+0} & \frac{\lambda}{\lambda+1} & \frac{\lambda}{\lambda+2} & \frac{3}{\lambda+3} & \frac{2}{\lambda+2} & \frac{1}{\lambda+1}
\end{array}
\qquad\qquad
\begin{array}{cccccc}
 & \nearrow & \searrow & \nearrow & \searrow & \\
\nearrow & & & & & \searrow \\
\frac{\lambda}{\lambda+0} & \frac{\lambda}{\lambda+1} & \frac{2}{\lambda+2} & \frac{\lambda}{\lambda+1} & \frac{2}{\lambda+2} & \frac{1}{\lambda+1}
\end{array}
$$

The sum of their probabilities gives the coefficient of z^2 in $T(z)$. Expanding these coefficients in powers of λ gives

$$\begin{aligned}
\Pr(N=1) &= 1-\lambda+\lambda^2+O(\lambda^3),\\
\Pr(N=2) &= \lambda-\frac{5}{2}\lambda^2+O(\lambda^3),\\
\Pr(N=3) &= \frac{3}{2}\lambda^2+O(\lambda^3).
\end{aligned}$$

This in turn gives

$$\mathrm{Ex}(N) = 1+\lambda+\frac{1}{2}\lambda^2+O(\lambda^3) \tag{3.5}$$

and

$$\mathrm{Var}(N) = \lambda+\frac{5}{2}\lambda^2+O(\lambda^3). \tag{3.6}$$

To evaluate $\mathrm{Ex}(N) = T'(1)+1$ exactly, we differentiate the formula $T(z)\,B(\lambda,z) = A(\lambda,z)$, then set $z=1$. This gives

$$\begin{aligned}
T'(1) &= A'(\lambda,1)-B'(\lambda,1)\\
&= \lambda-\lambda\sum_{j\geq 1}\frac{\lambda^j}{(j+1)\,j\,j!}+\lambda\sum_{j\geq 1}\frac{\lambda^j}{(j+1)\,j\,j!}\\
&= \lambda+\lambda\sum_{j\geq 1}\frac{\lambda^j}{(j+1)!}\\
&= e^{\lambda}-1.
\end{aligned}$$

This gives $\mathrm{Ex}(N) = e^{\lambda} = 1+\lambda+\frac{1}{2}\lambda^2+O(\lambda^3)$, which agrees with (3.5).

To evaluate $\mathrm{Var}(N) = T''(1)+T'(1)-T'(1)^2$ exactly, we differentiate the formula $T(z)\,B(\lambda,z) = A(\lambda,z)$ twice, then set $z=1$. This time we cannot simplify the sums as much as before, and we get

$$\mathrm{Var}(N) = 2\lambda E(\lambda)e^{\lambda}-e^{2\lambda}+2\lambda e^{\lambda}+e^{\lambda},$$

where

$$E(\lambda) = \sum_{j\geq 1}\frac{\lambda^j}{j\,j!} = \int_0^{\lambda}\frac{e^x-1}{x}\,dx.$$

This gives $\mathrm{Var}(N) = \lambda+\frac{5}{2}\lambda^2+O(\lambda^3)$, which agrees with (3.6).

We could continue in this way to work out higher moments of N in terms of increasingly complicated sums or integrals. We shall instead consider the asymptotic behavior of these moments, which has a particularly simple expression. We shall show the following.

Proposition 3.1: For each fixed k,

$$\mathrm{Ex}(N^k) \sim k!\, e^{k\lambda}$$

as $\lambda \to \infty$.

We shall need a lemma that says that certain functions with power series that are similar to that of the exponential function, $e^\lambda = \sum_{j\geq 0} \frac{\lambda^j}{j!}$, have the same asymptotic behavior as the exponential function.

Lemma 3.2: For any fixed natural numbers a and b,

$$\sum_{j\geq 1+a+b} \frac{\lambda^j}{j^a (j+1)^b (j-a-b)!} \sim e^\lambda \tag{3.7}$$

as $\lambda \to \infty$.

Proof: The proof proceeds in three steps. First, we subtract from the left-hand side of (3.7) the terms with $j < \lambda/2$. The sum over these terms is

$$\begin{aligned} \sum_{1+a+b\leq j<\lambda/2} \frac{\lambda^j}{j^a (j+1)^b (j-a-b)!} &\leq \sum_{1+a+b\leq j<\lambda/2} \frac{\lambda^j}{j!} \\ &\leq (\lambda/2)(2e)^{\lambda/2}. \end{aligned}$$

Here we have bounded the sum by the number of terms times the largest term, then used the inequality $j! \geq (j/e)^j$. Since $(2e)^{1/2} < e$, this sum is negligible compared with the right-hand side e^λ.

The second step is to observe that for the remaining terms, with $j \geq \lambda/2$, the denominator is close to $j!$. This yields

$$\sum_{j\geq\lambda/2} \frac{\lambda^j}{j!} \leq \sum_{j\geq\lambda/2} \frac{\lambda^j}{j^a (j+1)^b (j-a-b)!} \leq \left(\frac{\lambda}{\lambda + 2(a+b+1)}\right)^{a+b} \sum_{j\geq\lambda/2} \frac{\lambda^j}{j!}.$$

The third step is to "complete the exponential" e^λ, by adding the terms $\lambda^j/j!$ for $j < \lambda/2$. The sum of these terms is

$$\sum_{0\leq j<\lambda/2} \frac{\lambda^j}{j!} \leq (1+\lambda/2)(2e)^{\lambda/2},$$

which is again negligible compared with e^λ. Combining these three estimates yields the lemma. $\triangle$

Next we shall need two lemmas giving the the asymptotic behavior of derivatives of the series $A(\lambda, z)$ and $B(\lambda, z)$.

Lemma 3.3: For each fixed m,

$$\frac{\partial^m}{\partial z^m} A(\lambda, z) \Big|_{z=1} \sim -m\lambda^{m-2} e^{\lambda}$$

as $\lambda \to \infty$.

Proof: We apply the multinomial version of Leibniz's rule to the series for $A(\lambda, z)$. We observe that the factor $(1-z)$ in front of the sum will prevent the sum from contributing to the result unless this factor is differentiated exactly once. Thus we have

$$\frac{\partial^m}{\partial z^m} A(\lambda, z) \Big|_{z=1} = m!\, \lambda^m - m\lambda \sum_{a,b} \binom{m-1}{p,q} \sum_{j\geq 1} \frac{\lambda^{j+p+q}}{j^{p+1}\,(j+1)^{q+1}\,(j-m+p+q+1)!}.$$

Applying Lemma 3.2 to each term in the sum over p and q, we see that the term with $p = q = 0$ makes the largest contribution, and that this contribution gives the asymptotic behavior $-m\lambda^{m-2} e^{\lambda}$, with the contributions of all other terms being negligible. Since the term $m!\,\lambda^m$ is clearly also negligible, we obtain the result of the lemma. $\triangle$

A similar and simpler argument gives the following lemma.

Lemma 3.4: For each fixed m,

$$\frac{\partial^m}{\partial z^m} B(\lambda, z) \Big|_{z=1} \sim -m\lambda^{m-1} e^{\lambda}$$

as $\lambda \to \infty$.

Proof of Proposition 3.1: We shall first establish the asymptotic relation

$$\mathrm{Ex}\big((N)_k\big) \sim k!\, e^{k\lambda} \tag{3.8}$$

for the factorial moments $\mathrm{Ex}\big((N)_k\big) = T^{(k)}(1)$. The proposition will then follow by induction on k, using the relation

$$N^k = \sum_j \left\{ {k \atop j} \right\} (N)_j$$

(where $\left\{ {k \atop j} \right\}$ is the Stirling number of the Second Kind), with $\left\{ {k \atop k} \right\} = 1$.

To establish (3.8), we proceed by induction on k, with the basis being the estimate $\mathrm{Ex}(N) = e^{\lambda}$ derived above. For the inductive step, we differentiate the identity $T(z)B(\lambda, z) = A(\lambda, z)$ using Leibniz's rule, then set $z = 1$. This gives

$$T^{(k)}(1) = A^{(k)}(\lambda, 1) - \sum_{j\geq 1} \binom{k}{j} T^{(k-j)}(1)\, B^{(j)}(\lambda, 1).$$

Applying Lemmas 3.3 and 3.4 and the inductive hypothesis to each term on the right-hand side, we see that the term with $j = 1$ makes the largest contribution, and that this contribution give the asymptotic behavior $k!\, e^{k\lambda}$, with the contributions of all other terms being negligible. $\triangle$

We are now ready to prove the following analogue of Theorem 2.1.

Theorem 3.5:

$$\Pr(N/e^{\lambda} > u) \to e^{-u}$$

as $\lambda \to \infty$ with $u \geq 0$ fixed.

This will imply the following analogue of Corollary 2.2.

Corollary 3.6:

$$\Pr\big((\log N)/\lambda > u\big) \to \begin{cases} 1, & 0 \leq u < 1, \\ 0, & 1 \leq u, \end{cases}$$

as $\lambda \to \infty$ with $u \geq 0$ fixed.

Proof of Theorem 3.5: By Proposition 3.1, the rescaled variable $U = N/e^{\lambda}$ has moments satisfying

$$\mathrm{Ex}(U^k) \to k!$$

as $\lambda \to \infty$. We next observe that the exponential distribution with mean 1 is the unique distribution with these limiting values for moments. To see this, let M have exponential distribution with mean 1. Then

$$\mathrm{Ex}(M^k) = \int_0^{\infty} s^k\, e^{-s}\, ds = k!.$$

To establish uniqueness, we use the classical criterion of Carleman [C], which says that the distribution of a random variable Q on $[0, \infty)$ with mean 1 is characterized by its moments if

$$\sum_{2 \leq k \leq l} \mathrm{Ex}\big((Q-1)^k\big)^{-1/2k} \to \infty$$

as $l \to \infty$, where $\mathrm{Ex}\big((Q-1)^k\big)$ are the central moments of Q. For the exponential distribution, the central moments are

$$\begin{aligned} \mathrm{Ex}\big((M-1)^k\big) &= \int_0^{\infty} (s-1)^k\, e^{-s}\, ds \\ &= \int_0^{\infty} \sum_j \binom{k}{j} s^j (-1)^{k-j}\, e^{-s}\, ds \\ &= \sum_j \binom{k}{j} j!\, (-1)^{k-j} \\ &= D_k, \end{aligned}$$

where D_k is the k-th derangement number, which is the number of fixed-point-free permutations of $\{1, \ldots, k\}$. Since we have $D_k \le k! \le k^k$, we obtain

$$\sum_{2 \le k \le l} \mathrm{Ex}\big((M-1)^k\big)^{-1/2k} \ge \sum_{2 \le k \le l} k^{-1/2} \ge \int_2^l s^{-1/2}\, ds = 2\sqrt{l} - 2\sqrt{2},$$

so that Carleman's criterion is satisfied. Thus the distribution of $U = N/e^\lambda$ converges weakly to an exponential distribution with mean 1 as $\lambda \to \infty$. $\triangle$

We shall now turn our attention to the maximum number K of customers present in the system simultaneously during a busy period. We shall show that with high probability K lies in the interval $[e\lambda - 2\log\lambda, e\lambda]$. This conclusion will follow from a remarkably simple exact formula for the distribution of K.

Proposition 3.7: For $k \ge 0$, we have

$$\Pr(K > k) = \frac{1}{\sum_{0 \le j \le k} \frac{j!}{\lambda^j}}.$$

Proof: For $0 \le j \le k+1$, let P_j denote the probability that the system arrives at state $k+1$ before arriving at state 0, when started in state j. Then $\Pr(K > k) = P_1$.

We have $P_0 = 0$, $P_{k+1} = 1$, and

$$P_j = \frac{j}{\lambda + j} P_{j-1} + \frac{\lambda}{\lambda + j} P_{j+1}$$

for $1 \le j \le k$. Multiplying through by $\lambda + j$ and rearranging yields

$$j(P_j - P_{j-1}) = \lambda(P_{j+1} - P_j).$$

Thus if we set $Q_j = P_{j+1} - P_j$, then we have

$$jQ_{j-1} = \lambda Q_j$$

for $1 \le j \le k$, and

$$Q_k + \cdots + Q_1 = (P_{k+1} - P_k) + \cdots + (P_1 - P_0) = P_{k+1} - P_0 = 1.$$

Thus the Q_j satisfy the same equations as the equilibrium probabilities for the following system.

$$0 \;\underset{\lambda}{\overset{1}{\rightleftharpoons}}\; 1 \;\underset{\lambda}{\overset{2}{\rightleftharpoons}}\; 2 \;\underset{\lambda}{\overset{3}{\rightleftharpoons}}\; \cdots \;\underset{\lambda}{\overset{k-1}{\rightleftharpoons}}\; k-1 \;\underset{\lambda}{\overset{k}{\rightleftharpoons}}\; k$$

The solution of these equations is

$$Q_i = \frac{\frac{i!}{\lambda^i}}{\sum_{0\le j\le k} \frac{j!}{\lambda^j}}$$

for $0 \le i \le k$. The probability we seek is $\Pr(K > k) = P_1 = Q_0$, and taking $i = 0$ completes the proof. $\triangle$

We remark that the preceding derivation made no use of the specific transition rates of the $M/M/\infty$ system, and thus it is applicable to an arbitrary birth-and-death process, with the following result.

Corollary 3.8: Consider the birth-and-death process with the transition rates indicated in the following diagram.

$$0 \underset{\mu_1}{\overset{\lambda_0}{\rightleftharpoons}} 1 \underset{\mu_2}{\overset{\lambda_1}{\rightleftharpoons}} 2 \underset{\mu_3}{\overset{\lambda_2}{\rightleftharpoons}} \cdots \underset{\mu_j}{\overset{\lambda_{j-1}}{\rightleftharpoons}} j \underset{\mu_{j+1}}{\overset{\lambda_j}{\rightleftharpoons}} \cdots$$

Then for $k \ge 0$, we have

$$\Pr(K > k) = 1 \;/ \sum_{0\le j\le k} \Big(\prod_{1\le i\le j} \mu_i \;/ \prod_{1\le i\le j} \lambda_{j-1} \Big).$$

We shall now use Proposition 3.7 to bound the tails of the distribution of K. For $k \ge e\lambda$, we bound the sum in the denominator from below by its last term:

$$\Pr(K > k) \le \frac{\lambda^k}{k!}.$$

For $k = \lambda e$, Stirling's formula yields

$$\Pr(K > k) \le \frac{1}{(2\pi e\lambda)^{1/2}} + O\left(\frac{1}{\lambda^{3/2}}\right).$$

Thus $\Pr(K > k) \to 0$ as $\lambda \to \infty$ with $k \ge e\lambda$. To bound the lower tail, we observe that $1/(1+x) \ge 1 - x$, so that

$$\Pr(K > k) \ge 1 - \frac{1}{\lambda} - \sum_{2\le j\le k} \frac{j!}{\lambda^j}.$$

To bound the sum from above, we observe that the successive terms decrease until $j > \lambda$, then increase, so that the largest term in the sum is either the first or the last. The first term is $2/\lambda^2$. For $k = e\lambda - 2\log\lambda$, Stirling's formula yields

$$\frac{k!}{\lambda^k} = \frac{1}{(2\pi e)^{1/2}\lambda^{3/2}} + O\left(\frac{1}{\lambda^{5/2}}\right)$$

for the last term, which is therefore larger. Thus we have

$$\begin{aligned}\Pr(K > k) &\geq 1 - \frac{1}{\lambda} - (k-2)\frac{k!}{\lambda^k} \\ &\geq 1 - \frac{e^{1/2}}{(2\pi\lambda)^{1/2}} + O\left(\frac{1}{\lambda}\right).\end{aligned}$$

This implies that $\Pr(K > k) \to 1$ as $\lambda \to \infty$ with $k \leq e\lambda - 2\log\lambda$. Combining these results we have

Theorem 3.9:

$$\Pr\bigl(K/(e\lambda) > u\bigr) \to \begin{cases} 1, & 0 \leq u < 1, \\ 0, & 1 \leq u, \end{cases}$$

as $\lambda \to \infty$ with $u \geq 0$ fixed.

4. Conclusion

We have obtained strikingly similar results, for both the number of vertices and the size of the largest clique, in two models for random interval graphs, namely those based on the busy periods of the $M/D/\infty$ and $M/M/\infty$ systems. This similarity suggest that these results may be more robust than is shown by these particular cases. In particular, one might try to establish them for the $M/G/\infty$ system with general (but independent) service times. For the distribution of the number of vertices in this case, the work of Kingman [K] might provide an avenue of approach, but for the distribution of the size of the largest clique, we have no suggestion as to how to proceed.

5. References

[B1] B. Bollobás, *Random Graphs*, Academic Press, New York, 1985.

[B2] B. Bollobás, ‘The Chromatic Number of Random Graphs”, *Combinatorica*, 8 (1988) 49–55.

[C1] T. Carleman, *Les fonctions quasi-analytiques*, Gauthier-Villars, Paris, 1925.

[E1] P. Erdős and A. Rényi, “On Random Graphs I”, *Publ. Math. Debrecen*, 6 (1959) 290–297.

[E2] P. Erdős and A. Rényi, “On the Evolution of Random Graphs”, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5 (1960) 17–61.

[F] P. C. Fishburn, *Interval Orders and Interval Graphs*, John Wiley and Sons, New York, 1985.

[G1] E. Godehardt and J. Jaworski, "On the Connectivity of a Random Interval Graph", *Random Structures and Algorithms*, 9 (1996) 137–161.

[G2] G. R. Grimmett and C. J. H. McDiarmid, "On Colouring Random Graphs", *Math. Proc. Cambridge Phil. Soc.*, 77 (1975) 313–324.

[G3] F. Guillemin and A. Simonian, "Transient Characteristics of an $M/M/\infty$ System", *Adv. Appl. Prob.*, 27 (1995) 862–888.

[K] J. F. C. Kingman, "An Application of the Theory of Regenerative Phenomena", *Proc. Cambridge Phil. Soc.*, 68 (1970) 697–701.

[L1] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, 1965.

[N] M. F. Neuts, "The Distribution of the Maximum Length of a Poisson Queue During a Busy Period", *Operations Research*, 12 (1964) 281–285.

[P] E. Palmer, *Graphical Evolution: An Introduction to the Theory of Random Graphs*, John Wiley and Sons, New York, 1985.

[R] F. S. Roberts, "Indifference Graphs", in: F. Harary (Ed.), *Proof Techniques in Graph Theory*, Academic Press, New York, 1969.

[S1] E. R. Scheinerman, "Random Interval Graphs", *Combinatorica*, 8 (1988) 357–371.

[S2] E. R. Scheinerman, "An Evolution of Interval Graphs", *Discrete Math.*, 82 (1990) 287–302.

[T1] L. Takács, "On a Probability Problem Arising in the Theory of Counters", *Proc. Cambridge Phil. Soc.*, 52 (1956) 488–498.

[W] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, London, 1927.

Appendix: Interval Graphs versus Indifference Graphs

In this appendix we shall show that the probability that an interval graph arising from the $M/M/\infty$ system is an indifference graph tends to zero as λ tends to infinity. This may be contrasted with the fact that a graph arising from the $M/D/\infty$ system is always an indifference graph. Roberts [R] has shown that an interval graph is an indifference graph if and only if it does not contain $K_{1,3}$ (in which one vertex, the center, is joined by edges to three other vertices, the tips, among which there are no further edges) as an induced subgraph. We shall show that for the $M/M/\infty$ system, any particular service interval (for example, the one that starts a busy period) intersects, with probability tending to 1 as λ tends to infinity, three other service intervals that are pairwise disjoint among themselves.

Let I_0 be a service interval in a realization of the $M/M/\infty$ process. We shall describe a procedure for finding three service intervals I_1, I_2 and I_3, that each intersect I_0, but which are pairwise disjoint among themselves. At various points we may have to abandon the procedure, but we shall keep track of the probabilities of having to do this, and in the end the sum of these probabilities will be an upper bound on the probability that such intervals do not exist.

Set $\varepsilon = \big((\log \lambda)/\lambda\big)^{1/2}$. First, we shall abandon the procedure unless the length of I_0 is at least 5ε. The probability of abandonment at this point is at most 5ε. Next, we shall let J_1, J_2, J_3, J_4 and J_5 be the first five successive subintervals, each of length ε, in I_0.

Let H be the number of service intervals that begin during J_1. The number H is distributed as a Poisson random variable with mean $\eta = \lambda\varepsilon$ and therefore with variance η. We shall abandon the procedure unless $H \geq \eta/2$. By Chebyshev's inequality, the probability of abandonment at this point is at most $4/\eta = 4/(\lambda \log \lambda)^{1/2}$. Next we shall seek one of these H service intervals that terminates before the end of J_2, and we shall call this interval I_1. We shall abandon the procedure unless there is at least one such interval. The probability of abandonment at this point is at most $e^{-\varepsilon\eta/2} = 1/\lambda^{1/2}$, since each of at least $\eta/2$ service intervals must continue though J_2 if we fail to find I_1. Thus we find, except with probability at most $4/(\lambda \log \lambda)^{1/2} + 1/\lambda^{1/2}$, a service interval $I_1 \subseteq J_1 \cup J_2$. By repeating this argument we find, except with probability at most $4/(\lambda \log \lambda)^{1/2} + 1/\lambda^{1/2}$, a service interval $I_2 \subseteq J_3 \cup J_4$. Finally, by repating the first part of this argument we find, except with probability at most $4/(\lambda \log \lambda)^{1/2}$, a service interval I_3 that begins in J_5 (in this case we do not care when it ends).

Unless we abandoned the procedure at some point, we found intervals I_1, I_2 and I_3, that each intersect I_0, but which are pairwise disjoint among themselves. Thus sum of the probabilities of abandonment is at most $5\big((\log \lambda)/\lambda\big)^{1/2} + 12/(\lambda \log \lambda)^{1/2} + 2/\lambda^{1/2} = O\Big(\big((\log \lambda)/\lambda\big)^{1/2}\Big)$. Thus the probability that the resulting graph does not contain a forbidden subgraph for indifference graphs tends to zero as λ tends to infinity.