Department of Computer Science
University of British Columbia
2366 Main Mall
Vancouver, B.C. Canada V6T 1Z4
e-mail: gilmore@cs.ubc.ca

# SOUNDNESS & CUT-ELIMINATION for NaDSyL

by
Paul C Gilmore

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## ABSTRACT

NaDSyL, a Natural Deduction based Symbolic Logic, like some earlier logics, is motivated by the belief that a confusion of use and mention is the source of the set theoretic paradoxes. However NaDSyL differs from the earlier logics in several important respects.

"Truth gaps", as they have been called by Kripke, are essential to the consistency of the earlier logics, but are absent from NaDSyL; the law of the excluded middle is derivable for all the sentences of NaDSyL. But the logic has an undecidable elementary syntax, a departure from tradition that is of little importance, since the semantic tree presentation of the proof theory can incorporate the decision process for the elementary syntax.

The use of the lambda calculus notation in NaDSyL, rather than the set theoretic notation of the earlier logics, reflects much more than a change of notation. For a second motivation for NaDSyL is the provision of a higher order logic based on the original term models of the lambda calculus rather than on the Scott models. These term models are the "natural" intepretation of the lambda calculus for the naive nominalist view that justifies the belief in the source of the paradoxes. They provide the semantics for the first order domain of the second order logic NaDSyL.

The elementary and logical syntax or proof theory of NaDSyL is fully described, as well as its semantics. Semantic proofs of the soundness of NaDSyL with cut and of the completeness of NaDSyL without cut are given. That cut is a redundant rule follows form these results. Some applications of the logic are also described.

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# 1. INTRODUCTION

This report provides an introduction to the semantics and proof theory of the logic NaDSyL. Semantic proofs of the soundness of NaDSyL with cut and of its completeness without cut are provided from which follows the redundancy of cut. A sketch of some of the applications of the logic is also given. [Gilmore97a] provides an abbreviated introduction to the logic with proofs of soundness and completeness with cut but not of the redundancy of cut, while [Gilmore97b] describes the applications more fully.

NaDSyL, a Natural Deduction based Symbolic Logic, like the logics described in [Gilmore71,80,86], is motivated by the belief that a confusion of use and mention is the source of the set theoretic paradoxes, a view also expressed in [Sellars63a,63b]. NaDSyL differs from the earlier logics in several important respects.

"Truth gaps", as they were called in [Kripke75], are essential to the consistency of the earlier logic, but are absent from NaDSyL; the law of the excluded middle is derivable for all the sentences of NaDSyL. But the logic has an undecidable elementary syntax, a departure from tradition that is of little importance since the semantic tree presentation of the proof theory can incorporate the decision process for the elementary syntax.

The use of the lambda calculus notation in NaDSyL, rather than the set theoretic notation of the earlier logics, reflects much more than a change of notation. For a second motivation for NaDSyL is the provision of a higher order logic based on the original term models of the lambda calculus rather than on the Scott models described for example in [Barendregt84]. These term models are the "natural" intepretation of the lambda calculus for the naive nominalist view of the logic that justifies the belief in the source of the paradoxes and that is sketched in [Gilmore80] . They provide the semantics for the first order domain of the second order logic NaDSyL.

Incidentally, that the source of the paradoxes is a confusion of use and mention is more competently argued in [Sellars63a,63b]. This view of the source of the paradoxes may have some relevance for logic programming. In §2 of [Nadathur&Miller94], titled "Motivating a Higher-Order Extension to Horn Clauses", predicate variables appearing in "extensional" positions in atomic formulas are distinguished from those appearing in "intensional" positions. For the latter, values can be found by a structural analysis. The distinction between extensional and intensional uses is exactly that of use and mention.

## 1.1. Summary of Paper

The elementary syntax is defined in §2 in two stages. First the syntax for an extended lambda calculus is defined as a set $\mathbb{S}$ of strings of characters and a lambda reduction relation $\geqslant$ is then defined on $\mathbb{S}$. The Church-Rosser theorem for $\geqslant$ over $\mathbb{S}$ is stated without proof. Then the set $\mathbb{F}$ of formulas of the logic is defined as a subset of $\mathbb{S}$. Although $\mathbb{S}$ includes strings that are not formulas of the pure lambda calculus, a proof that $\mathbb{F}$ is not a decidable subset of $\mathbb{S}$ is sketched in §2.5 based on the concept of head normal form [Barendregt84]. A discussion of the significance of this result is deferred to §4.4.

The semantics for the logic is described in §3. Parameters are used in the logic in place of free quantification variables. The first order domain $\mathbb{d}$ for interpretations and models of NaDSyL consists of the members of $\mathbb{S}$ in which no parameter occurs and no variable has a free occurrence; $\mathbb{d}$ is closed under $\gg$.

The logical syntax or proof theory, described in §4, is presented as a theory of semantic trees which is the tree version of the original semantic tableaux of [Beth55]. The close connection with the Gentzen sequent calculus [Gentzen34-35] is apparent: Derivations in NaDSyL are derivations of Gentzen sequents. Another tree version of semantic tableaux is described in [Smullyan68]. In §5 semantic proofs of the soundness of the logic with cut and its completeness without cut are provided from which the redundancy of cut follows.

### 1.2. Related Work

The single most important inspiration for [Gilmore71] was §21 of [Church41], where a logic is defined within the $\lambda$–$\delta$ calculus. Related papers cited in [Fitch52] and in [Schütte60,77] suggest that they had a similar inspiration. The logics described in [Gilmore71,80,86], as well as NaDSyL, differs from those of Fitch and Schütte in allowing some second order terms to also be first order terms. Those second order terms that are also first order are those for which the distinction between use and mention can be maintained; see §2.4 where the set $\mathbb{t}$ of first order terms is defined.

Theories described in [Cocchiarella79,85] also allow some second order terms to be first order. How NaDSyL relates to the these theories is complicated by their presentation as axiomatic theories, in contrast to the natural deduction presentation of NaDSyL. One of the motivations for the presentation of NaDSyL is the desire to treat abstraction in the same manner as logical connectives and quantification.

The most recent related work is that described in [Apostoli94,95]. The logic G described in [Apostoli94] was motivated by the first order logic NaDSet 1 described in [Gilmore86]. G remains first order but, unlike NaDSet 1, arithmetic can be formalized within it, although with the addition of axioms that are instances of the law of the excluded middle. [Apostoli95] describes another first order theory LPL based on the theory of pairs and formalized in the manner of NaDSet 1 and with a similarly defined semantics. However again unlike NaDSyL 1, LPL has axioms; one of its axiom schemes asserts for example that the representative for the natural number 0 is not an ordered pair. The paper [Apostoli&Kanda96] argues for the importance of the logic LPL for computer science, while the monograph [Apostoli&Kanda97] argues that LPL is a consistent replacement for Frege's inconsistent logic.

### 1.3. Some Applications of NaDSyL

Here a sketch of the existing and proposed contents of the monograph [Gilmore97b] will be described.

Chapter 1 is a leisurely presentation of the contents of this paper together with some additional topics. These include other formulations of the logic including a natural deduction presentation in the style of [Prawitz65] for an intutionistic version of the logic.

The sparce notation of NaDSyL as it is described in this paper is not suitable for many of its applications. Chapter 2 describes how the logic can be extended by definitions of intensional and extensional identity, ordered pairs and the natural numbers. These definitions permit the formalization of second order arithmetic within NaDSyL, in both its classical and intuitionistic forms.

Also in Chapter 2 a notation for partial first order functions is added, with defined domains as their "type". Two rules of deduction are introduced for reasoning about partial functions. They provide a conservative extension of NaDSyL. An advantage of the semantic tree presentation of the proof theory of NaDSyL is demonstrated here. Although the partial function notation used is closely related to definite descriptions, the "waste cases" that complicate the formalization of definite descriptions in [Quine51] can be ignored in NaDSyL.

Since axiomatic theories play such a large role in mathematics, Chapter 2 describes how such theories can be formalized within NaDSyL. The results of [Gilmore&Tsiknis93a], where a formalization of category theory in an earlier unsuccessful logic NaDSet were described, are revisited and revised for NaDSyL.

The last topic dealt with in Chapter 2 is Cantor's diagonal argument. A rule of deduction is derived which distinguishes between correct and incorrect uses of Cantor's diagonal argument within NaDSyL. Cantor's use to prove that there are more subsets of the natural numbers than there are numbers cannot be justified by the rule. But non-controversial uses can be; for example, the use of the argument to prove that the Turing computable real numbers cannot be enumerated by a Turing machine, can be justified.

An important advantage of a logic like NaDSyL, over a logic in which an axiom of infinity must be added, is the ease with which recursively defined sets can be defined and reasoned about. This is demonstrated in Chapter 3. There very general methods are developed in parallel for defining recursively both well-founded and non-well-founded sets. This is followed by some applications in this and later chapters. One demonstration of this advantage is given in [Gilmore&Tsiknis93b] where the semantics for a programming language is defined in the unsuccessful logic NaDSet; the results of that paper are repeated and enlarged for NaDSyL. The lambda notation of NaDSyL and the semantic tree form of its proof theory makes the results easier to state and prove.

Sets of domain equations can be used to provide a denotational description of programming languages [Gordon79]. In [Gilmore&Tsiknis92] solutions of such equations are defined in NaDSet. The results are repeated and extended for NaDSyL.

Further topics to be discussed include grammars, temporal logics, and applications of non-well-founded sets. It is expected that one chapter will be devoted to a description of an interactive computer system for assisting in the construction of NaDSyL derivations.

In light of [Nadathur&Miller94], higher-order logic programming using NaDSyL in place of the version of Church's Simple Theory of Types described in [Andrews71] will be explored.

### 1.4. Acknowledgements

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## 2. ELEMENTARY SYNTAX

Seven different denumerable sets of strings of characters form the basis for the elementary syntax of NaDSyL: Constants, quantification variables, and parameters, both first and second order, and abstraction variables. As noted before, parameters play the role of free quantification variables. The characters from which the strings are formed consist of upper and lower case Latin letters and the numerals. The particular strings that are members of these sets and the notation used to name the sets is described in §2.1. In §2.2 the set $\mathbb{S}$ of strings is defined using additional primitive characters and a substitution operator defined for it. A reduction relation $\gtrdot$ between members of $\mathbb{S}$ is defined in §2.3. Finally in §2.4 the formulas and terms of NaDSyL are defined. A sketch is given in §2.5 of a proof that the set $\mathbb{F}$ of formulas of NaDSyL is undecidable.

### 2.1. Notation for Constants, Variables, & Parameters

The following notation is used for the sets of strings used in the elementary syntax of NaDSyL: $\mathbb{c}$, $\mathbb{qv}$, and $\mathbb{p}$ are the sets of first order constants, quantification variables, and parameters. $\mathbb{C}(n)$, $\mathbb{QV}(n)$, and $\mathbb{P}(n)$ are the corresponding sets for the second order terms of arity n, n ≥ 0, and $\mathbb{C}$, $\mathbb{QV}$, and $\mathbb{P}$ are the sets of all second order constants, quantification variables, and parameters of any arity. $\mathbb{av}$ is the set of abstraction variables used with the λ abstraction operator. Additional primitive symbols are '(', ')', '.', '[', ']', '↓', 'λ', and '∀'. '↓' is the joint denial logical operator, for which [**F**↓**G**] is <u>true</u> if and only if both **F** and **G** are false. This single logical connective is used in order to reduce the number of cases that have to be considered in many proofs.

The strings that are members of $\mathbb{c}$, $\mathbb{p}$, and $\mathbb{qv}$ begin with lower case letters, respectively, one of 'a' through 'c', one of 'p', 'q', and 'r', and with one of 'x', 'y', or 'z'. The strings that are members of $\mathbb{av}$ begin with one of the letters 'u', 'v', and 'w'.

The strings that are members of $\mathbb{C}(n)$, $\mathbb{P}(n)$, and $\mathbb{QV}(n)$ begin with upper case letters, respectively, one of 'A' through 'C', one of 'P', 'Q', and 'R', and with one of 'X', 'Y', or 'Z'. It is assumed that each occurrence of one of these strings has an arity associated with it, although the arity need not be explicitly indicated, but may be inferred from its context.

### 2.2. The Set $\mathbb{S}$

In the following inductive definition of the set of strings $\mathbb{S}$, bold letters **R** and **S** are used as variables over $\mathbb{S}$, and **v** and **x** as variables over respectively $\mathbb{av}$ and $\mathbb{qv} \cup \mathbb{QV}$:

***Definition of $\mathbb{S}$***

1. $\mathbb{c} \cup \mathbb{p} \cup \mathbb{qv} \cup \mathbb{av} \cup \mathbb{C} \cup \mathbb{P} \cup \mathbb{QV} \subseteq \mathbb{S}$.

An occurrence of a variable from $\mathbb{qv} \cup \mathbb{av} \cup \mathbb{QV}$ is a *free* occurrence in itself.

2. $\mathbf{R}, \mathbf{S} \in \mathbb{S} \Rightarrow (\mathbf{R}.\mathbf{S}) \in \mathbb{S}$

   A free occurrence of a variable in **R** or **S** is a *free* occurrence in $(\mathbf{R}.\mathbf{S})$.
3. $\mathbf{R}, \mathbf{S} \in \mathbb{S} \Rightarrow [\mathbf{R}\!\downarrow\!\mathbf{S}] \in \mathbb{S}$

   A free occurrence of a variable in **R** or **S** is a *free* occurrence in $[\mathbf{R}\!\downarrow\!\mathbf{S}]$.
4. $\mathbf{R} \in \mathbb{S}$ & $\mathbf{v} \in \mathbb{av} \Rightarrow (\lambda\mathbf{v}.\mathbf{R}) \in \mathbb{S}$

   A free occurrence of a variable other than **v** in **R** is a *free* occurrence in $(\lambda\mathbf{v}.\mathbf{R})$. A free occurrence of **v** in **R** is an occurrence *bound by* the *abstraction operator* $\lambda\mathbf{v}$ in $(\lambda\mathbf{v}.\mathbf{R})$. **R** is the *scope* of the abstraction operator.
5. $\mathbf{R} \in \mathbb{S}$ & $\mathbf{x} \in \mathbb{qv} \cup \mathbb{QV} \Rightarrow (\forall\mathbf{x}.\mathbf{R}) \in \mathbb{S}$

   A free occurrence of a variable other than **x** in **R** is a *free* occurrence in $(\forall\mathbf{x}.\mathbf{R})$. A free occurrence of **x** in **R** is an occurrence *bound by* the *quantifier* $\forall\mathbf{x}$ in $(\forall\mathbf{x}.\mathbf{R})$. **R** is the *scope* of the quantifier.

***End of definition***

The functional application notation $(\mathbf{R}.\mathbf{S})$ of the λ calculus does not conform with the more usual notation of the predicate logic. However, the latter notation can be introduced by an abbreviating definition: For $\mathbf{R}, \mathbf{S}_1, \dots, \mathbf{S}_n \in \mathbb{S}$

$\mathbf{R}(\mathbf{S}_1, \dots, \mathbf{S}_n)$ **for** $(\dots((\mathbf{R}.\mathbf{S}_1).\mathbf{S}_2) \dots .\mathbf{S}_n)$

Multiple abstraction operators can be similarly abbreviated:

$(\lambda\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n.\mathbf{R})$ **for** $(\lambda\mathbf{u}_1.(\lambda\mathbf{u}_2. \dots .(\lambda\mathbf{u}_n.\mathbf{R}) \dots))$

When there is no risk of confusion, parenthesis '(' and ')' in contexts $(\lambda\mathbf{v}.\mathbf{R})$ and $(\forall\mathbf{x}.\mathbf{R})$ will be omitted.

Let **u** be any variable; that is a member of $\mathbb{av} \cup \mathbb{qv} \cup \mathbb{QV}$. Let **v** be a variable that is a member of the same set as **u**, and of the same arity as **u** if $\mathbf{u} \in \mathbb{QV}$. The variable **v** is said to be *free to replace* **u** in $\mathbf{S} \in \mathbb{S}$ provided no free occurrence of **u** in **S** is within the scope of an abstraction operator $\lambda\mathbf{v}$ or of a quantifier $\forall\mathbf{v}$. An $\mathbf{R} \in \mathbb{S}$ in which no variable has a free occurrence is said to be *closed*. $c\mathbb{S}$ is the set of all closed members of $\mathbb{S}$.

### 2.2.1. *Substitution*

It is assumed that there is an enumeration of each of the sets $\mathbb{av}$, $\mathbb{qv}$, and $\mathbb{QV}$ so that it is meaningful to speak of the first member of such a set with a given property.

Let **u** be any variable and let $\mathbf{R1}, \mathbf{S1} \in \mathbb{S}$. The substitution operator $[\mathbf{R1}/\mathbf{u}]$ when applied to **S1** produces a member $[\mathbf{R1}/\mathbf{u}]\mathbf{S1}$ of $\mathbb{S}$ defined as follows:

***Definition of Substitution Operator***

1. $[\mathbf{R1}/\mathbf{u}]\mathbf{S1}$ is **R1** if **S1** is **u**;

   **S1** if **S1** is a variable other than **u** or a member of $\mathbb{c} \cup \mathbb{p} \cup \mathbb{C} \cup \mathbb{P}$.
2. $[\mathbf{R1}/\mathbf{u}](\mathbf{R}.\mathbf{S})$ is $([\mathbf{R1}/\mathbf{u}]\mathbf{R}.[\mathbf{R1}/\mathbf{u}]\mathbf{S})$

3. $[\mathbf{R1}/\mathbf{u}][\mathbf{R}{\downarrow}\mathbf{S}]$ is $[[\mathbf{R1}/\mathbf{u}]\mathbf{R}{\downarrow}[\mathbf{R1}/\mathbf{u}]\mathbf{S}]$
4. $[\mathbf{R1}/\mathbf{u}](\lambda\mathbf{v}.\mathbf{R})$ is $(\lambda\mathbf{v}.\mathbf{R})$ if **u** is **v**;
   $(\lambda\mathbf{v}.[\mathbf{R1}/\mathbf{u}]\mathbf{R})$ if **v** has no free occurrences in **R1**; and
   $(\lambda\mathbf{w}.[\mathbf{R1}/\mathbf{u}][\mathbf{w}/\mathbf{v}]\mathbf{R})$ if **v** has a free occurrence in **R1**; here **w** is the first abstraction variable without a free occurrence in **R1** that is free to replace **v** in **R**.
5. $[\mathbf{R1}/\mathbf{u}](\forall\mathbf{x}.\mathbf{R})$ is $(\forall\mathbf{x}.\mathbf{R})$ if **u** is **x**;
   $(\forall\mathbf{x}.[\mathbf{R1}/\mathbf{u}]\mathbf{R})$ if **x** has no free occurrences in **R1**; and
   $(\forall\mathbf{y}.[\mathbf{R1}/\mathbf{u}][\mathbf{y}/\mathbf{x}]\mathbf{R})$ if **x** has a free occurrence in **R1**; here **y** is the first quantification variable of the same order and arity as **x** without a free occurrence in **R1** that is free to replace **x** in **R**.

For $\mathbf{p} \in \mathbb{p} \cup \mathbb{P}$ and $\mathbf{R} \in \mathrm{c}\mathbb{S}$, $[\mathbf{R}/\mathbf{p}]\mathbf{S}$ is defined to be the string obtained from $\mathbb{S}$ by replacing each occurrence of **p** by **R**.

***End of definition***

## 2.3. Reductions

A relation $\geqslant$ between members of $\mathbb{S}$ is defined inductively as follows:

***Definition of*** $\geqslant$

1. $\mathbf{R} \geqslant \mathbf{S}$, when **S** is an α-, β-, or η-contractum of **R**, where
   .1. $(\lambda\mathbf{u}.[\mathbf{u}/\mathbf{v}]\mathbf{S})$ is an *α-contractum* of $(\lambda\mathbf{v}.\mathbf{S})$, provided **u** is free to replace **v** in **S**, and $(\forall\mathbf{y}.[\mathbf{y}/\mathbf{x}]\mathbf{S})$ is an *α-contractum* of $(\forall\mathbf{x}.\mathbf{S})$, provided **y** is free to replace **x** in **S**.
   .2. $[\mathbf{R}/\mathbf{v}]\mathbf{S}$ is a *β-contractum* of $(\lambda\mathbf{v}.\mathbf{S})(\mathbf{R})$.
   .3. **S** is an *η-contractum* of $(\lambda\mathbf{v}.\mathbf{S}(\mathbf{v}))$, provided **v** has no free occurrence in **S**.
2. Let $\mathbf{R} \geqslant \mathbf{S}$:
   .1. $\mathbf{T} \in \mathbb{S} \Rightarrow (\mathbf{T}.\mathbf{R}) \geqslant (\mathbf{T}.\mathbf{S})$ & $(\mathbf{R}.\mathbf{T}) \geqslant (\mathbf{S}.\mathbf{T})$.
   .2. $\mathbf{T} \in \mathbb{S} \Rightarrow [\mathbf{R}{\downarrow}\mathbf{T}] \geqslant [\mathbf{S}{\downarrow}\mathbf{T}]$ & $[\mathbf{T}{\downarrow}\mathbf{R}] \geqslant [\mathbf{T}{\downarrow}\mathbf{S}]$.
   .3. $(\lambda\mathbf{v}.\mathbf{R}) \geqslant (\lambda\mathbf{v}.\mathbf{S})$.
   .4. $(\forall\mathbf{x}.\mathbf{R}) \geqslant (\forall\mathbf{x}.\mathbf{S})$

***End of definition***

The α-, β-, and η-contractum terminology is that of [Curry58]; see also [Barendregt84]. That $\forall\mathbf{y}.[\mathbf{y}/\mathbf{x}]\mathbf{S}$ is an *α-contractum* of $\forall\mathbf{x}.\mathbf{S}$ is not part of the lambda calculus tradition but has been added because it too results from a change of bound variable.

Clause (1) of the definition defines a replacement of all of **R** with **S**; **R** can be said to be a part of **R** at depth 0. Repetitions of the inductive steps (2.1) - (2.4) has the effect of replacing a part of greater depth in **R** by an α-, β-, or η-contractum of it.

Two elementary properties of the $>$ relation follow from its definition:

***Lemma 2.3***

1. If **R** $\in$ c$\mathbb{S}$ and **R** $>$ **S**, then **S** $\in$ c$\mathbb{S}$.
2. Let **x** $\in \mathbb{qv}$, **t** $\in$ c$\mathbb{S}$, and **R** $>$ **S**. Then [**t**/**x**]**R** $>$ [**t**/**x**]**S**.

### 2.3.1. *The Relation $\gg$ and the Church-Rosser Theorem*

The $\gg$ relation on $\mathbb{S}$ is the transitve closure of $>$:

1. **R** $\gg$ **R**, for **R** $\in \mathbb{S}$; and
2. **R** $\gg$ **S** and **S** $>$ **T** => **R** $\gg$ **T**, for **R**, **S**, **T** $\in \mathbb{S}$.

The most important property of the $\gg$ relation is expressed in the following theorem:

***Theorem (Church-Rosser)***

Let **R** $\gg$ **S** and **R** $\gg$ **T**. Then there is an **R**' for which **S** $\gg$ **R**' and **T** $\gg$ **R**'.

The theorem gets its name from the paper [Church&Rosser36] where a similar result for the pure lambda calculus was first established. Other proofs are given or cited in [Curry58] or in [Barendregt84] and may be adapted to the extended syntax of $\mathbb{S}$.

For **R**, **S** $\in \mathbb{S}$, **R**$\equiv$**S** is defined to mean that **R** $\gg$ **T** and **S** $\gg$ **T** for some **T** $\in \mathbb{S}$. The notation for the relation is justified by the following corollary to the theorem:

***Corollary***

The relation $\equiv$ is an equivalence relation on $\mathbb{S}$.

***Proof of Corollary***

That $\equiv$ is reflexive and symmetric follows immediately from its definition. That it is also transitive follows from the Church-Rosser theorem. For let **R**$\equiv$**S** and **S**$\equiv$**T**. Then for some $\mathbf{T}_1$ and $\mathbf{T}_2$, **R** $\gg \mathbf{T}_1$, **S** $\gg \mathbf{T}_1$, **S** $\gg \mathbf{T}_2$, and **T** $\gg \mathbf{T}_2$. Therefore by the theorem there is a $\mathbf{T}_3$ for which $\mathbf{T}_1 \gg \mathbf{T}_3$, and $\mathbf{T}_2 \gg \mathbf{T}_3$. But since $\gg$ is transitive, it follows that **R** $\gg \mathbf{T}_3$ and **T** $\gg \mathbf{T}_3$; that is **R** $\equiv$ **T**.

***End of proof***

## 2.4. Definitions of Formulas, Degrees, & Terms

The set $\mathbb{t}$ of *first order terms* is the set of **R** $\in \mathbb{S}$ for which no member of $\mathbb{P}$ has an occurrence in **R** and no member of $\mathbb{QV}$ has a free occurrence. Since first order terms are understood to be *mentioned* and thus to be implicitly within single quotes, this restriction on the membership of $\mathbb{t}$ is necessary to avoid the error described in footnote 136 of [Church56].

An *atomic formula* is a string **CPV**($\mathbf{t}_1$, ... , $\mathbf{t}_n$) for which

**CPV** $\in \mathbb{C}(n) \cup \mathbb{P}(n) \cup \mathbb{QV}(n) \cup \mathbb{av}$, and $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathbb{t}$, for $n \geq 0$. $\mathbb{AF}$ is the set of atomic formulas. The set $\mathbb{F}$ of *formulas* is defined inductively:

***Definition of*** $\mathbb{F}$

1. $\mathbb{AF} \subset \mathbb{F}$.
2. $\mathbf{F}, \mathbf{G} \in \mathbb{F} \Rightarrow [\mathbf{F}\!\downarrow\!\mathbf{G}] \in \mathbb{F}$.
3. $[\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n) \in \mathbb{F}$ & $\mathbf{v} \in \mathbb{av} \Rightarrow (\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n) \in \mathbb{F}$, for $n \geq 0$.
4. $\mathbf{F} \in \mathbb{F}$ & $\mathbf{x} \in \mathbb{qv} \cup \mathbb{QV} \Rightarrow (\forall\mathbf{x}.\mathbf{F}) \in \mathbb{F}$.

***End of definition***

Clause (3) of this definition ensures that $\mathbb{F}$ is undecidable because of its relationship to head normal form reductions in the lambda calculus [Barendregt84]; this connection is discussed at greater length in §2.5.

The degree deg(**F**) of a formula **F** is defined inductively:

***Definition of*** *deg(**F**)*

1. $\mathbf{F} \in \mathbb{AF} \Rightarrow \deg(\mathbf{F}) = 0$.
2. $\deg(\mathbf{F}) = d1$ & $\deg(\mathbf{G}) = d2 \Rightarrow \deg([\mathbf{F}\!\downarrow\!\mathbf{G}]) = \max\{d1, d2\}+1$.
3. $\deg([\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n)) = d \Rightarrow \deg((\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)) = d+1$.
4. $\deg([\mathbf{p}/\mathbf{x}]\mathbf{F}) = d \Rightarrow \deg(\forall\mathbf{x}.\mathbf{F}) = d+1$, where **p** is a parameter of the same order and arity as **x** that does not occur in **F**.

***End of definition***

The members of the sets $c\mathbb{t}$, $c\mathbb{F}$, and $c\mathbb{AF}$ are the members of the cited sets in which no variable has a free occurrence. *Sentences* are members of $c\mathbb{F}$. The set $c\mathbb{T}(n)$ is the set of $\mathbf{T} \in c\mathbb{S}$ for which $\mathbf{T}(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in c\mathbb{F}$, where $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{p}$, $n \geq 0$, are distinct from each other and from any parameter occurring in **T**. The degree deg(**T**) of $\mathbf{T} \in c\mathbb{T}(n)$ is $\deg(\mathbf{T}(\mathbf{p}_1, \ldots, \mathbf{p}_n))$.

***Lemma 2.4.***

1. Let $\mathbf{px} \in \mathbb{p} \cup \mathbb{qv}$, $\mathbf{t} \in c\mathbb{t}$, and $\mathbf{F} \in \mathbb{F}$. Then $[\mathbf{t}/\mathbf{px}]\mathbf{F} \in \mathbb{F}$.
2. Let $\mathbf{PX} \in \mathbb{P}(n) \cup \mathbb{QV}(n)$ and $\mathbf{T} \in c\mathbb{T}(n)$, for some n, $n \geq 0$, and let $\mathbf{F} \in \mathbb{F}$. Then $[\mathbf{T}/\mathbf{PX}]\mathbf{F} \in \mathbb{F}$.

***Proof***

A proof of (1) by induction on the definition of $\mathbb{F}$ is immediate. A proof of (2) by induction on the definition of $\mathbb{F}$ makes use of (1): If **F** is $\mathbf{PX}(\underline{\mathbf{t}})$, where $\underline{\mathbf{t}}$ is a sequence of n members of $\mathbb{t}$, then $[\mathbf{T}/\mathbf{PX}]\mathbf{F}$ is $\mathbf{T}(\underline{\mathbf{t}})$. Since $\mathbf{T}(\underline{\mathbf{p}}) \in \mathbb{F}$, where $\underline{\mathbf{p}}$ is a sequence of n distinct first order parameters without an occurrence in **T**, $\mathbf{T}(\underline{\mathbf{t}}) \in \mathbb{F}$ by (1). The remaining cases for (2) are immediate.

***End of proof***

## 2.5. $\mathbb{F}$ is Undecidable

Let **St** $\in \mathbb{S}$. **St**' is a *formula part of* **St** under the following circumstances corresponding to the clauses (2) - (4) of the definition of $\mathbb{F}$: **St** is $[\mathbf{F}{\downarrow}\mathbf{G}]$, $(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$, or $(\forall\mathbf{x}.\mathbf{F})$, where $\mathbf{v} \in \mathbb{av}$ and $\mathbf{x} \in \mathbb{qv} \cup \mathbb{QV}$, and **St**' is one of **F** and **G** in the first two case, is $[\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n)$ in the second case, and is **F** in the third. A sequence $\mathbf{St}_0, \mathbf{St}_1, \ldots, \mathbf{St}_n$ of members of $\mathbb{S}$ is a *chain of formula parts* of **St** if **St** is $\mathbf{St}_0$ and $\mathbf{St}_{i+1}$ is a formula part of $\mathbf{St}_i$ for $0 \le i < n$. A *maximal* chain $\mathbf{St}_0, \mathbf{St}_1, \ldots, \mathbf{St}_n$ is one for which $\mathbf{St}_n$, $0 \le n$, has no formula part.

The following lemma follows immediately from these definitions and the definition of $\mathbb{F}$:

***Lemma 2.5.***

Let **St** $\in \mathbb{S}$. Then **St** $\in \mathbb{F}$ if and only if each maximal chain of formula parts of **St** ends in a member of $\mathbb{AF}$.

Should there be a bound on the length of the chains of formula parts of **St**, then it is possible to decide of **St** whether or not it is in $\mathbb{F}$. But there are **St** $\in \mathbb{S}$ for which no such bound exists; for example, $((\lambda u.u(u))(\lambda u.u(u)))$ which is a formula part of itself.

The string $((\lambda u.u(u))(\lambda u.u(u)))$ is a pure lambda calculus term; that is a member of the subset $\mathbb{L}$ of $\mathbb{S}$ defined:

1. $\mathbb{av} \subset \mathbb{L}$
2. $\mathbf{R}, \mathbf{S} \in \mathbb{L} \Rightarrow (\mathbf{R}.\mathbf{S}) \in \mathbb{L}$
3. $\mathbf{R} \in \mathbb{L}$ & $\mathbf{v} \in \mathbb{av} \Rightarrow (\lambda\mathbf{v}.\mathbf{R}) \in \mathbb{L}$

A chain of formula parts of a member **St** of $\mathbb{L}$ is a head reduction path as defined §8.3 of [Barendregt84]. Thus **St** has a head normal form if and only if the chain of formula parts of **St** ends in lambda calculus term that is in head normal form. Since there can be no decision procedure for head normal form, there can be no decision procedure for membership in $\mathbb{F}$.

## 3. SEMANTICS

Preliminary to the definition of models, a definition of an interpretation $\mathbb{I}$ of NaDSyL is given in §3.1. Then in §3.2 a set $\Omega[\mathbb{I}]$ of *signed* sentences $\pm\mathbf{F}$ is defined that records the sentences that are true and false in $\mathbb{I}$. As is the case with the models described in [Henkin50], not all interpretations are models. A definition in §3.3 provides the basis for the definition of models in §3.4..

### 3.1. Interpretations of NaDSyL

The set $\mathbb{d}$ is defined to have as its members the $\mathbf{t} \in \mathrm{c}\mathbb{t}$ in which no parameter occurs. Thus the members of $(\mathrm{c}\mathbb{t}-\mathbb{d})$ are the members of $\mathrm{c}\mathbb{t}$ in which first order parameters occur. Clearly $\mathbb{d}$ is closed under $>$; that is $\mathbf{d} \in \mathbb{d}$ & $\mathbf{d} > \mathbf{d}' \Rightarrow \mathbf{d}' \in \mathbb{d}$.

The set $\mathbb{D}(0)$ has as its members the two truth values <u>true</u> and <u>false</u>. For $n > 0$, $\mathbb{D}(n)$ is the set of functions $f$: $\mathbb{d}^n \to \mathbb{D}(0)$ satisfying the condition: $f(\mathbf{d}_1, \ldots, \mathbf{d}_i, \ldots, \mathbf{d}_n)$ is $f(\mathbf{d}_1, \ldots, \mathbf{d}'_i, \ldots, \mathbf{d}_n)$, whenever $\mathbf{d}_i > \mathbf{d}'_i$ and $\mathbf{d}'_i, \mathbf{d}_1, \ldots, \mathbf{d}_n \in \mathbb{d}$. The effect of this condition is to make $f$ a function of the equivalence classes of $\mathbb{d}$ under $\equiv$.

A *base* $\mathbb{B}$ is a sequence of nonempty sets $\mathbb{B}(n)$ for which $\mathbb{B}(0)$ is $\mathbb{D}(0)$ and $\mathbb{B}(n) \subseteq \mathbb{D}(n)$, for $n > 0$. An *interpretation* with base $\mathbb{B}$ is a pair of functions $\Phi_1$ and $\Phi_2$ for which $\Phi_1$: $\mathbb{d} \cup \mathbb{p} \to \mathbb{d}$, where $\Phi_1[\![\mathbf{d}]\!]$ is $\mathbf{d}$ for $\mathbf{d} \in \mathbb{d}$, and $\Phi_2$: $\mathbb{C}(n) \cup \mathbb{P}(n) \to \mathbb{B}(n)$, $n \geq 0$.

The domain of $\Phi_1$ is extended to include $\mathrm{c}\mathbb{t}$ as follows: For $\mathbf{t} \in \mathrm{c}\mathbb{t}$, $\Phi_1[\![\mathbf{t}]\!]$ is the result of replacing each occurrence of a first order parameter $\mathbf{p}$ in $\mathbf{t}$ by $\Phi_1[\![\mathbf{p}]\!]$. Note that $\mathbf{t} > \mathbf{t}' \Rightarrow \Phi_1[\![\mathbf{t}]\!] > \Phi_1[\![\mathbf{t}']\!]$, for $\mathbf{t}, \mathbf{t}' \in \mathrm{c}\mathbb{t}$. Similarly the definition of $\Phi_2$ is extended: $\Phi_2[\![\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_n)]\!]$ is defined to be $\Phi_2[\![\mathbf{CP}]\!](\Phi_1[\![\mathbf{t}_1]\!], \ldots, \Phi_1[\![\mathbf{t}_n]\!])$ when $\mathbf{CP} \in \mathbb{C}(n) \cup \mathbb{P}(n)$ and $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathrm{c}\mathbb{t}$.

***Lemma 3.1***

If $\mathbf{CP} \in \mathbb{C}(n) \cup \mathbb{P}(n)$, $\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_n, \mathbf{t}_i' \in \mathrm{c}\mathbb{t}$, and $\mathbf{t}_i > \mathbf{t}_i'$ for some i, $1 \leq i \leq n$, then $\Phi_2[\![\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_n)]\!]$ is $\Phi_2[\![\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i', \ldots, \mathbf{t}_n)]\!]$

***Proof***

Let $\Phi_2[\![\mathbf{CP}]\!]$ be $f \in \mathbb{B}(n)$. Then $\Phi_2[\![\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_n)]\!]$ and $\Phi_2[\![\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i', \ldots, \mathbf{t}_n)]\!]$ are respectively $f(\Phi_1[\![\mathbf{t}_1]\!], \ldots, \Phi_1[\![\mathbf{t}_i]\!], \ldots, \Phi_1[\![\mathbf{t}_n]\!])$ and $f(\Phi_1[\![\mathbf{t}_1]\!], \ldots, \Phi_1[\![\mathbf{t}_i']\!], \ldots, \Phi_1[\![\mathbf{t}_n]\!])$. Since $\Phi_1[\![\mathbf{t}]\!] > \Phi_1[\![\mathbf{t}']\!]$, the conclusion follows from the definition of $\mathbb{D}(n)$.

***end of proof***

Let $\mathbb{I}$ be an interpretation with functions $\Phi_1$ and $\Phi_2$, and let $\mathbf{p}$ be a parameter, first or second order. An interpretation $\mathbb{I}^*$ is a ***p** variant* of $\mathbb{I}$ if it has the same base as $\mathbb{I}$ and its functions $\Phi_1^*$ and $\Phi_2^*$ satisfy the conditions:

1. If $\mathbf{p}$ is first order, then $\Phi_2^*$ is $\Phi_2$ and $\Phi_1^*[\![\mathbf{q}]\!]$ differs from $\Phi_1[\![\mathbf{q}]\!]$ only if $\mathbf{q}$ is $\mathbf{p}$.
2. If $\mathbf{p}$ is second order, then $\Phi_1^*$ is $\Phi_1$ and $\Phi_2^*[\![\mathbf{q}]\!]$ differs from $\Phi_2[\![\mathbf{q}]\!]$ only if $\mathbf{q}$ is $\mathbf{p}$.

### 3.2. The set $\Omega[\mathbb{I}]$

An interpretation $\mathbb{I}$ assigns a single truth value $\Phi_2[\![\mathbf{A}]\!]$ to each $\mathbf{A} \in \mathrm{c}\mathbb{AF}$. This assignment will be extended to an assignment of a single truth value $\Phi_2[\![\mathbf{F}]\!]$ to each $\mathbf{F} \in \mathrm{c}\mathbb{F}$. This assignment of truth values is recorded as a set $\Omega[\mathbb{I}]$ of *signed* sentences for which $+\mathbf{F} \in \Omega[\mathbb{I}]$ records that $\Phi_2[\![\mathbf{F}]\!]$ is <u>true</u> and $-\mathbf{F} \in \Omega[\mathbb{I}]$ that $\Phi_2[\![\mathbf{F}]\!]$ is <u>false</u>.

The set $\Omega[\mathbb{I}]$ is defined to be $\cup\{\Omega_k[\mathbb{I}] \mid k \geq 0\}$ where $\Omega_k[\mathbb{I}]$ is defined for $k \geq 0$ as follows:

***Definition of $\Omega_k[\mathbb{I}]$***

1. $\Omega_0[\mathbb{I}]$ is the set of signed atomic sentences $\pm\mathbf{A}$ for which $\Phi_2[\![\mathbf{A}]\!]$ is <u>true</u>, respectively <u>false</u>.
2. Assuming $\Omega_k[\mathbb{I}]$ is defined for all interpretations $\mathbb{I}$, $\Omega_{k+1}[\mathbb{I}]$ consists of all members of $\Omega_k[\mathbb{I}]$ together with the sentences
   .1. $+[\mathbf{F}\downarrow\mathbf{G}]$ for which $-\mathbf{F} \in \Omega_k[\mathbb{I}]$ and $-\mathbf{G} \in \Omega_k[\mathbb{I}]$; and
   $-[\mathbf{F}\downarrow\mathbf{G}]$ for which $+\mathbf{F} \in \Omega_k[\mathbb{I}]$ or $+\mathbf{G} \in \Omega_k[\mathbb{I}]$.
   .2. $\pm(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$ for which $\pm[\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n) \in \Omega_k[\mathbb{I}]$,
   where $n \geq 0$ and $\mathbf{v}$ is an abstraction variable.
   .3. $+\forall\mathbf{x}.\mathbf{F}$ for which $+[\mathbf{p}/\mathbf{x}]\mathbf{F} \in \Omega_k[\mathbb{I}^*]$ for every $\mathbf{p}$ variant $\mathbb{I}^*$ of $\mathbb{I}$; and
   $-\forall\mathbf{x}.\mathbf{F}$ for which $-[\mathbf{p}/\mathbf{x}]\mathbf{F} \in \Omega_k[\mathbb{I}^*]$ for some $\mathbf{p}$ variant $\mathbb{I}^*$ of $\mathbb{I}$,
   where $\mathbf{F}$ is a formula in which at most the quantification variable $\mathbf{x}$ has a free occurrence, and $\mathbf{p}$ is any parameter that does not occur in $\mathbf{F}$ and is of the same order and arity as $\mathbf{x}$.

***End of definition***

***Lemma 3.2***

For each interpretation $\mathbb{I}$,

1. $+\mathbf{F} \notin \Omega[\mathbb{I}]$ or $-\mathbf{F} \notin \Omega[\mathbb{I}]$, and $+\mathbf{F} \in \Omega[\mathbb{I}]$ or $-\mathbf{F} \in \Omega[\mathbb{I}]$, for $\mathbf{F} \in \mathrm{c}\mathbb{F}$.
2. $\pm[\mathbf{r}/\mathbf{x}]\mathbf{F} \in \Omega[\mathbb{I}]$ and $\mathbf{r} > \mathbf{t} \Rightarrow \pm[\mathbf{t}/\mathbf{x}]\mathbf{F} \in \Omega[\mathbb{I}]$, where $\mathbf{F} \in \mathbb{F}$, $\mathbf{x} \in \mathbb{qv}$, and $\mathbf{r}, \mathbf{t} \in \mathrm{c}\mathbb{t}$.

***Proof***

Since $\deg(\mathbf{F})$ is defined in §2.4.1 for every $\mathbf{F} \in \mathbb{F}$, to prove (1) and (2) it is sufficient to prove by induction on k that if $\deg(\mathbf{F}) \leq k$ then

a) $+\mathbf{F} \notin \Omega_k[\mathbb{I}]$ or $-\mathbf{F} \notin \Omega_k[\mathbb{I}]$,
b) $+\mathbf{F} \in \Omega_k[\mathbb{I}]$ or $-\mathbf{F} \in \Omega_k[\mathbb{I}]$, and
c) $\pm[\mathbf{r}/\mathbf{x}]\mathbf{F} \in \Omega_k[\mathbb{I}]$ and $\mathbf{r} > \mathbf{t} \Rightarrow \pm[\mathbf{t}/\mathbf{x}]\mathbf{F} \in \Omega_k[\mathbb{I}]$.

That each of (a), (b), and (c) hold when k=0 is immediate. Let them now (i) hold for all $\mathbf{F}$ and all $\mathbb{I}$ when $\deg(\mathbf{F}) \leq k$, and consider an $\mathbf{F}$ for which $\deg(\mathbf{F}) = k+1$. The three cases to be considered are: $\mathbf{F}$ is $[\mathbf{G}\downarrow\mathbf{H}]$, where $\deg(\mathbf{G})$, $\deg(\mathbf{H}) \leq k$; $\mathbf{F}$ is $(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$, where $\deg([\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n)) = k$; and $\mathbf{F}$ is $\forall\mathbf{x}.\mathbf{G}$, where $\deg([\mathbf{p}/\mathbf{x}]\mathbf{G}) = k$ for some parameter **p** that does not occur in **G** and is of the same order and arity as **x**. That each of (a), (b), and (c) hold in these cases can be concluded from the induction assumption.

***End of proof***

### 3.2.1. *Sequents &Satisfaction*

A *sequent* is an expression of the form

a) $\mathbf{F}_1, \ldots, \mathbf{F}_m \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$

where the sentences $\mathbf{F}_i$, $0 \le i \le m$, form the *antecedent* of the sequent and the sentences $\mathbf{G}_j$, $0 \le j \le n$, the *succedent*. Sequents were first introduced in [Gentzen34-5] and for that reason are sometimes called Gentzen sequents. If $\Gamma$ is the sequence $\mathbf{F}_1, \ldots, \mathbf{F}_m$ and $\Theta$ the sequence $\mathbf{G}_1, \ldots, \mathbf{G}_n$, then (a) can be written $\Gamma \vdash \Theta$.

A sequent $\Gamma \vdash \Theta$ is said to be *satisfied by an interpretation* $\mathbb{I}$ if there is a sentence **F** for which $\mathbf{F} \in \Gamma$ and $-\mathbf{F} \in \Omega[\mathbb{I}]$, or $\mathbf{F} \in \Theta$ and $+\mathbf{F} \in \Omega[\mathbb{I}]$.

### 3.3 Definition of $\Phi_2[\![\mathbf{T}]\!]$

Let $\mathbf{T} \in c\mathbb{T}(n)$, $n \ge 0$, and $\mathbb{I}$ be an interpretation with functions $\Phi_1$ and $\Phi_2$. Let $\underline{\mathbf{d}}$ be a sequence of n members of $\mathbb{d}$. By (1) of lemma 2.4, $\mathbf{T}(\underline{\mathbf{d}})$ is a sentence so that by (1) of lemma 3.2 exactly one of $+\mathbf{T}(\underline{\mathbf{d}})$ and $-\mathbf{T}(\underline{\mathbf{d}})$ is in $\Omega[\mathbb{I}]$. Define $\Phi_2[\![\mathbf{T}]\!]$ to be the function $f$ for which $f(\underline{\mathbf{d}})$ is <u>true</u>, respectively <u>false</u>, if $+\mathbf{T}(\underline{\mathbf{d}})$, respectively $-\mathbf{T}(\underline{\mathbf{d}})$, is in $\Omega[\mathbb{I}]$.
Thus $\Phi_2[\![\mathbf{T}]\!] \in \mathbb{D}(n)$ by (2) of lemma 3.2.

***Lemma 3.3***

Let **H** be a formula in which at most the quantification variable **x** has a free occurrence and let **p** be a parameter of the same order and arity as **x** not occurring in **H**. Let $\mathbf{t} \in c\mathbb{t}$ if **x** is first order, and $\mathbf{t} \in c\mathbb{T}(n)$ if **x** is second order of arity n. Let $\mathbb{I}^*$ be a **p** variant of an interpretation $\mathbb{I}$ for which $\Phi_1{}^*[\![\mathbf{p}]\!]$ is $\Phi_1[\![\mathbf{t}]\!]$ if **x** is first order, and $\Phi_2{}^*[\![\mathbf{p}]\!]$ is $\Phi_2[\![\mathbf{t}]\!]$ if **x** is second order. Then for $k \ge 0$,

a) $\pm[\mathbf{p}/\mathbf{x}]\mathbf{H} \in \Omega_k[\mathbb{I}^*] \Rightarrow \pm[\mathbf{t}/\mathbf{x}]\mathbf{H} \in \Omega_k[\mathbb{I}]$.

***Proof***

The proof of the lemma will be by induction on k. Let k be 0. If **x** is first order, then **H** takes the form $\mathbf{CP}(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ where **x** may have a free occurrence in $\mathbf{r}_1, \ldots, \mathbf{r}_n \in \mathbb{t}$. In this case $\Phi_2{}^*[\![[\mathbf{p}/\mathbf{x}]\mathbf{H}]\!]$ is $\Phi_2[\![\mathbf{CP}]\!](\Phi_1{}^*[\![[\mathbf{p}/\mathbf{x}]\mathbf{r}_1]\!], \ldots, \Phi_1{}^*[\![[\mathbf{p}/\mathbf{x}]\mathbf{r}_n]\!])$. But $\Phi_1{}^*[\![[\mathbf{p}/\mathbf{x}]\mathbf{r}_i]\!]$ is $[\Phi_1[\![\mathbf{p}]\!]/\mathbf{x}]\Phi_1[\![[\mathbf{r}_i]\!]$, since **p** does not occur in $\mathbf{r}_i$, and this is $[\Phi_1[\![\mathbf{t}]\!]/\mathbf{x}]\Phi_1[\![[\mathbf{r}_i]\!]$, which is $\Phi_1[\![[\mathbf{t}/\mathbf{x}]\mathbf{r}_i]\!]$. Thus $\Phi_2{}^*[\![[\mathbf{p}/\mathbf{x}]\mathbf{H}]\!]$ is $\Phi_2[\![[\mathbf{t}/\mathbf{x}]\mathbf{H}]\!]$. If **x** is second order, then **H** takes the form $\mathbf{x}(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ where $\mathbf{t}_1, \ldots, \mathbf{t}_n \in c\mathbb{t}$, and **t** is a second order parameter or constant **CP**. Thus $\Phi_2{}^*[\![\mathbf{p}]\!]$ is $\Phi_2[\![\mathbf{CP}]\!]$ and $\Phi_1{}^*[\![\mathbf{t}_i]\!]$ is $\Phi_1[\![\mathbf{t}_i]\!]$ from which (a) follows immediately.

Assume now that (a) holds and let $\pm[\mathbf{p}/\mathbf{x}]\mathbf{H} \in \Omega_{k+1}[\mathbb{I}^*]$. Consider first the case that **H** is $[\mathbf{F}{\downarrow}\mathbf{G}]$ and let $+[\mathbf{p}/\mathbf{x}][\mathbf{F}{\downarrow}\mathbf{G}] \in \Omega_{k+1}[\mathbb{I}^*]$. Then $+[\mathbf{p}/\mathbf{x}][\mathbf{F}{\downarrow}\mathbf{G}] \in \Omega_{k+1}[\mathbb{I}^*] \Rightarrow$
$-[\mathbf{p}/\mathbf{x}]\mathbf{F}, -[\mathbf{p}/\mathbf{x}]\mathbf{G} \in \Omega_k[\mathbb{I}^*] \Rightarrow -[\mathbf{t}/\mathbf{x}]\mathbf{F}, -[\mathbf{t}/\mathbf{x}]\mathbf{G} \in \Omega_k[\mathbb{I}] \Rightarrow +[\mathbf{t}/\mathbf{x}][\mathbf{F}{\downarrow}\mathbf{G}] \in \Omega_{k+1}[\mathbb{I}]$.
The arguments for the – case, as well as for the cases when **H** is $(\lambda\mathbf{u}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$, are similar.

Let now **H** be $\forall\mathbf{y}.\mathbf{F}$, where it may be assumed that **x** is distinct from **y** and has a free occurrence in **F**, so that $[\mathbf{p}/\mathbf{x}]\forall\mathbf{y}.\mathbf{F}$ is $\forall\mathbf{y}.[\mathbf{p}/\mathbf{x}]\mathbf{F}$ and $[\mathbf{t}/\mathbf{x}]\forall\mathbf{y}.\mathbf{F}$ is $\forall\mathbf{y}.[\mathbf{t}/\mathbf{x}]\mathbf{F}$. Let **q** be of the same order and arity as **y** and not occur in $[\mathbf{p}/\mathbf{x}]\mathbf{F}$ or in **t**. Then $+\forall\mathbf{y}.[\mathbf{p}/\mathbf{x}]\mathbf{F} \in \Omega_{k+1}[\mathbb{I}^*] \Rightarrow$
$+[\mathbf{q}/\mathbf{y}][\mathbf{p}/\mathbf{x}]\mathbf{F} \in \Omega_k[\mathbb{I}^{**}]$ for every **q** variant $\mathbb{I}^{**}$ of $\mathbb{I}^*$. For each $\mathbb{I}^{**}$ there is a **q** variant $\mathbb{I}^{*'}$ of $\mathbb{I}$ for which $\mathbb{I}^{**}$ is a **p** variant of $\mathbb{I}^{*'}$. Thus by the induction assumption
$+\forall\mathbf{y}.[\mathbf{p}/\mathbf{x}]\mathbf{F} \in \Omega_{k+1}[\mathbb{I}^*] \Rightarrow +[\mathbf{q}/\mathbf{y}][\mathbf{t}/\mathbf{x}]\mathbf{F} \in \Omega_k[\mathbb{I}^{*'}]$ for every **q** variant $\mathbb{I}^{*'}$ of $\mathbb{I}$
$\Rightarrow +\forall\mathbf{y}.[\mathbf{t}/\mathbf{x}]\mathbf{F} \in \Omega_{k+1}[\mathbb{I}]$. The $-\forall$ case can be similarly argued.
***End of proof***

### 3.4. Models & Validity

For every term $\mathbf{t} \in c\mathbb{t}$ and interpretation $\mathbb{I}$, $\Phi_1[\![\mathbf{t}]\!] \in \mathbb{d}$. But although $\Phi_2[\![\mathbf{T}]\!] \in \mathbb{D}(n)$, it does not follow that $\Phi_2[\![\mathbf{T}]\!] \in \mathbb{B}(n)$, where $\{\mathbb{B}(n) \mid n \geq 0\}$ is the base of $\mathbb{I}$. $\mathbb{I}$ is a *model* if $\Phi_2[\![\mathbf{T}]\!] \in \mathbb{B}(n)$ for every $\mathbf{T} \in c\mathbb{T}(n)$, $n > 0$. Clearly there exists a model: Every interpretation for which $\mathbb{B}(n)$ is $\mathbb{D}(n)$ for $n > 0$ is a model. Following [Henkin50] such an intepretation is called a *standard* model.

A sequent $\Gamma \vdash \Theta$ is *valid* if it is satisfied by every model. A *counter-example* for the sequent is a model $\mathbb{M}$ for which $+\mathbf{F} \in \Omega[\mathbb{M}]$ for each $\mathbf{F} \in \Gamma$, and $-\mathbf{F} \in \Omega[\mathbb{M}]$ for each $\mathbf{F} \in \Theta$.

The logical syntax or proof theory of NaDSyL to be defined in §4 can be understood to be a method for attempting the construction of a counter-example for a given sequent. It will be proved in §5 that if the method fails to find a derivation for a given sequent, then a counter-example for the sequent can be constructed.

## 4. LOGICAL SYNTAX

The logical syntax, or proof theory, of NaDSyL defines derivations for sequents. A derivation of a sequent is a finite binary tree with nodes consisting of signed sentences that are related by semantic rules and that satisfy special conditions. The semantic rules are described in §4.1 and the special conditions in §4.2. Some terminology for and transformations of derivations are described in §4.3.

### 4.1. Semantic Rules

$+\downarrow$ $\quad +[\mathbf{F}\downarrow\mathbf{G}]$ $\quad +[\mathbf{F}\downarrow\mathbf{G}]$ $\qquad -\downarrow$ $\quad -[\mathbf{F}\downarrow\mathbf{G}]$

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$-\mathbf{F}$ $\quad -\mathbf{G}$ $\qquad\qquad +\mathbf{F} \quad +\mathbf{G}$

$+\lambda 1$ $\quad +\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_n)$ $\qquad -\lambda 1$ $\quad -\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_n)$

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$+\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i', \ldots, \mathbf{t}_n)$ $\qquad -\mathbf{CP}(\mathbf{t}_1, \ldots, \mathbf{t}_i', \ldots, \mathbf{t}_n)$

where $\mathbf{CP} \in \mathbb{C}(n) \cup \mathbb{P}(n)$, $\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_n, \mathbf{t}_i' \in c\mathbb{t}$, and $\mathbf{t}_i > \mathbf{t}_i'$ with $1 \leq i \leq n$.

$+\lambda 2$ $\quad +(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$ $\qquad -\lambda 2$ $\quad -(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$

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$+[\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n)$ $\qquad -[\mathbf{R}/\mathbf{v}]\mathbf{T}(\mathbf{S}_1, \ldots, \mathbf{S}_n))$

$+\forall$ $\quad +\forall\mathbf{x}.\mathbf{F}$ $\qquad -\forall$ $\quad -\forall\mathbf{x}.\mathbf{F}$

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$+[\mathbf{t}/\mathbf{x}]\mathbf{F}$ $\qquad -[\mathbf{p}/\mathbf{x}]\mathbf{F}$

Here $\mathbf{F} \in \mathbb{F}$, and either $\mathbf{x} \in \mathbb{qv}$, $\mathbf{p} \in \mathbb{p}$ and $\mathbf{t} \in c\mathbb{t}$, or $\mathbf{x} \in \mathbb{QV}(n)$, $\mathbf{p} \in \mathbb{P}(n)$ and $\mathbf{t} \in c\mathbb{T}(n)$. In addition, **p** does not occur in **F** or in any node above the premiss of the application of the rule introducing it. The parameter **p** of an application of $-\forall$ is called the *eigen* parameter, or *e-par*, of the application, and the term **t** of $+\forall$ is called the *eigen* term, or *e-term*, of the application.

The last semantic rule has a character different from the above *logical* rules. It is a rule without premiss and with two conclusions:

Cut

______________

$+\mathbf{F} \qquad -\mathbf{F}$

The sentence **F** is called the *cut* sentence of an application.

It is proved in §5 that the cut rule is not needed; that is, that any derivation of a sequent in which cut is used can be replaced with a derivation in which it is not used. Nevertheless, cut is a useful rule; in [Gilmore97b] it is used in a variety of ways to permit the reuse of previously constructed derivations in the construction of a new derivation.

Although these are the only rules of deduction that will be assumed to exist in this report, rules for the more usual logical connectives ¬, →, ∧, ∨, and ↔ and the existential quantifier ∃ can be derived from the rules ±↓ and ±∀.

## 4.2. Derivations

Given a sequent $\mathbf{F}_1, \ldots, \mathbf{F}_m \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$, a semantic tree *based on* the sequent consists of a tree of signed sentences defined inductively as follows:

***Definition of a Semantic Tree Based on a Sequent***

1. Any tree with a single branch consisting of some nodes $+\mathbf{F}_i$ and $-\mathbf{G}_j$ is a tree based on the sequent.
2. Let τ be a tree based on the sequent, and let τ' be obtained from τ by adding to the end of a branch of τ either
   - .1. a signed sentence that is the single conclusion of an application of one of the rules +↓, ±λ1, ±λ2, or ±∀ with premiss a signed sentence on the given branch; or
   - .2. two signed sentences on separate branches that are the conclusions of cut, or of an application of –↓ with premiss a signed sentence on the given branch.

   Then τ' is a tree based on the sequent.

***End of definition***

Note that not all the sentences in the antecedent or succedent of a sequent need be signed and added as nodes of a semantic tree based on the sequent. The nodes that are so added are called the *initial* nodes of the semantic tree.

A branch of a semantic tree is *closed* if there is an **A** ∈ c$\mathbb{AF}$ for which both +**A** and –**A** are nodes of the branch. A semantic tree is *closed* if each of its branches is closed. A *derivation* of a sequent is a closed semantic tree based on the sequent.

An example derivation is given for the sequent

∀X.[X((λ.w.w(p, a))(λu,v.u))↓∀x.D(x)] ⊢ [∀x.C(x)↓∀x.(λw.w(x))(D)].

| | |
|---|---|
| +∀X.[X((λ.w.w(p, a))(λu,v.u))↓∀x.D(x)] | initial node |
| –[∀x.C(x)↓∀x.(λw.w(x))(D)] | initial node |
| +[C((λ.w.w(p, a))(λu,v.u))↓∀x.D(x)] | +∀ |
| –C((λ.w.w(p, a))(λu,v.u)) | +↓ |
| –C((λu,v.u)(p, a)) | –λ1 |
| –C(p) | –λ1 |
| –∀x.D(x) | +↓ |
| –D(q) | –∀ |

---

| | | | |
|---|---|---|---|
| +∀x.C(x) | –↓ | +∀x.(λw.w(x))(D) | –↓ |
| +C(p) | +∀ | +(λw.w(q))(D) | +∀ |
| ===== | | +D(q) | +λ2 |
| | | ===== | |

Apart from the two initial nodes, the rule cited to the right of a node is the rule of which the node is a conclusion; the premiss for the rule is a node above the conclusion. The double lines at the bottom of the two branches indicate that the branches are closed.

### 4.2.1. Terminology

The derivation above has been illustrated with its root at the top and with its branches spreading downward. The terminology used in discussing trees reflects this orientation. Thus a node $\eta 1$ is *above* a node $\eta 2$ if they are both on the same branch and $\eta 1$ is closer to the root of the tree than $\eta 2$; it is *below* $\eta 2$ if $\eta 2$ is above it. The *height* of a node *on a given branch* is the number of nodes below it on the branch; the *height* of a node in a tree is the maximum of its heights on the branches on which it occurs. A *leaf* node of a tree is a node of height zero.

### 4.2.2. Eliminable Rules

The two $\lambda 1$ rules can be generalized to the following rules:

| | | | |
|---|---|---|---|
| $+\gg$ | $+[\mathbf{r}/\mathbf{x}]\mathbf{F}$ | $-\gg$ | $-[\mathbf{r}/\mathbf{x}]\mathbf{F}$ |
| | - - - - - - | | - - - - - - |
| | $+[\mathbf{t}/\mathbf{x}]\mathbf{F}$ | | $-[\mathbf{t}/\mathbf{x}]\mathbf{F}$ |

where $\mathbf{x} \in \mathbb{qv}$, $\mathbf{r}$, $\mathbf{t} \in \mathrm{c}\mathbb{t}$, and $\mathbf{r} \gg \mathbf{t}$.

These rules are eliminable in the sense that a derivation of a sequent in which they are used can be replaced by a derivation in which they are not used. For let $\pm[\mathbf{t}/\mathbf{x}]\mathbf{F}$ be the conclusion of an application of one of the rules $\pm\gg$. Let $\pm[\mathbf{t}/\mathbf{x}]\mathbf{F}$ be at the same time a premiss of an application of one of the logical rules other than $\pm\lambda 1$. Then $\pm[\mathbf{r}/\mathbf{x}]\mathbf{F}$ can equally well be the premiss of the latter rule when occurrences of $\mathbf{t}$ in its conclusion or conclusions are replaced by $\mathbf{r}$. Then an application of $\pm\gg$ to a single conclusion, or applications of $\pm\gg$ to each of the two conclusions of an application of $-\downarrow$, restores the derivation. In this way applications of the $\pm$ rules can be postponed until they become applications of the $\pm\lambda 1$ rules.

## 4.3. The Undecidability of the Elementary Syntax

A sketch of the undecidability of $\mathbb{F}$ was given in §2.4.3. The undecidability of $\mathbb{F}$ need have surprisingly little effect on the construction of derivations, because of the similarity of the clauses (2), (3), and (4) of the definition of $\mathbb{F}$ in §2.4.1 with the semantic rules $\pm\downarrow$, $\pm\lambda 2$, and $\pm\forall$. These are emphasized in the proof of the following lemma:

***Lemma 4.3***

$\mathbf{F} \vdash \mathbf{F}$ is a derivable sequent for each $\mathbf{F} \in \mathrm{c}\mathbb{F}$.

***Proof***

The lemma will be proved by induction on deg(**F**). If deg(**F**)=0, then **F** ∈ $\mathbb{AF}$ so that if **F** ∈ c$\mathbb{AF}$, then **F** ⊢ **F** has a derivation.

Let now deg(**F**)=k+1. Let **F** be [**G**↓**H**] so that deg(**F**), deg(**F**) ≤ k. By the induction assumption both **G** ⊢ **G** and **H** ⊢ **H** are derivable sequents. A derivation of [**G**↓**H**] ⊢ [**G**↓**H**] can be constructed from derivations of these sequents by one application of –↓ and one application of each of the two +↓ rules.

The other two cases when **F** is (λ**v**.**T**)(**R**, $\mathbf{S}_1$, ... , $\mathbf{S}_n$) and **F** is ∀**x**.**G** can be proved in a similar way. In the latter case [**p**/**x**]**G** ⊢ [**p**/**x**]**G** is derivable since deg([**p**/**x**]**G**)=k; here **p** is a parameter of the same order and arity as **x** that does not occur in **G**, . From a derivation of that sequent, a derivation of the sequent ∀**x**.**G** ⊢ ∀**x**.**G** can be obtained by one application of –∀ with e-par **p** followed by one application of +∀ with e-term **p**.

***end of proof***

The derivation for the sequent **F** ⊢ **F** described in the proof is in essence a method for displaying all the maximal chains of formula parts of **F** as these are defined in §2.4.3. However, for some sequent-like strings not of the form of the law of the excluded middle, it is possible to produce a tree that appear to be derivation but is not. Consider the following example:

i) [C(a)↓(λu.u(u))(λu.u(u))], C(a) ⊢

The following tree records a search for a derivation:

+[C(a)↓(λu.u(u))(λu.u(u))]
+C(a)
–C(a)
====

But this is not a derivation because [C(a)↓(λu.u(u))(λu.u(u))] ∉ c$\mathbb{F}$. Because the derivation applies only the first of the two rules with premiss +[C(a)↓(λu.u(u))(λu.u(u))], it does not test whether all maximal chains of formula parts of [C(a)↓(λu.u(u))(λu.u(u))] terminates in a member of c$\mathbb{AF}$. This simple example illustrates the main consequence of the undecidability of $\mathbb{F}$: A sequent like string may have a derivation when only one of the +↓ rules is applied to a given premiss, even though the string is not a sequent. But if necessary this effect can be compensated for by requiring a proof of membership in c$\mathbb{F}$ for these cases by requiring a derivation for **F** ⊢ **F** for the unused formula parts **F**.

## 5. SOUNDNESS & CUT-ELIMINATION

A semantic proof of the consistency or soundness of NaDSyL is given in §5.1. A proof of the completeness of NaDSyL without the cut rule is given in §5.2 to §5.4. It is an adaptation of the proof for the second order predicate logic given in [Prawitz67]. That cut is a redundant rule of deduction is a corollary of the completeness theorem.

### 5.1. Soundness Theorem

A derivable sequent is valid.

***Proof***

Consider a derivation for a sequent $\Gamma \vdash \Theta$. Let $\eta$ be any node of the derivation which does not have an initial node below it. Define $\Gamma[\eta]$ and $\Theta[\eta]$ to be the sets of sentences **F** for which +**F**, respectively –**F**, is $\eta$ itself or is a node above $\eta$. Thus if $\eta$ is the last of the initial nodes of the derivation, $\Gamma[\eta] \subseteq \Gamma$ and $\Theta[\eta] \subseteq \Theta$; hence if $\Gamma[\eta] \vdash \Theta[\eta]$ is satisfied by a model $\mathbb{M}$, so is $\Gamma \vdash \Theta$.

By induction on the height $h(\eta)$ of $\eta$, $\Gamma[\eta] \vdash \Theta[\eta]$ will be shown to be valid. If $h(\eta)=0$, then $\eta$ is a leaf node of a branch of the derivation. Since the branch is closed, $\Gamma[\eta] \vdash \Theta[\eta]$ is valid. Assume therefore that $h(\eta) > 0$, and that there is a model $\mathbb{M}$ that does not satisfy $\Gamma[\eta] \vdash \Theta[\eta]$. Necessarily $\eta$ is immediately above a conclusion $\eta 1$ or conclusions $\eta 1$ and $\eta 2$ of one of the rules of deduction. There are therefore two main cases to consider corresponding to the single conclusion rules $+\downarrow$, $\pm\lambda 1$, $\pm\lambda 2$, and $\pm\forall$, and to the two conclusion rules $-\downarrow$ and cut.

For the single conclusion rules it is sufficient to illustrate the argument with the second order $\pm\forall$ rules with premiss $\pm\forall\mathbf{X}.\mathbf{F}$ and conclusion respectively $+[\mathbf{T}/\mathbf{X}]\mathbf{F}$ and $-[\mathbf{P}/\mathbf{X}]\mathbf{F}$, where $\mathbf{X} \in \mathbb{QV}(n)$, $\mathbf{T} \in c\mathbb{T}(n)$, and $\mathbf{P} \in \mathbb{P}(n)$ with **P** not occurring in **F**. For the + case $\forall\mathbf{X}.\mathbf{F} \in \Gamma[\eta]$, $\Gamma[\eta 1]$ is $\Gamma[\eta] \cup \{[\mathbf{T}/\mathbf{X}]\mathbf{F}\}$, and $\Theta[\eta 1]$ is $\Theta[\eta]$. Since $\mathbb{M}$ satisfies $\Gamma[\eta 1] \vdash \Theta[\eta 1]$ but does not satisfy $\Gamma[\eta] \vdash \Theta[\eta]$ it follows that $+\forall\mathbf{X}.\mathbf{F}, -[\mathbf{T}/\mathbf{X}]\mathbf{F} \in \Omega[\mathbb{M}]$. Thus for every **P** variant $\mathbb{M}^*$ of $\mathbb{M}$, $+[\mathbf{P}/\mathbf{X}]\mathbf{F} \in \Omega[\mathbb{M}^*]$. Consider the **P** variant for which $\Phi_2^*[\![\mathbf{P}]\!]$ is $\Phi_2[\![\mathbf{T}]\!]$. By lemma 3.3 it follows that $+[\mathbf{T}/\mathbf{X}]\mathbf{F} \in \Omega[\mathbb{M}]$, contradicting $-[\mathbf{T}/\mathbf{X}]\mathbf{F} \in \Omega[\mathbb{M}]$.

For the – case, $\Gamma[\eta 1]$ is $\Gamma[\eta]$, $\forall\mathbf{X}.\mathbf{F} \in \Theta[\eta]$, and $\Theta[\eta 1]$ is $\Theta[\eta] \cup \{[\mathbf{P}/\mathbf{X}]\mathbf{F}\}$. Since $\Gamma[\eta 1] \vdash \Theta[\eta 1]$ is valid, it follows that it is satisfied by every **P** variant $\mathbb{M}^*$ of $\mathbb{M}$. Further, since $\mathbb{M}$ does not satisfy $\Gamma[\eta] \vdash \Theta[\eta]$ it follows that $-[\mathbf{P}/\mathbf{X}]\mathbf{F} \in \Omega[\mathbb{M}]$ while $+[\mathbf{P}/\mathbf{X}]\mathbf{F} \in \Omega[\mathbb{M}]$, since $\mathbb{M}$ is a **P** variant of itself.

Consider now the two conclusion rules. Let the premiss of an application of $-\downarrow$ be $-[\mathbf{F}\downarrow\mathbf{G}]$ and the conclusions +**F** and +**G.** Thus $[\mathbf{F}\downarrow\mathbf{G}] \in \Theta[\eta]$, $\Gamma[\eta 1]$ is $\Gamma[\eta] \cup \{\mathbf{F}\}$, $\Gamma[\eta 2]$ is $\Gamma[\eta] \cup \{\mathbf{G}\}$, $\Theta[\eta 1]$ is $\Theta[\eta]$, and $\Theta[\eta 2]$ is $\Theta[\eta]$. As before it follows that $-\mathbf{F}, -\mathbf{G} \in \Omega[\mathbb{M}]$

and therefore that $+[\mathbf{F}{\downarrow}\mathbf{G}] \in \Omega[\mathbb{M}]$ again leading to a contradiction. For the case of cut let the cut sentence be **F**. In this case $\Gamma[\eta 1]$ is $\Gamma[\eta] \cup \{\mathbf{F}\}$, $\Theta[\eta 1]$ is $\Theta[\eta]$, $\Gamma[\eta 2]$ is $\Gamma[\eta]$, and $\Theta[\eta 2]$ is $\Theta[\eta] \cup \{\mathbf{F}\}$. It follows therefore that $+\mathbf{F}, -\mathbf{F} \in \Omega[\mathbb{M}]$ which is impossible by (1) of lemma 3.2.

***End of proof***

## 5.2. Derivable & Underivable Sets

It is convenient to now represent sequents as sets of signed sentences. The members of a set $\mathbb{Sq}$ representing a sequent consist of all the potential initial nodes of a derivation of the sequent. Thus the sequent $\mathbf{F}_1, \ldots, \mathbf{F}_m \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$ is represented by the set $\{+\mathbf{F}_1, \ldots, +\mathbf{F}_m, -\mathbf{G}_1, \ldots, -\mathbf{G}_n\}$.

A finite set $\mathbb{Sq}$ is said to be *derivable* if there is a derivation with initial nodes selected from the set in which no application of cut appears and in which the eliminable rules $\pm\gg$ are used in place of the $\pm\lambda 1$ rules. An infinite set is *derivable* if a finite subset of it is derivable. A set is said to be *underivable* if it is not derivable. A set of signed sentences is said to be *consistent* if not both $\pm\mathbf{F}$ are members for some sentence **F**, and is said to be *inconsistent* otherwise. An inconsistent set is necessarily derivable since a derivation of $\mathbf{F} \vdash \mathbf{F}$ without its initial nodes can be appended to any branch on which both $\pm\mathbf{F}$ are nodes.

By systematically applying the rules of deduction to a branch of a semantic tree based on a set $\mathbb{Sq}$, a *downward closure* of $\mathbb{Sq}$ can be constructed as defined here:

***Definition of a Downward Closure for a Set $\mathbb{Sq}$***

A set of signed sentences $\mathrm{dc}[\mathbb{Sq}]$ is a *downward closure* of a set $\mathbb{Sq}$ if it satisfies the following conditions:

1. $\mathbb{Sq} \subset \mathrm{dc}[\mathbb{Sq}]$.
2. $\pm[\mathbf{r}/\mathbf{x}]\mathbf{F} \in \mathrm{dc}[\mathbb{Sq}]$ & $\mathbf{r} \gg \mathbf{t} \Rightarrow \pm[\mathbf{t}/\mathbf{x}]\mathbf{F} \in \mathrm{dc}[\mathbb{Sq}]$, when $\mathbf{x} \in \mathbb{qv}$ and $\mathbf{r}, \mathbf{t} \in \mathrm{c}\mathbb{t}$.
3. $+[\mathbf{F}{\downarrow}\mathbf{G}] \in \mathrm{dc}[\mathbb{Sq}] \Rightarrow -\mathbf{F}, -\mathbf{G} \in \mathrm{dc}[\mathbb{Sq}]$
   $-[\mathbf{F}{\downarrow}\mathbf{G}] \in \mathrm{dc}[\mathbb{Sq}] \Rightarrow +\mathbf{F}$ or $+\mathbf{G} \in \mathrm{dc}[\mathbb{Sq}]$
4. $\pm(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \underline{\mathbf{S}}) \in \mathrm{dc}[\mathbb{Sq}] \Rightarrow \pm[\mathbf{R}/\mathbf{v}]\mathbf{T}(\underline{\mathbf{S}}) \in \mathrm{dc}[\mathbb{Sq}]$
5. $+\forall\mathbf{x}.\mathbf{F} \in \mathrm{dc}[\mathbb{Sq}] \Rightarrow +[\mathbf{t}/\mathbf{x}]\mathbf{F} \in \mathrm{dc}[\mathbb{Sq}]$ for all **t** of the same order and arity as **x** with parameters occurring in members of $\mathrm{dc}[\mathbb{Sq}]$.
   $-\forall\mathbf{x}.\mathbf{F} \in \mathrm{dc}[\mathbb{Sq}] \Rightarrow -[\mathbf{p}/\mathbf{x}]\mathbf{F} \in \mathrm{dc}[\mathbb{Sq}]$ for some parameter of the same order and arity as **x** not occurring in **F**.

***End of definition***

***Lemma 5.2***

If a set $\mathbb{Sq}$ is underivable then there is a downward closure dc[$\mathbb{Sq}$] of $\mathbb{Sq}$ that is consistent.

***Proof***

If $\mathbb{Sq}$ is underivable, then any semantic tree based on it must have a branch that cannot be closed no matter how the branch is extended. By systematically applying the logical rules to the nodes of such a branch, a downward closure of $\mathbb{Sq}$ can be constructed. Details will be left to the reader.

***End of proof***

Following [Hintikka55] a consistent downward closure of a set $\mathbb{Sq}$ is called a *model set*. Necessarily if $\mathbb{Sq}$ has a model set, then $\mathbb{Sq}$ is underivable. In the remaining sections an interpretation $\mathbb{I}$[md[$\mathbb{Sq}$]] will first be constructed from a model set md[$\mathbb{Sq}$] of an underivable set $\mathbb{Sq}$; this is then followed by the construction of a model of NaDSyL that is a counter-example for $\mathbb{Sq}$.

## 5.3. An Interpretation Defined from a Model Set

Let md[$\mathbb{Sq}$] be a consistent downward closure for an underivable set $\mathbb{Sq}$. Here an interpretation $\mathbb{I}$[md[$\mathbb{Sq}$]] is defined from md[$\mathbb{Sq}$ ]. To this end the notation $\mathbf{T}^\dagger$ is introduced for any term or formula $\mathbf{T}$.

Let the sets $\mathbb{c}$ and $\mathbb{p}$ be enumerated $c_1$, $c_2$, ... and $p_1$, $p_2$, ... . $\mathbf{T}^\dagger$ is obtained from $\mathbf{T}$ by first replacing each occurrence of $c_i$ by $c_{2i}$, and then replacing each occurrence of $p_i$ by $c_{2i-1}$, for $i \geq 1$. Thus no $\mathbf{p} \in \mathbb{p}$ occurs in $\mathbf{T}^\dagger$. In particular, for $\mathbf{t} \in c\mathbb{t}$, $\mathbf{t}^\dagger \in \mathbb{d}$. Should $\underline{\mathbf{t}} \in c\mathbb{t}^n$ be $\mathbf{t}_1, \ldots, \mathbf{t}_n$, then by $\underline{\mathbf{t}}^\dagger$ is meant $\mathbf{t}_1{}^\dagger, \ldots, \mathbf{t}_n{}^\dagger$, and should also $\underline{\mathbf{r}} \in c\mathbb{t}^n$ by $\underline{\mathbf{r}} \gg \underline{\mathbf{t}}$ is meant $\mathbf{r}_i \gg \mathbf{t}_i$ and by $\underline{\mathbf{r}} \ggg \underline{\mathbf{t}}$ is meant $\mathbf{r}_i \ggg \mathbf{t}_i$ for $1 \leq i \leq n$.

For each $\mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n)$, $n > 0$, a function $f^+[\mathbf{PC}]$: $\mathbb{d}^n \rightarrow \mathbb{D}(0)$ is defined:
$f^+[\mathbf{PC}](\underline{\mathbf{r}}^\dagger)$ is <u>true</u> <=> $+\mathbf{PC}(\underline{\mathbf{t}}) \in$ md[$\mathbb{Sq}$] for some $\underline{\mathbf{t}} \in c\mathbb{t}^n$ for which $\underline{\mathbf{r}} \ggg \underline{\mathbf{t}}$.

***Lemma 5.3.1***

For each $\mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n)$, $n \geq 0$, $f^+[\mathbf{PC}]$ has the following properties:

1. $\pm\mathbf{PC}(\underline{\mathbf{r}}) \in$ md[$\mathbb{Sq}$] => $f^+[\mathbf{PC}](\underline{\mathbf{r}}^\dagger)$ is <u>true</u>, respectively <u>false</u>, for $\underline{\mathbf{r}} \in c\mathbb{t}^n$.
2. $f^+[\mathbf{PC}] \in \mathbb{D}_n$.

***Proof***

1) Since $\mathbf{r}_i \ggg \mathbf{r}_i$ for $1 \leq i \leq n$, the + case of the implication follows immediately from the definition of $f^+[\mathbf{PC}]$. Consider now the – case. Let $f^+[\mathbf{PC}](\underline{\mathbf{r}}^\dagger)$ be <u>true</u>. Then for some $\underline{\mathbf{t}} \in c\mathbb{t}^n$, $\underline{\mathbf{r}} \ggg \underline{\mathbf{t}}$ and $+\mathbf{PC}(\underline{\mathbf{t}}) \in$ md[$\mathbb{Sq}$ ]. But from (2) of of the definition of md[$\mathbb{Sq}$] it follows that if $-\mathbf{PC}(\underline{\mathbf{r}}) \in$ md[$\mathbb{Sq}$] then $-\mathbf{PC}(\underline{\mathbf{t}}) \in$ md[$\mathbb{Sq}$] so that $-\mathbf{PC}(\underline{\mathbf{r}}) \notin$ md[$\mathbb{Sq}$].

2) It is sufficient to prove that if $\mathbf{r}_i \gg \mathbf{r}_i'$ then

$f^+[\mathbf{PC}](\mathbf{r}_1{}^\dagger, \ldots, \mathbf{r}_i{}^\dagger, \ldots, \mathbf{r}_n{}^\dagger)$ is <u>true</u>, <=> $f^+[\mathbf{PC}](\mathbf{r}_1{}^\dagger, \ldots, \mathbf{r}_i{}'^\dagger, \ldots, \mathbf{r}_n{}^\dagger)$ is <u>true</u>.
Let $\mathbf{r}_i > \mathbf{r}_i'$ and $f^+[\mathbf{PC}](\mathbf{r}_1{}^\dagger, \ldots, \mathbf{r}_i{}^\dagger, \ldots, \mathbf{r}_n{}^\dagger)$ be <u>true</u>. There are therefore $\mathbf{t}_1, \ldots, \mathbf{t}_n$ for which $\mathbf{r}_j \gg \mathbf{t}_j$ for $1 \le j \le n$ and $+\mathbf{PC}(\mathbf{t}_1, \ldots, \mathbf{t}_j, \ldots, \mathbf{t}_n) \in \mathrm{md}[\mathbb{S}\mathbb{q}]$. Thus $\mathbf{r}_i > \mathbf{r}_i'$ and $\mathbf{r}_i \gg \mathbf{t}_i$ so that by the Church-Rosser theorem there is an $\mathbf{s}_i$ for which $\mathbf{t}_i \gg \mathbf{s}_i$ and $\mathbf{r}_i' \gg \mathbf{s}_i$. Hence again from (2) of of the definition of $\mathrm{md}[\mathbb{S}\mathbb{q}]$ $+\mathbf{PC}(\mathbf{t}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{t}_n) \in \mathrm{md}[\mathbb{S}\mathbb{q}]$ so that $f^+[\mathbf{PC}](\mathbf{r}_1{}^\dagger, \ldots, \mathbf{r}_i{}'^\dagger, \ldots, \mathbf{r}_n{}^\dagger)$ is <u>true</u>.

Let now $\mathbf{r}_i > \mathbf{r}_i'$ and $f^+[\mathbf{PC}](\mathbf{r}_1{}^\dagger, \ldots, \mathbf{r}_i{}'^\dagger, \ldots, \mathbf{r}_n{}^\dagger)$ be <u>true</u>. For some $\mathbf{t}_1, \ldots, \mathbf{t}_n$ for which $\mathbf{r}_j \gg \mathbf{t}_j$ for $j \neq i$ and $\mathbf{r}_i' \gg \mathbf{t}_i$, $+\mathbf{PC}(\mathbf{t}_1, \ldots, \mathbf{t}_n) \in \mathrm{md}[\mathbb{S}\mathbb{q}]$. But then since $\mathbf{r}_i > \mathbf{r}_i'$ it follows that $\mathbf{r}_j \gg \mathbf{t}_j$ for $1 \le j \le n$ so that $f^+[\mathbf{PC}](\mathbf{r}_1{}^\dagger, \ldots, \mathbf{r}_i{}^\dagger, \ldots, \mathbf{r}_n{}^\dagger)$ is <u>true</u>.

***End of proof***

The base set $\mathbb{B}_n$, $n > 0$, for $\mathbb{I}[\mathrm{md}[\mathbb{S}\mathbb{q}]]$ is defined to be $\{f^+[\mathbf{PC}] \mid \mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n)\}$. Thus $\mathbb{B}_n \subseteq \mathbb{D}_n$ as required. The functions $\Phi_1$ and $\Phi_2$ for $\mathbb{I}[\mathrm{md}[\mathbb{S}\mathbb{q}]]$ are defined as follows: $\Phi_1[\![p_i]\!]$ is $c_{2i-1}$ for all $i \ge 1$, so that $\Phi_1[\![\mathbf{t}]\!]$ is $\mathbf{t}^\dagger$ for $\mathbf{t} \in \mathrm{c}\mathbb{t}$; and $\Phi_2[\![\mathbf{PC}]\!]$ is $f^+[\mathbf{PC}]$, for $\mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n)$, $n \ge 0$.

***Lemma 5.3.2***

$\pm\mathbf{F} \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow \pm\mathbf{F}^\dagger \in \Omega[\mathbb{I}[\mathrm{md}[\mathbb{S}\mathbb{q}]]]$

***Proof***

By induction on the degree k of **F** the following result will be proved:

i) $\pm\mathbf{F} \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow \pm\mathbf{F}^\dagger \in \Omega_k[\mathbb{I}]$

Here $\mathbb{I}$ abbreviates $\mathbb{I}[\mathrm{md}[\mathbb{S}\mathbb{q}]]$.

Let deg(**F**)=0, so that **F** is $\mathbf{PC}(\underline{\mathbf{r}})$, for some $\mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n)$, $n \ge 0$, and $\underline{\mathbf{r}} \in \mathrm{c}\mathbb{t}^n$. $\mathbf{F}^\dagger$ is then $\mathbf{PC}(\underline{\mathbf{r}}^\dagger)$. By (1) of lemma 5.3.1, $\pm\mathbf{PC}(\underline{\mathbf{r}}) \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow f^+[\mathbf{PC}](\underline{\mathbf{r}}^\dagger)$ is <u>true</u>, respectively <u>false</u>, for $\underline{\mathbf{r}} \in \mathrm{c}\mathbb{t}^n \Rightarrow \pm\mathbf{PC}(\underline{\mathbf{r}}^\dagger) \in \Omega_0[\mathbb{I}]$.

Assume (i) for $0 \le k \le m$. Let deg(**F**)=m+1 and consider the forms that **F** can take:
**F** is $[\mathbf{G}{\downarrow}\mathbf{H}]$. By (2) of the definition of $\mathrm{md}[\mathbb{S}\mathbb{q}]$ and the definition of $\Omega_{m+1}[\mathbb{I}]$:
$+[\mathbf{G}{\downarrow}\mathbf{H}] \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow -\mathbf{G}, -\mathbf{H} \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow -\mathbf{G}^\dagger, -\mathbf{H}^\dagger \in \Omega_m[\mathbb{I}] \Rightarrow +[\mathbf{G}{\downarrow}\mathbf{H}]^\dagger \in \Omega_{m+1}[\mathbb{I}]$.
Similarly, $-[\mathbf{G}{\downarrow}\mathbf{H}] \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow +\mathbf{G}$ or $+\mathbf{H} \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow +\mathbf{G}^\dagger$or $+\mathbf{H}^\dagger \in \Omega_m[\mathbb{I}] \Rightarrow$ $-[\mathbf{G}{\downarrow}\mathbf{H}]^\dagger \in \Omega_{m+1}[\mathbb{I}]$. The case where **F** is $(\lambda\mathbf{v}.\mathbf{T})(\mathbf{R}, \underline{\mathbf{S}})$ can be similarly argued.

**F** is $\forall\mathbf{x}.\mathbf{F}$, where $\mathbf{x} \in \mathbb{Q}\mathbb{V}(n)$. By (5) of the definition of $\mathrm{md}[\mathbb{S}\mathbb{q}]$ and the definition of $\Omega_{m+1}[\mathbb{I}]$: $+\forall\mathbf{x}.\mathbf{F} \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow +[\mathbf{PC}/\mathbf{x}]\mathbf{F} \in \mathrm{md}[\mathbb{S}\mathbb{q}]$ for all $\mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n) \Rightarrow$ $+[\mathbf{PC}/\mathbf{x}]\mathbf{F}^\dagger \in \Omega_m[\mathbb{I}]$ for all $\mathbf{PC} \in \mathbb{P}(n) \cup \mathbb{C}(n) \Rightarrow +[\mathbf{P}/\mathbf{x}]\mathbf{F}^\dagger \in \Omega_m[\mathbb{I}^*]$ for all **P** variants $\mathbb{I}^*$ of $\mathbb{I}$, where **P** does not occur in $\mathbf{F} \Rightarrow +\forall\mathbf{x}.\mathbf{F}^\dagger \in \Omega_{m+1}[\mathbb{I}]$. Similarly, $-\forall\mathbf{x}.\mathbf{F} \in \mathrm{md}[\mathbb{S}\mathbb{q}] \Rightarrow$ $-[\mathbf{P}/\mathbf{x}]\mathbf{F} \in \mathrm{md}[\mathbb{S}\mathbb{q}]$ for some $\mathbf{P} \in \mathbb{P}(n)$ not in $\mathbf{F} \Rightarrow -[\mathbf{P}/\mathbf{x}]\mathbf{F}^\dagger \in \Omega_m[\mathbb{I}] \Rightarrow$ $-\forall\mathbf{x}.\mathbf{F}^\dagger \in \Omega_{m+1}[\mathbb{I}]$. The case where $\mathbf{x} \in \mathbb{q}\mathbb{v}$ can be similarly argued.

***End of proof***

## 5.4. A Model that is a Counter-Example

Here a model of NaDSyL will be defined that is a counter-example for the sequent represented by $\mathbb{Sq}$. To this end the sets $\mathbb{P}(n)$ of second order parameters are enlarged for each arity n, $n \geq 0$. For each ordinal $\alpha$ of a class to be specified later, a set $\mathbb{P}_\alpha(n)$ of new parameters is added. $\mathbb{P}_0(n)$ is $\mathbb{P}(n)$ and $\mathbb{P}_\alpha(n) \cap \mathbb{P}_\beta(n)$ is empty when $\alpha \neq \beta$. Essential to the definition of the counter-example is the fact that the sets $\mathbb{t}$ and $\mathbb{d}$ are not affected by the introduction of the new second order parameters.

The set of sentences $c\mathbb{F}_\beta$ is defined exactly like $c\mathbb{F}$ except that in place of the set $\mathbb{P}(n)$ the set $\cup\{\mathbb{P}_\alpha(n) \mid 0 \leq \alpha \leq \beta\}$ is used for each n. The sets $c\mathbb{T}_\beta(n)$ are defined from $c\mathbb{F}_\beta$ in the same way that the sets $c\mathbb{T}(n)$ were defined from $c\mathbb{F}$. The set $\cup\{c\mathbb{T}_\beta(n) \mid n \geq 0\}$ is denoted by $c\mathbb{T}_\beta$. For each $\beta$ and n, $n \geq 0$, it is assumed that there is a single member $\mathbf{P}[\mathbf{T}]$ of $\mathbb{P}_{\beta+1}(n)$ assigned to each $\mathbf{T} \in c\mathbb{T}_\beta(n)$, with distinct $\mathbf{T}$ assigned distinct members of $\mathbb{P}_{\beta+1}(n)$.

***Definition of $\mathbb{K}_\beta$***

A set $\mathbb{K}_\beta$ of signed sentences is defined for each $\beta$:

1. $\mathbb{K}_0$ is md[$\mathbb{Sq}$].
2. $\mathbb{K}_{\beta+1}$ is $\mathbb{K}_\beta$ together with all signed sentences $\pm[\mathbf{P}[\mathbf{T}_1]/\mathbf{Y}_1] \ldots [\mathbf{P}[\mathbf{T}_m]/\mathbf{Y}_m]\mathbf{F}$ for which $\pm[\mathbf{T}_1/\mathbf{Y}_1] \ldots [\mathbf{T}_m/\mathbf{Y}_m]\mathbf{F} \in \mathbb{K}_\beta$. Here $\mathbf{Y}_1, \ldots, \mathbf{Y}_m \in \mathbb{QV}$ and $\mathbf{T}_1, \ldots, \mathbf{T}_m \in c\mathbb{T}_\beta$.
3. $\mathbb{K}_\beta$ is $\cup\{\mathbb{K}_\alpha \mid \alpha < \beta\}$ for a limit ordinal $\beta$.

***End of definition***

***Lemma 5.4.1***

For each $\beta$, $\mathbb{K}_\beta$ is a model set for $\mathbb{Sq}$.

***Proof***

It will be proved by transfinite induction on $\beta$ that $\mathbb{K}_\beta$ is a consistent downward closure of $\mathbb{Sq}$. The case $\beta=0$ follows from the definition of $\mathbb{K}_0$. Assume the lemma for $\beta$ and consider the case $\beta+1$ and the clauses of the definition of a downward closure. That $\mathbb{K}_{\beta+1}$ is consistent and that (1) holds is immediate. Since no $\mathbf{Y} \in \mathbb{QV}$ can have a free occurrence in a $\mathbf{t} \in \mathbb{t}$, no occurrence of a $\mathbf{T}$ in $\mathbf{t}$ is replaced by $\mathbf{P}[\mathbf{T}]$; therefore (2) holds for $\mathbb{K}_{\beta+1}$ if it holds for $\mathbb{K}_\beta$. (3), (4) and the first order case of (5) hold for $\mathbb{K}_{\beta+1}$ if they hold for $\mathbb{K}_\beta$. Consider now the second order case of (5). The – case is immediate since a parameter in $\mathbb{P}_{\beta+1}(n)$ is distinct from any parameter in $\{\mathbb{P}_\alpha(n) \mid \alpha < \beta\}$.

Let $+\forall\mathbf{X}.\mathbf{F} \in \mathbb{K}_{\beta+1}$ where $\mathbf{X} \in \mathbb{QV}(n)$. To prove that (5) holds in this case it is necessary to prove that $+[\mathbf{R}/\mathbf{X}]\mathbf{F} \in \mathbb{K}_{\beta+1}$ for each $\mathbf{R} \in c\mathbb{T}_{\beta+1}(n)$. If $[\mathbf{R}/\mathbf{X}]\mathbf{F} \in c\mathbb{F}_\beta$ then necessarily $+\forall\mathbf{X}.\mathbf{F} \in \mathbb{K}_\beta$ and $+[\mathbf{R}/\mathbf{X}]\mathbf{F} \in \mathbb{K}_\beta$ so that $+[\mathbf{R}/\mathbf{X}]\mathbf{F} \in \mathbb{K}_{\beta+1}$. Assume therefore that $[\mathbf{R}/\mathbf{X}]\mathbf{F} \in c\mathbb{F}_{\beta+1}$. For some $\mathbf{T}_1, \ldots, \mathbf{T}_m \in c\mathbb{T}_\beta$, and some $\mathbf{Y}_1, \ldots, \mathbf{Y}_m \in \mathbb{QV}$, there is an $\mathbf{S} \in \mathbb{T}_\beta(n)$ and a $\mathbf{G} \in \mathbb{F}_\beta$ for which $\mathbf{F}$ is $[\mathbf{P}[\mathbf{T}_1]/\mathbf{Y}_1] \ldots [\mathbf{P}[\mathbf{T}_m]/\mathbf{Y}_m]\mathbf{G}$ and $\mathbf{R}$ is

$[\mathbf{P}[\mathbf{T}_1]/\mathbf{Y}_1] \ldots [\mathbf{P}[\mathbf{T}_m]/\mathbf{Y}_m]\mathbf{S}$. Thus $+\forall\mathbf{X}.[\mathbf{T}_1/\mathbf{Y}_1] \ldots [\mathbf{T}_m/\mathbf{Y}_m]\mathbf{G} \in \mathbb{K}_\beta$ and $[\mathbf{T}_1/\mathbf{Y}_1] \ldots [\mathbf{T}_m/\mathbf{Y}_m]\mathbf{S} \in c\mathbb{T}_\beta(n)$ so that $+[\mathbf{T}_1/\mathbf{Y}_1] \ldots [\mathbf{T}_m/\mathbf{Y}_m]([\mathbf{S}/\mathbf{X}]\mathbf{G}) \in \mathbb{K}_\beta$ and therefore $+[\mathbf{P}[\mathbf{T}_1]/\mathbf{Y}_1] \ldots [\mathbf{P}[\mathbf{T}_m]/\mathbf{Y}_m]([\mathbf{R}/\mathbf{X}]\mathbf{F}) \in \mathbb{K}_{\beta+1}$ as required.

***End of proof***

***Definition of $\mathbb{I}_\beta$***

An interpretation $\mathbb{I}_\beta$ is defined for each $\beta$. The first order function for each $\mathbb{I}_\beta$ is the function $\Phi_1$ for $\mathbb{I}_0$, $\Phi_{2,\beta}$ denotes the second order function, and $\mathbb{B}_\beta(n)$ denotes the arity n base set.

1. $\mathbb{I}_0$ is $\mathbb{I}[md[\mathbb{S}q]]$.
2. Let $\mathbb{I}_\beta$ be defined. $\mathbb{I}_{\beta+1}$ is defined as follows:
   .1. $\mathbb{B}_{\beta+1}(n)$ is $\mathbb{B}_\beta(n) \cup \{\Phi_{2,\beta}[\![\mathbf{T}]\!] \mid \mathbf{T} \in c\mathbb{T}_\beta(n)\}$, for $n > 0$.
   .2. $\Phi_{2,\beta+1}[\![\mathbf{PC}]\!]$ is $\Phi_{2,\beta}[\![\mathbf{PC}]\!]$, for $\mathbf{PC} \in \cup\{\mathbb{P}_\alpha(n) \mid \alpha < \beta\} \cup \mathbb{C}(n)$, $n \geq 0$; and
   $\Phi_{2,\beta+1}[\![\mathbf{P}[\mathbf{T}]]\!]$ is $\Phi_{2,\beta}[\![\mathbf{T}]\!]$ for $\mathbf{T} \in c\mathbb{T}_\beta$.
3. Let $\mathbb{I}_\alpha$ be defined for $\alpha < \beta$, where $\beta$ is a limit ordinal. Then
   .1. $\Phi_{2,\beta}[\![\mathbf{C}]\!]$ is $\Phi_{2,0}[\![\mathbf{C}]\!]$, for $\mathbf{C} \in \mathbb{C}(n)$, $n \geq 0$; and
   $\Phi_{2,\beta}[\![\mathbf{P}]\!]$ is $\Phi_{2,\alpha}[\![\mathbf{P}]\!]$, where $\mathbf{P} \in \mathbb{P}_\alpha(n)$, $n \geq 0$ and $\alpha < \beta$.
   .2. $\mathbb{B}_\beta(n)$ is $\cup\{\mathbb{B}_\alpha(n) \mid \alpha < \beta\}$.

***End of definition***

Note that since $\mathbb{P}_\alpha(n) \subset c\mathbb{T}_\beta(n)$ when $\alpha < \beta$, $\mathbb{B}_{\beta+1}(n)$ is $\{\Phi_{2,\beta+1}[\![\mathbf{P}]\!] \mid \mathbf{P} \in \mathbb{P}_{\beta+1}(n)\}$.

***Lemma 5.4.2***

For all ordinals $\beta$ and all $\mathbf{F} \in c\mathbb{F}_\beta$, $\pm\mathbf{F} \in \mathbb{K}_\beta \Rightarrow \pm\mathbf{F}^\dagger \in \Omega[\mathbb{I}_\beta]$

***Proof***

The proof is by transfinite induction on $\beta$. The case $\beta$=0 follows from (1) of the definition and lemma 5.3.2. Assume the lemma for $\beta$ and consider the case $\beta$+1. By induction on the degree k of $\mathbf{F}$, $\pm\mathbf{F} \in \mathbb{K}_{\beta+1} \Rightarrow \pm\mathbf{F}^\dagger \in \Omega_k[\mathbb{I}_{\beta+1}]$ can be proved in much the same way as lemma 5.3.2.

Consider now a limit ordinal $\beta$. Assume that the lemma holds for all $\alpha$, $\alpha < \beta$. The lemma will be proved for $\beta$ by induction on deg($\mathbf{F}$). The result is immediate when deg($\mathbf{F}$) is 0. Of the forms that $\mathbf{F}$ can take when deg($\mathbf{F}$) is k+1, only the form $+\forall\mathbf{X}.\mathbf{G}$ presents any new difficulties. Let $+\forall\mathbf{X}.\mathbf{G} \in \mathbb{K}_\beta$, where $\mathbf{X} \in \mathbb{QV}(n)$. By lemma 5.4.1 it follows that $+[\mathbf{T}/\mathbf{X}]\mathbf{G} \in \mathbb{K}_\beta$ for all $\mathbf{T} \in c\mathbb{T}_\beta(n)$. In particular it follows that $+[\mathbf{P}[\mathbf{T}]/\mathbf{X}]\mathbf{G} \in \mathbb{K}_\beta$ for all $\mathbf{P}[\mathbf{T}] \in \mathbb{P}_\beta(n)$. By the induction assumption therefore $+[\mathbf{P}[\mathbf{T}]/\mathbf{X}]\mathbf{G}^\dagger \in \Omega[\mathbb{I}_\beta]$ for the same $\mathbf{P}$. But this means that $+[\mathbf{P}/\mathbf{X}]\mathbf{G}^\dagger \in \Omega[\mathbb{I}_\beta{}^*]$ for all $\mathbf{P}$ variants $\mathbb{I}_\beta{}^*$ of $\mathbb{I}_\beta$, where $\mathbf{P}$ is not in $\mathbf{G}$. Hence $+\forall\mathbf{X}.\mathbf{G}^\dagger \in \Omega[\mathbb{I}_\beta]$.

***End of proof***

***Theorem 5.4***

There is an ordinal $\beta$ for which $\mathbb{I}_\beta$ is a counter-example for $\mathbb{S}\mathbb{q}$.

***Proof***

For each $\beta \geq 0$ and each $n \geq 0$, $\mathbb{B}_\beta(n) \subseteq \mathbb{B}_{\beta+1}(n) \subseteq \mathbb{D}(n)$. Further $\mathbb{d}$ is denumerable. Thus the cardinal of an ordinal $\beta$ for which $\mathbb{B}_\beta(n) \not\subset \mathbb{D}(n)$ cannot exceed the cardinal of $\mathbb{D}(n)$. Therefore there is an ordinal $\beta$ for which $\mathbb{B}_{\beta+1}(n) \subseteq \mathbb{B}_\beta(n)$, for $n \geq 0$. But this can only be the case if $\Phi_{2,\beta}[\mathbf{T}](\underline{\mathbf{t}}^\dagger) \in \mathbb{B}_\beta(n)$ for each $\mathbf{T} \in c\mathbb{T}_\beta(n)$, $n \geq 0$; that is if $\mathbb{I}_\beta$ is a model. It is necessarily a counter-example for the sequent represented by $\mathbb{S}\mathbb{q}$ from lemma 5.4.2.

***End of proof.***

***Corollary***

The logical syntax of NaDSyL is complete without the cut rule.

***Proof***

It is not possible for there to be a set $\mathbb{S}\mathbb{q}$ that is derivable with cut but not without. For let $\mathbb{M}$ be the counter-example for $\mathbb{S}\mathbb{q}$. If $\mathbb{S}\mathbb{q}$ were derivable with cut it would be valid by soundness theorem 5.1, and therefore satisfied by the model $\mathbb{M}$.

***End of proof***.

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