Topological Aspects of Regular Languages

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Abstract: We establish a number of new results (and rederive some old results) concerning regular languages, using essentially topological methods. Our development is based on the duality (established by Stone) between Boolean algebras and certain topological spaces (which are now called "Stone spaces"). (This duality does not seem to have been recognized in the literature on regular languages, even though it is well known that the regular languages over a fixed alphabet form a Boolean algebra and that the "implicit operations" with a fixed number of operands form a Stone space!) By exploiting this duality, we are able to obtain a much more accessible account of the Galois correspondence between varieties of regular languages (in the sense of Eilenberg) and certain sets of "implicit identities". The new results include an analogous Galois correspondence for a generalization of varieties, and an explicit characterization by means of closure conditions of the sets of implicit identities involved in these correspondences.

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1. Introduction

A watershed in the development of the theory of regular languages was the definition and characterization of "varieties" of regular languages by Eilenberg (announced in [E73] and presented fully in [E76]). Eilenberg's work put scattered results on diverse classes of languages into a general setting, and the greater part of subsequent work on regular languages can be properly viewed as taking place within this setting. A *variety* is a set of languages closed under Boolean operations, taking inverse homomorphic images, and taking (left and right) quotients by words. Though these operations might at first seem to be chosen arbitrarily, they in fact correspond to natural operations on finite automata: Boolean operations to parallel connections with Boolean post-processing, taking inverse homomorphic images to pre-processing by homomorphisms, and taking quotients to certain operations on initial and final states.

Eilenberg's main result is that varieties of regular languages are characterized by their "syntactic semigroups", and that the corresponding sets of finite semigroups are those that are closed under taking sub-semigroups, taking quotient semigroups, and taking finite products of semigroups. Eilenberg called these sets "varieties" of finite semigroups. This correspondence is a "direct" one: larger varieties of languages correspond to larger varieties of semigroups.

Eilenberg's definition of a variety of finite semigroups is similar to, but subtly different from, the definition of a "family" of algebraic structures given by Birkhoff [B35]. (What Birkhoff called a "family" of structures is now usually called a "variety" in the literature on universal algebra. In this paper we revert to Birkhoff's original terminology to avoid conflict with Eilenberg's. This is not put forward as a principled choice, but merely as an expedient one: it preserves the usage with which computer scientists are familiar.) A family of structures is defined as being closed under taking sub-structures, taking quotient structures, and taking arbitrary (not necessarily finite) products of structures. (In the definition of a variety of finite semigroups, one can take the elements of the semigroups to be natural numbers without any loss of generality. The definitionof a family, with its reference to "arbitrary" products, probes the limitations of the underlying set theory in which it is developed. We shall not discuss these set-theoretic complications in this paper, since families appear only in a motivating role, and we shall not need any specific results of their theory.)

Birkhoff proved two main results. First, he showed that families of structures are characterized by the sets of "identities" that they satisfy. (For example, semigroups are characterized among structures with one binary operation by the associative identity

$(xy)z = x(yz)$ and its consequences; commutative semigroups are characterized among semigroups by the further identity $xy = yx$ and its consequences.) Second, he gave a set of closure conditions (the "deductive closure" conditions of equational logic) that characterize those sets of identities that correspond to families of structures. This correspondence is a "Galois" correspondence: larger families of structures correspond to smaller sets of identities. (Galois correspondences get their name from the archetypal situation in Galois theory, in which larger field extensions correspond to smaller groups of automorphisms. The modern use of the term includes any reciprocal correspondence between dual lattices defined by closure conditions; see Ward [W42], Ore [O44] and Everett [E44].)

Families of structures play a central role in universal algebra, and it was thus natural to seek an analogous connection between varieties of finite semigroups and identities. The weaker closure conditions for varieties (only finite products rather than arbitrary products) lead, however, to much richer possibilities for varieties than for families, and it soon became apparent that identities are too crude an instrument to distinguish them all. Eilenberg recognized, however, that varieties could be characterized by infinite sequences of identities, with each semigroup satisfying all but finitely many identities in each sequence. (Thus for example the aperiodic finite semigroups (see Schützenberger [S65] and McNaughton and Papert [M71]) are those that satisfy the identity $x^k = x^{k+1}$ for all sufficiently large k.)

Decisive progress was made by Reiterman [R82], who showed that varieties are characterized by the "implicit identities" that they satisfy, that is, by identities between "implicit terms", which can be viewed as limits of sequences of the ordinary terms that appear in ordinary identities. (For example, the aperiodic finite semigroups are charactererized among finite semigroups by the implicit identity $x^\omega = x^\omega x$, where the implicit term x^ω is the limit of the sequence $x^1, x^2, x^6, \ldots, x^{k!}, \ldots$ of ordinary terms, and can be thought of as representing the unique idempotent element in the cyclic finite semigroup generated by x.) Almeida [A90] made further progress by characterizing the sets of implicit identities that correspond to varieties (though he did not succeed in giving these in the form of closure conditions like those of Birkhoff).

With this background, we can now explain the contributions of the present paper. One contribution is to give explicit closure conditions for implicit identities, thereby completing the description of the Galois correspondence between varieties and sets of implicit identities. These closure conditions take the form of Birkhoff's deductive closure conditions, supplemented by some new infinitary closure conditions of a topological character. In the course of doing this, we obtain a parallel theory for a generalization of varieties of regular languages that we call "strains", for which the requirement of being closed under left and right quotients by words is dropped. This again corresponds to a natural distinction for finite automata: Boolean operations and inverse homomorphic images correspond to interconnections of automata with pre- or post-processing, whereas quotients correspond to tampering with the internal structure of the automata (their initial and final states).

A greater contribution than these, however, is our redevelopment of the theory in a new way, based on exploiting the duality established by Stone [S36] between Boolean algebras and certain topological spaces (which are now called "Stone spaces"). This development makes the derivation of the Galois correspondence simpler and more elegant, and reveals the true mathematical underpinnings of the theory. Central to this development is a completely topological characterization of the regular languages that should be of independent interest. (This characterization is a simplification of one due to Almeida [A88].)

In the following sections we shall describe these contributions in logical order, starting with the topological characterization of regular languages and culminating in the Galois correspondence for varieties. We assume familiarity with basic terminology and results from algebra (mainly semigroups) and topology (mainly metric spaces). We omit all proofs in this abstract; in all cases the proofs use standard methods from algebra and topology, whose use in traditional areas of mathematics would be regarded as routine.

2. Topological Characterization of Regular Languages

For now, let us work with a fixed finite alphabet, say $\mathbf{B}_n = \{0, 1, \ldots, n-1\}$. We shall denote by $\mathbf{E}_n = \mathbf{B}_n^+$ the set of all non-empty finite words over $\mathbf{B}_n$, which has the structure of a semigroup under concatenation. (We have chosen to restrict attention to non-empty words, since the resulting theory of varieties is capable of making more delicate distinctions. There is, however, a parallel theory that includes the empty word and is based on monoids rather than semigroups.) An equivalence relation $\equiv$ on $\mathbf{E}_n$ is a *congruence* if, for all $w, x, y, z \in \mathbf{E}_n$, $w \equiv y$ and $x \equiv z$ imply $wx \equiv yz$. If $\equiv$ is a congruence on $\mathbf{E}_n$, the $\equiv$-classes themselves form a semigroup, which we denote $\mathbf{E}_n/\equiv$. A congruence $\equiv$ will be called *finite* if there are finitely many $\equiv$-classes, so that $\mathbf{E}_n/\equiv$ is a finite semigroup.

A *language* over the alphabet $\mathbf{B}_n$ is a a set of words $L \subseteq \mathbf{E}_n$. Associated with any language L over $\mathbf{B}_n$ is a congruence $\cong_L$ on $\mathbf{E}_n$, called the *syntactic congruence* of L, defined as follows: $x \cong_L y$ if and only if, for all $w, z \in \mathbf{E}_n$ we have (1) $x \in L$ if and only if $y \in L$, (2) $wx \in L$ if and only if $wy \in L$, (3) $xz \in L$ if and only if $yz \in L$, and (4) $wxz \in L$ if and only if $wyz \in L$. (These conditions, which would be much simpler if the empty word were allowed, say that x and y behave the same way in all possible syntactic contexts insofar as determining whether a word belongs to L is concerned.) The semigroup $\mathbf{E}_n/\cong_L$, called the *syntactic semigroup* of L, will be denoted $\mathrm{Syn}(L)$. The following well known algebraic characterization of regular languages is due to J. R. Myhill (see Rabin and Scott [R59], Theorem 1).

Proposition 2.1: For every $L \subseteq \mathbf{E}_n$, the following three conditions are equivalent.

(i) L is regular.

(ii) L is a union of $\equiv$-classes for some finite congruence $\equiv$ on $\mathbf{E}_n$.

(iii) $\mathrm{Syn}(L)$ is a finite semigroup.

Our goal in this section is to derive an alternative topological characterization of the regular languages. The first step is to form an appropriate topological space by taking the completion of $\mathbf{E}_n$ with respect to a certain metric.

We shall say that a congruence $\equiv$ *separates* two words x and y if $x \not\equiv y$. If x and y are distinct, they are separated by a finite congruence, for example the syntactic congruence of the regular language $\{x\}$. We shall define the *distance* $d(x,y)$ between x and y to be 0 if $x = y$, and to be $1/k$, where k is the smallest number of $\equiv$-classes of a congruence $\equiv$ separating x and y, if $x \neq y$. The distance $d(x,y)$ forms a metric on the space $\mathbf{E}_n$. In fact, it forms an "ultrametric", since the triangle inequality $d(x,z) \leq d(x,y) + d(x,z)$ holds in the stronger form $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ (if a congruence separates x and z, then it must also separate y from either x or z). Since a congruence that separates wx and yz must also separate either w and y or x and z, it follows that $d(wx,yz) \leq \max\{d(w,y), d(x,z)\}$, and thus that the operation of multiplication (viewed as a map from $\mathbf{E}_n \times \mathbf{E}_n$ to $\mathbf{E}_n$) is continuous.

Let S be a finite semigroup. We endow the set S^{S^n} of maps from S^n to S with the structure of a semigroup by defining

$$(fg)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)\, g(x_1, \ldots, x_n)$$

for $f, g \in S^{S^n}$ and $x_1, \ldots, x_n \in S$. Define the map $H_S : \mathbf{B}_n \to S^{S^n}$ by

$$\big(H_S(a)\big)(x_1, \ldots, x_n) = x_a$$

for $a \in \mathbf{B}_n$. Since $\mathbf{E}_n$ is the free semigroup generated by $\mathbf{B}_n$, this map extends to a unique homomorphism $H_S : \mathbf{E}_n \to S^{S^n}$.

If $x \neq y$ are distinct words in $\mathbf{E}_n$, then there exists a finite semigroup S such that $H_S(x) \neq H_S(y)$. To see this, let $S = \mathrm{Syn}(\{x\})$, and let $h : \mathbf{E}_n \to S$ be the canonical homomorphism, so that $h(x) \neq h(y)$. Then we have

$$\big(H_S(x)\big)\big(h(0), \ldots h(n-1)\big) = h(x) \neq h(y) = \big(H_S(y)\big)\big(h(0), \ldots h(n-1)\big).$$

We shall now introduce another metric on $\mathbf{E}_n$. We shall say that a semigroup S such that $H_S(x) \neq H_S(y)$ *separates* x and y. If $x \neq y$ are distinct words in $\mathbf{E}_n$, we define the *distance* $d'(x, y)$ to be $1/k$, where k is the smallest possible cardinality $\#S$ of a finite semigroup S that separates x and y. If $x = y$, we define $d'(x, y)$ to be 0. The distance d' is a metric on $\mathbf{E}_n$.

In fact, these two metrics coincide: $d = d'$. For if $h : \mathbf{E}_n \to S$ is a homomorphism, there is a unique homomorphism $h' : S^{S^n} \to S$ such that $h = h' \circ H_S$. If $h(x) \neq h(y)$, then we must have $H_S(x) \neq H_S(y)$. Conversely, if $H_S(x) \neq H_S(y)$, these two functions must differ for some argument values, say

$$H_S(x)(z_0, \ldots, z_{n-1}) \neq H_S(y)(z_0, \ldots, z_{n-1}).$$

Letting h be the unique homomorphism such that $h(a) = z_a$ for each $a \in \mathbf{B}_n$, we conclude that $h(x) \neq h(y)$. In what follows, we shall write d for this metric, and follow whichever definition is more convenient in a given situation.

We shall denote by $\mathbf{I}_n = \hat{\mathbf{E}}_n$ the "completion" of $\mathbf{E}_n$ with respect to the metric d. (This may be viewed as follows: we consider "Cauchy sequences" of elements of $\mathbf{E}_n$; we call two such sequences "equivalent" if their interleaving is also a Cauchy sequence; and we take $\mathbf{I}_n$ to be the set of equivalence classes of Cauchy sequences. In this respect, the process of passing from $\mathbf{E}_n$ to $\mathbf{I}_n$ is the same as that of passing from the rational numbers $\mathbf{Q}$ to the real numbers $\mathbf{R}$.)

In general, the completion $\hat{X}$ of a metric space X is a complete metric space having X as a dense subspace. The completion is essentially unique, in the sense that any two complete metric spaces having a common dense subspace are homeomorphic. The completion has the universal property that any continuous map between metric spaces has a unique extension to a continuous map between their completions. In particular, the continuous operation of multiplication on $\mathbf{E}_n$ extends to a unique continuous operation on $\mathbf{I}_n$. This operation is associative, and thus endows $\mathbf{I}_n$ with the structure of a continuous semigroup.

If $\{x_i\}_{i \in \mathbf{N}}$ is a Cauchy sequence in $\mathbf{E}_n$ and S is a finite semigroup, then the sequence $\{H_S(x_i)\}_{i \in \mathbf{N}}$ be constant for all sufficiently large i. Furthermore, if $\{x_i\}_{i \in \mathbf{N}}$ and $\{y_i\}_{i \in \mathbf{N}}$

are equivalent Cauchy sequences, then this constant must be the same for both sequences. Thus we may extend H_S to a map $H_S : \mathbf{I}_n \to S^{S^n}$. This map is continuous (when S^{S^n} is given the discrete topology), since if $H_S^{-1}(u)$ is empty, it is open, and if it contains any point $x \in \mathbf{I}_n$, then it also contains the open ball about x with radius $1/\#S$, and thus is again open. Furthermore, $H_S : \mathbf{I}_n \to S^{S^n}$ is a homomorphism, since $H_S : \mathbf{E}_n \to S^{S^n}$ is a homomorphism and all of the multiplications involved are continuous. Finally, if $x \neq y$ are distinct elements of $\mathbf{I}_n$, there exists a finite semigroup separating x and y. To see this, let v and w be words of $\mathbf{E}_n$ such that $d(v,x) \leq d(x,y)/4$ and $d(w,y) \leq d(x,y)/4$, so that $d(v,w) \geq d(x,y)/2$. Then v and w can be separated by a semigroup of cardinality at most $2/d(x,y)$. This semigroup must also separate x and y, since it is too small to separate v from x or w from y.

A metric space is *compact* if every infinite sequence has a convergent subsequence. We say that a set in a topological space is *crisp* if it is both open and closed. A pair $x \neq y$ of distinct points are separated by a set K if K contains one, but not both, of x and y. A topological space is *totally disconnected* if every pair of distinct points is separated by a crisp set.

In the following lemma, the assertion that $\mathbf{I}_n$ is compact was proved by Reiterman [R82], and the assertion that $\mathbf{I}_n$ is totally disconnected was proved by Almeida [A88]. Our proof is simpler than either of theirs.

Lemma 2.2: The space $\mathbf{I}_n$ is compact and totally disconnected.

Proof: Suppose that $\{x_i\}_{i\in\mathbf{N}}$ is an arbitrary infinite sequence in $\mathbf{I}_n$. Let $S_0, S_1, \ldots$ be an infinite sequence of finite semigroups that contains just one member isomorphic to any finite semigroup. We define a doubly indexed sequence $\{x_{i,j}\}_{i,j\in\mathbf{N}}$ by induction on j. Let $\{x_{i,0}\}_{i\in\mathbf{N}} = \{x_i\}_{i\in\mathbf{N}}$. Then, if $\{x_{i,j}\}_{i\in\mathbf{N}}$ has been defined, we take $\{x_{i,j+1}\}_{i\in\mathbf{N}}$ to be an infinite subsequence of $\{x_{i,j}\}_{i\in\mathbf{N}}$ such that $H_{S_j}(x_{i,j+1})$ is constant for $i \in \mathbf{N}$ (which we can do because $H_{S_j}(x_{i,j})$ assumes only finitely many distinct values). The diagonal sequence $\{x_{i,i}\}_{i\in\mathbf{N}}$ is Cauchy (since to ensure $d(x_{i,i}, x_{j,j}) \leq 1/k$ it suffices to take j and j larger than the index in the sequence $S_0, S_1, \ldots$ of any finite semigroup with at most k elements). Thus the sequence $\{x_{i,i}\}_{i\in\mathbf{N}}$ converges to a limit y in $\mathbf{I}_n$. Since the sequence $\{x_{i,i}\}_{i\in\mathbf{N}}$ is a convergent subsequence of the original sequence $\{x_i\}_{i\in\mathbf{N}}$, $\mathbf{I}_n$ is compact.

To see that $\mathbf{I}_n$ is totally disconnected, suppose that $x \neq y$ are distinct elements of $\mathbf{I}_n$, and let S be a finite semigroup separating x and y. Since $H_S : \mathbf{I}_n \to S^{S^n}$ is continuous, and since every subset of the discrete space S^{S^n} is crisp, the set $H_S^{-1}(H_S(x)$ is crisp, and contains x but not y. Thus $\mathbf{I}_n$ is totally disconnected. $\triangle$

The following lemma is due to Hunter [H88]. We note that the definition of a syntactic congruence makes sense for a subset in any semigroup, and thus may be used for subsets of $\mathbf{I}_n$ as well as those of $\mathbf{E}_n$.

Lemma 2.3: A set $L \subseteq \mathbf{I}_n$ is crisp if and only if every $\cong_L$-class is open.

Proof: (if) Suppose that every $\cong_L$-class is open. Since they form a disjoint cover of the compact space $\mathbf{I}_n$, they must be finite in number. Since the complement of each is a finite union of open sets, and is therefore open, they are also closed, and thus crisp. Since L is a finite union of these crisp $\cong_L$-classes, it is also crisp.

(only if) Suppose that L is crisp, but that M is a $\cong_L$-class that is not open. Then there exists a point $x \in M$ and a sequence $y_0, y_1, \ldots \notin M$ converging to x. Since $x \not\cong_L y_i$, we can find $w_i, z_i \in \mathbf{I}_n$ such that either (1) $w_i x z_i \in L$ and $w_i y_i z_i \notin L$, or (2) $w_i x z_i \notin L$ and $w_i y_i z_i \in L$. Furthermore, one of these two possibilities must occur for infinitely many i. By transfering attention from L to its complement in $\mathbf{I}_n$ (which is also crisp and has the same syntactic congruence) if necessary, we may assume without loss of generality that (1) occurs for infinitely many i, and by restricting attention to an infinite subsequence, we may assume that (1) occurs for all i. Since $\mathbf{I}_n$ is compact, by further restricting attention to infinite subsequences, we may assume that w_i and z_i each converge to elements of $\mathbf{I}_n$, say u and v, respectively. Since multiplication is continuous, the sequences $w_i x z_i$ and $w_i y_i z_i$ each converge to uxv. Since uxv is the limit of $w_i x z_i \in L$ and L is closed, we have $uxv \in L$. Since uxv is the limit of $w_i y_i z_i \notin L$ and the complement of L is closed, we have $uxv \notin L$. This contradiction completes the proof. $\triangle$

In the following lemma, $\mathrm{cl}(X)$ denotes the topological closure of the set X in $\mathbf{I}_n$.

Lemma 2.4: If $X \subseteq \mathbf{I}_n$ is crisp, then

$$\mathrm{cl}(X \cap \mathbf{E}_n) = X.$$

Proof: Since X is closed and $X \cap \mathbf{E}_n \subseteq X$, we have $\mathrm{cl}(X \cap \mathbf{E}_n) \subseteq X$. Thus it remains to show that $\mathrm{cl}(X \cap \mathbf{E}_n) \supseteq X$. Suppose $x \in X$. Since $\mathbf{E}_n$ is dense in $\mathbf{I}_n$, there exists a sequence $w_0, w_1, \ldots \in \mathbf{E}_n$ converging to x. Since X is open, all but finitely many elements of this sequence belong to X, and by restricting attention to an infinite subsequence we may assume that all elements belong to X. Thus the sequence $w_0, w_1, \ldots$ lies in $X \cap \mathbf{E}_n$, and its limit x lies in $\mathrm{cl}(X \cap \mathbf{E}_n)$. $\triangle$

The main result of this section is the following, which is a simplification of a criterion due to Almeida [A88]. In it, $\mathrm{cl}(X)$ denotes the topological closure of the set X in $\mathbf{I}_n$.

Theorem 2.5: A language $L \subseteq \mathbf{E}_n$ is regular if and only if $\mathrm{cl}(L)$ is open.

Proof: (if) Suppose that $\mathrm{cl}(L)$ is open. Then $\mathrm{cl}(L)$ is crisp and, by Lemma 2.3, each $\cong_{\mathrm{cl}(L)}$-class is open. Since they form a disjoint cover of the compact set $\mathbf{I}_n$, they must be finite in number. Thus $S = \mathrm{Syn}\big(\mathrm{cl}(L)\big)$ is finite, and if $h : \mathbf{I}_n \to S$ is the canonical homomorphism, then $\mathrm{cl}(L) = h^{-1}(K)$ for some $K \subseteq S$. Since $L = \mathrm{cl}(L) \cap \mathbf{E}_n$, we have $L = h_0^{-1}(K)$, where the homomorphism $h_0 : \mathbf{E}_n \to S$ is the restriction of h to $\mathbf{E}_n$. If we define the congruence $\equiv$ on $\mathbf{E}_n$ by $x \equiv y$ if and only if $h_0(x) = h_0(y)$, then $\equiv$ is finite and L is the union of $\equiv$-classes, so L is regular by Proposition 2.1.

(only if) Suppose the L is regular. Then $S = \mathrm{Syn}(L)$ is finite by Proposition 2.1. Let $h : \mathbf{E}_n \to S$ be the canonical homomorphism, and take $K \subseteq S$ such that $L = h^{-1}(K)$. We can factor h through S^{S^n} by writing $h = h' \circ H_S$, where $h' : S^{S^n} \to S$ is the evaluation map defined by

$$h'(f) = f\big(h(0), \ldots, h(n-1)\big)$$

for $f \in S^{S^n}$ (so that $f : S^n \to S$). The map h' is continuous, since S^{S^n} has the discrete topology, and is a homomorphism. Since H_S is also a continuous homomorphism, their composition h is a continuous homomorphism. Since $\mathbf{E}_n$ is dense in $\mathbf{I}_n$, h has a unique extension to a continuous homomorphism $h_1 : \mathbf{I}_n \to S$. Since K is crisp in S, $h_1^{-1}(K)$ is crisp in $\mathbf{I}_n$. By Lemma 2.4, $\mathrm{cl}\big(h_1^{-1}(K) \cap \mathbf{E}_n\big) = h_1^{-1}(K)$, and thus $\mathrm{cl}\big(h_1^{-1}(K) \cap \mathbf{E}_n\big)$ is crisp. But $h_1^{-1}(K) \cap \mathbf{E}_n = h^{-1}(K) = L$, and thus $\mathrm{cl}(L)$ is crisp. $\triangle$

3. Stone Duality

In 1936, Stone [S36] established a duality between Boolean algebras and totally disconnected compact Hausdorff spaces (which are thus now usually called "Stone spaces"). The gist of this duality is as follows. If we start with a Boolean algebra $\mathbf{A}$, the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}_2$ (regarded as the two-element Boolean algebra), endowed with the topology induced by the product topology on the set of all functions from $\mathbf{A}$ to $\mathbf{B}_2$, forms a Stone space that will be denoted $\mathbf{A}^*$. Conversely, if we start with a Stone space $\mathbf{S}$, the collection of all crisp sets in $\mathbf{S}$, with Boolean operations defined in the usual way, forms a Boolean algebra that will be denoted $\mathbf{S}^*$. Furthermore, these processes are reciprocal, so that $\mathbf{A}^{**} = \mathbf{A}$ and $\mathbf{S}^{**} = \mathbf{S}$. (An expository account of the theory is given by Halmos [H63].)

After the development in the preceding section, the following theorem should come as no surprise.

Theorem 3.1: The Boolean algebra $\mathbf{L}_n$ of regular languages over $\mathbf{B}_n$ is dual to the Stone space $\mathbf{I}_n$ (that is, $\mathbf{L}_n^*$ is isomorphic to $\mathbf{I}_n$, and $\mathbf{I}_n^*$ is homeomorphic to $\mathbf{L}_n$).

Proof: It will suffice (see Halmos [H63], pp. 79–80) to construct a *pairing* $\phi : \mathbf{L}_n \times \mathbf{I}_n \to \mathbf{B}_2$ satisfying the following four conditions. (1) For every $L \in \mathbf{L}_n$, the map $g : \mathbf{I}_n \to \mathbf{B}_2$ defined by $g(t) = \phi(L, t)$ is continuous. (2) Every continuous map $g : \mathbf{I}_n \to \mathbf{B}_2$ is of the form $g(t) = \phi(L, t)$ for some $L \in \mathbf{L}_n$. (2) For every $t \in \mathbf{I}_n$, the map $f : \mathbf{L}_n \to \mathbf{B}_2$ defined by $f(L) = \phi(L, t)$ is a homomorphism. (4) Every homomorphism $f : \mathbf{L}_n \to \mathbf{B}_2$ is of the form $f(L) = \phi(L, t)$ for some $t \in \mathbf{I}_n$.

We begin by constructing the pairing ϕ. Define

$$\phi(L, t) = \begin{cases} 1, & \text{if } t \in \mathrm{cl}(L); \\ 0, & \text{otherwise.} \end{cases}$$

We now proceed to verfy conditions (1) through (4).

(1) Suppose that $L \in \mathbf{L}_n$ and define g by $g(t) = \phi(L, t)$. We must show that g is continuous. Since L is regular, $\mathrm{cl}(L)$ is crisp in $\mathbf{I}_n$ by Theorem 2.5. Thus $\mathrm{cl}(L)$ and its complement are both open in $\mathbf{I}_n$. It follows that all sets of the form $g^{-1}(B)$ for $B \subseteq \mathbf{B}_2$ are open in $\mathbf{I}_n$, and thus that g is continuous. This completes the verification of (1).

(2) Suppose that $g : \mathbf{I}_n \to \mathbf{B}_2$ is continuous. We must show that g is of the form $g(t) = \phi(L, t)$ for some regular langauge $L \in \mathbf{L}_n$. Since g is continuous, the set $g^{-1}(1)$ and its complement $g^{-1}(0)$ are both open, whence $g^{-1}(1)$ is crisp in $\mathbf{I}_n$. Take $L = g^{-1}(1) \cap \mathbf{E}_n$. By Lemma 2.4, $\mathrm{cl}(L) = g^{-1}(1)$, and thus $\mathrm{cl}(L)$ is crisp in $\mathbf{I}_n$. Thus by Theorem 2.5 L is regular. This completes the verification of (2).

(3) Suppose that $t \in \mathbf{I}_n$ and define f by $f(L) = \phi(L, t)$. We must show that f is a homomorphism. To do this, it will suffice to show that $f(L \cup M) = f(L) \vee f(M)$ and $f(\mathbf{E}_n \setminus L) = \neg f(L)$ for regular languages L and M, since all Boolean operations can be expressed in terms of join and complement. Since $\mathrm{cl}(L \cup M) = \mathrm{cl}(L) \cup \mathrm{cl}(M)$, we have $f(L \cup M) = f(L) \vee f(M)$ by the definitions of f and ϕ. Since L is regular, $\mathrm{cl}(L)$ is crisp in $\mathbf{I}_n$ by Lemma 2.4. Thus we have $\mathbf{I}_n \setminus \mathrm{cl}(L) = \mathrm{cl}(\mathbf{E}_n \setminus L)$, and $f(\mathbf{E}_n \setminus L) = \neg f(L)$ follows from the definitions of f and ϕ. This completes the verification of (3).

(4) Suppose that $f : \mathbf{L}_n \to \mathbf{B}_2$ is a homomorphism. We must show that f is of the form $f(L) = \phi(L, t)$ for some $t \in \mathbf{I}_n$. Set $\mathbf{K} = f^{-1}(1)$. Then $\mathbf{K}$ is a maximal filter in $\mathbf{L}_n$. That is, (1) if $L \in \mathbf{K}$ and $M \in \mathbf{L}_n$, then $L \cup M \in \mathbf{K}$; (2) if $L \in \mathbf{K}$ and $M \in \mathbf{K}$, then $L \cap M \in \mathbf{K}$; and (3) for every $L \in \mathbf{L}_n$, either L or or its complement (but not both) belongs to $\mathbf{K}$. These conditions imply that the collection

$$\mathbf{H} = \{\mathrm{cl}(L) : L \in \mathbf{K}\}$$

is an ultrafilter basis in $\mathbf{I}_n$. Since $\mathbf{I}_n$ is a compact Hausdorff space, the intersection of theultrafilter basis $\mathbf{H}$ comprises a single point, say t. It remains to verify that $f(L) = \phi(L,t)$ for all regular languages $L \in \mathbf{E}_n$. If $f(L) = 1$, then $L \in \mathbf{K}$ and $\mathrm{cl}(L) \in \mathbf{H}$. Thus $\phi(L,t) = 1$ by the definition of ϕ. On the other hand, if $f(L) = 0$, then applying the same argument to the complement of L yields the conclusion that $\phi(L,t) = 0$. This completes the verification of (4), and thus the proof of Theorem 3.1. $\triangle$

The duality between the Boolean algebra $\mathbf{L}_n$ of regular languages over $\mathbf{B}_n$ and the Stone space $\mathbf{I}_n$ in fact extends to a one-to-one correspondence between *subalgebras* of $\mathbf{L}_n$ (that is, sets of regular languages over $\mathbf{B}_n$ that are closed under Boolean operations, including the empty language and the full language), and quotients of the Stone space $\mathbf{I}_n$ (that is, quotient spaces of $\mathbf{I}_n$ that are totally disconnected). Since we shall be interested in Boolean algebras of regular languages, we are led to investigate the quotients of $\mathbf{I}_n$. Our goal in this section is to characterize the equivalence relations $\equiv$ on $\mathbf{I}_n$ for which the quotient $\mathbf{I}_n/\equiv$ is a totally disconnected compact metric space. We shall do this by giving closure conditions on the graph of $\equiv$.

For $\equiv$ to be an equivalence relation on $\mathbf{I}_n$, it is necessary and sufficient that its graph be reflexive, symmetric and transitive. Thus we have the following closure conditions.

I. For all $x \in \mathbf{I}_n$, $x \equiv x$.

II. For all $x, y \in \mathbf{I}_n$, $x \equiv y$ implies $y \equiv x$.

III. For all $x, y, z \in \mathbf{I}_n$, $x \equiv y$ and $y \equiv z$ imply $x \equiv z$.

For the $\equiv$-classes to form a metric space, it is necessary and sufficient that they be topologically closed. We shall not pause to cast this requirement in terms of closure conditions, however, since it will follow automatically from a requirement that we shall impose later.

When $\mathbf{I}_n/\equiv$ forms a metric space, the projection $\pi : \mathbf{I}_n \to \mathbf{I}_n/\equiv$ is continuous, so that $\mathbf{I}_n/\equiv$ (being a continuous image of a compact space) is compact. Thus it remains to consider the requirement that $\mathbf{I}_n/\equiv$ be totally disconnected. This requirement can be expressed by saying that the crisp unions of $\equiv$-classes must separate the $\equiv$-classes. This expression includes the requirement that the $\equiv$-classes be closed, for they are then the intersections of all the crisp unions of $\equiv$-classes containing them. It also implies that the graph of $\equiv$ is closed in the product topology on $\mathbf{I}_n \times \mathbf{I}_n$, for this graph is intersection of the closed relations $(X \times X) \cup \big((\mathbf{I}_n \setminus X) \times (\mathbf{I}_n \setminus X)\big)$ over all crisp sets $X \subseteq \mathbf{I}_n$. We shall say that an equivalence $\equiv$ on $\mathbf{I}_n$ is *clean* if the crisp unions of $\equiv$-classes separate the $\equiv$-classes.

To cast this requirement in terms of closure conditions, we shall need a few definitions. Let F denote the set of all continuous maps from $\mathbf{I}_n$ to $\mathbf{B}_2$ (regarded as the two-element space with the discrete topology). (These are just the maps f such that $f^{-1}(0)$ and $f^{-1}(1)$ are both crisp. Thus these maps are finite objects, and the set F is countable, since the crisp sets in $\mathbf{I}_n$ are just the closures of the regular languages.) We shall say that sequences $\{x_f\}_{f\in F}$ and $\{y_f\}_{f\in F}$ are *good* for x and y if, for each $f \in F$, $f(x_f) = f(y_f)$ implies $f(x) = f(y)$. We can now formulate the relevant closure condition.

IV. For all sequences $\{x_f\}_{f\in F}$ and $\{y_f\}_{f\in F}$ that are good for x and y, the conditions $x_f \equiv y_f$ for all $f \in F$ imply $x \equiv y$.

The main result of this section is the following.

Theorem 3.2: Let $\equiv$ be an equivalence relation on $\mathbf{I}_n$ (that is, let $\equiv$ satisfy I, II and III). Then $\equiv$ is clean if and only if $\equiv$ satisfies IV.

Proof: (if) Suppose that $\equiv$ satisfies IV. Suppose further, for the sake of contradiction, that $\equiv$ is not clean. Let $x, y \in \mathbf{I}_n$ be such that that $x \not\equiv y$, but no crisp union of $\equiv$-classes separates x and y. Then for every continuous $f : \mathbf{I}_n \to \mathbf{B}_2$ with $f(x) \neq f(y)$, there exists some $\equiv$-class that meets both $f^{-1}(0)$ and $f^{-1}(1)$, say at points x_f and y_f, respectively. Then, if we define $x_f = y_f$ arbitrarily for continuous $f : \mathbf{I}_n \to \mathbf{B}_2$ with $f(x) = f(y)$, the sequences $\{x_f\}_{f\in F}$ and $\{y_f\}_{f\in F}$ are good for x and y. Furthermore, we have $x_f \equiv y_f$ for all $f \in F$. Thus by IV we conclude $x \equiv y$. This contradiction completes the proof of the "if" part. $\triangle$

(only if) Suppose that $\equiv$ is clean. Suppose further, for the sake of contradiction, that $\{x_f\}_{f\in F}$ and $\{y_f\}_{f\in F}$ are good for x and y, and that $x_f \equiv y_f$ for all $f \in F$, but that $x \not\equiv y$. Let X be a crisp union of $\equiv$-classes that separates the $\equiv$-classes containing x and y. Then the function $f : \mathbf{I}_n \to \mathbf{B}_2$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in X, \\ \\ 0, & \text{if } x \notin X, \end{cases}$$

is continuous. Since $f(x) \neq f(y)$, we must have $f(x_f) \neq f(y_f)$. Since $f^{-1}(0)$ and $f^{-1}(1)$ are unions of $\equiv$-classes, we conclude that $x_f \not\equiv y_f$. This contradiction completes the proof of the "only if" part. $\triangle$

4. Strains and Varieties of Languages

Often we wish to deal, not just with the regular languages over a fixed alphabet (which form a Boolean algebra), but simultaneously with languages over various finite alphabets (which do not form a Boolean algebra, since the notion of complement is not well defined). One way of dealing with this problem is to shift attention to "locally regular" languages over an infinite alphabet (which form a Boolean algebra; this is approach is described by Almeida [A90]. A somewhat simpler alternative, which we shall employ, is to consider "strains" of regular languages. A *strain* $\mathbf{K}$ of regular languages is a sequence $\mathbf{K}_1, \mathbf{K}_2, \ldots,$ where each $\mathbf{K}_n$ is a Boolean subalgebra of $\mathbf{L}_n$, and where the sequence satisfies the following condition.

Str. For every homomorphism $h : \mathbf{E}_m \to \mathbf{E}_n$, $L \in \mathbf{K}_n$ implies $h^{-1}(L) \in \mathbf{K}_m$.

(Thus the set of languages in a strain is closed under taking inverse homomorphic images. The homomorphisms in question are finite objects, since they are determined by their values for words of length 1.)

Each of the Boolean algebras $\mathbf{K}_n$ of the strain $\mathbf{K}$ corresponds to a Stone quotient $\mathbf{I}_n/\equiv_n$ (where we have put the subscript n on the equivalence, since there is a separate equivalence for each value of n). We shall call such a sequence of equivalences $\equiv_1, \equiv_2, \ldots$ (each satisfying conditions I, II, III and IV of the preceding section) a "global equivalence" if they satisfy the following additional closure condition.

V. For every homomorphism $h : \mathbf{I}_m \to \mathbf{I}_n$ and every $x, y \in \mathbf{I}_m$, $x \equiv_m y$ implies $h(x) \equiv_n h(y)$.

Our first goal in this section is to establish a one-to-one correspondence between strains and global equivalences.

Theorem 4.1: A sequence $\mathbf{K}_n$ of subalgebras of $\mathbf{L}_n$ forms a strain if and only if their duals $\mathbf{I}_n/\equiv_n$ are obtained from a global equivalence.

We shall need the following lemma.

Lemma 4.2: If $g : \mathbf{E}_m \to \mathbf{E}_n$ is continuous, $f : \mathbf{I}_m \to \mathbf{I}_n$ is the the extension of g, $f(\mathbf{E}_m) \subseteq \mathbf{E}_n$, and $L \subseteq \mathbf{E}_n$ is regular, then

$$\mathrm{cl}\big(g^{-1}(L)\big) = f^{-1}\big(\mathrm{cl}(L)\big).$$

Proof: First we show that

$$\mathrm{cl}(L) \cap \mathbf{E}_n = L. \tag{1}$$

If $x \in L$, then $x \in \mathrm{cl}(L)$ and $x \in \mathbf{E}_n$, so we have $L \subseteq \mathrm{cl}(L) \cap \mathbf{E}_n$. On the other hand, if $x \notin L$, but $x \in \mathbf{E}_n$, then x belongs to the open set $\mathrm{cl}(\mathbf{E}_n \setminus L) = \mathbf{I}_n \setminus \mathrm{cl}(L)$. Since this open set is disjoint from L, we have $x \notin \mathrm{cl}(L)$, so we also have $\mathrm{cl}(L) \cap \mathbf{E}_n \subseteq L$. This establishes (1).

Next we show that

$$f^{-1}\big(\mathrm{cl}(L)\big) \cap \mathbf{E}_m = g^{-1}(L). \tag{2}$$

Indeed, we have

$$\begin{aligned} f^{-1}\big(\mathrm{cl}(L)\big) \cap \mathbf{E}_m &= f^{-1}\big(\mathrm{cl}(L)\big) \cap f^{-1}(\mathbf{E}_n) \cap \mathbf{E}_m \\ &= f^{-1}\big(\mathrm{cl}(L) \cap \mathbf{E}_n\big) \cap \mathbf{E}_m \\ &= f^{-1}(L) \cap \mathbf{E}_m \\ &= g^{-1}(L). \end{aligned}$$

This establishes (2).

Finally, we show that

$$\mathrm{cl}\big(g^{-1}(L)\big) = f^{-1}\big(\mathrm{cl}(L)\big).$$

Since $\mathrm{cl}(L)$ is crisp, $f^{-1}\big(\mathrm{cl}(L)\big)$ is also crisp. Thus $f^{-1}\big(\mathrm{cl}(L)\big)$ is a closed set containing $g^{-1}(L)$, and therefore also containing $\mathrm{cl}\big(g^{-1}(L)\big)$. It remains to show that $f^{-1}\big(\mathrm{cl}(L)\big) \subseteq \mathrm{cl}\big(g^{-1}(L)\big)$. Suppose that $x \in f^{-1}\big(\mathrm{cl}(L)\big)$, and let X be any open set containing x. We shall show that X meets $g^{-1}(L)$, and thus that $x \in \mathrm{cl}\big(g^{-1}(L)\big)$. Since X is open and $f^{-1}\big(\mathrm{cl}(L)\big)$ is open, $X \cap f^{-1}\big(\mathrm{cl}(L)\big)$ is open. Since $\mathbf{E}_m$ is dense in $\mathbf{I}_m$, we have $X \cap f^{-1}\big(\mathrm{cl}(L)\big) \cap \mathbf{E}_m \neq \emptyset$. But by (2), this implies $X \cap g^{-1}(L) \neq \emptyset$, which completes the proof. $\triangle$

Proof of Theorem 4.1: (if) Suppose that $\{\equiv_n\}_{n \in \mathbf{N}}$ is a global equivalence. Suppose further that $L \in \mathbf{K}_n$ and that $g : \mathbf{E}_m \to \mathbf{E}_n$ is a homomorphism. We first observe that if a congruence $\equiv$ separates $g(x)$ and $g(y)$, then the congruence $\equiv'$ (defined by $v \equiv' w$ if and only $g(v) \equiv g(w)$) is a congruence that has no more classes than $\equiv$ and separates x and y. We conclude that $d\big(g(x), g(y)\big) \leq d(x, y)$, and therefore that g is continuous.

Let $f : \mathbf{I}_m \to \mathbf{I}_n$ be the continuous extension of g. Since L is regular, $\mathrm{cl}(L)$ is crisp by Theorem 2.5. Since f is continuous, $f^{-1}\big(\mathrm{cl}(L)\big)$ is crisp. If $f^{-1}\big(\mathrm{cl}(L)\big)$ is not a union of $\equiv_m$-classes, then there exist $x \equiv_m y$ separated by $f^{-1}\big(\mathrm{cl}(L)\big)$. Since $\{\equiv_n\}_{n \in \mathbf{N}}$ is a global equivalence, we have $f(x) \equiv_n f(y)$. Furthermore, $f(x)$ and $f(y)$ are separated by $\mathrm{cl}(L)$. Thus $\mathrm{cl}(L)$ is not a union of $\equiv_n$-classes, contradicting the assumption that $L \in \mathbf{K}_n$. This contradiction shows that $f^{-1}\big(\mathrm{cl}(L)\big)$ is a crisp union of $\equiv_m$-classes. By Lemma 4.2, $f^{-1}\big(\mathrm{cl}(L)\big) = \mathrm{cl}\big(g^{-1}(L)\big)$, so $\mathrm{cl}\big(g^{-1}(L)\big)$ is a crisp union of $\equiv_m$-classes. Thus $g^{-1}(L) \in \mathbf{K}_m$. This concludes the proof of the "if" part.

(only if) Suppose that $\{\mathbf{K}_n\}_{n\in\mathbf{N}}$ is a strain of regular languages. Suppose further that $x \equiv_m y$ and that $h : \mathbf{I}_m \to \mathbf{I}_n$ is a homomorphism. For each $i \in \mathbf{N}$, choose $w_0, \ldots, w_{m-1} \in \mathbf{E}_n$ such that $d\big(w_a, h(a)\big) \le 1/(i+1)$ for each $a \in \mathbf{B}_m$, then define $g_i : \mathbf{E}_m \to \mathbf{E}_n$ to be the unique homomorphism such that $g_i(a) = w_a$ for each $a \in \mathbf{B}_m$. By virtue of the inequality $d(wx, yz) \le \max\{d(w,y), d(x,z)\}$, we have $d\big(g_i(v), h(v)\big) \le 1/(i+1)$ for all $v \in \mathbf{E}_m$. Each homomorphism g_i is continuous (as in the proof of the "if" part), and thus has a unique extension to a continuous map $f_i : \mathbf{I}_m \to \mathbf{I}_n$. Since d and f_i are continuous, $\mathbf{I}_m$ is compact and $\mathbf{E}_m$ is dense in $\mathbf{I}_m$, we have $d\big(f_i(v), h(v)\big) \le 1/(i+1)$ for all $v \in \mathbf{I}_m$.

Suppose, for the sake of contradiction, that $f_i(x) \not\equiv_n f_i(y)$. Let X be a crisp union of $\equiv_n$-classes separating $f_i(x)$ and $f_i(y)$. Take $L = X \cap \mathbf{E}_n$. Then $\mathrm{cl}(L) = X$ is crisp by Lemma 2.4, so $L \in \mathbf{K}_n$. Since $\{\mathbf{K}_n\}_{n\in\mathbf{N}}$ is a strain, we have $g_i^{-1}(L) \in \mathbf{K}_m$. It follows that $\mathrm{cl}\big(g_i^{-1}(L)\big)$ is a crisp union of $\equiv_m$-classes separating x and y, and thus that $x \not\equiv_m y$. This contradiction shows that $f_i(x) \equiv_n f_i(y)$. Since the sequences $\{f_i(x)\}_{i\in\mathbf{N}}$ and $\{f_i(y)\}_{i\in\mathbf{N}}$ converge to $h(x)$ and $h(y)$, respectively, and since the graph of the relation $\equiv_n$ is closed in $\mathbf{I}_n \times \mathbf{I}_n$ with the product topology, we conclude that $h(x) \equiv_n h(y)$. $\triangle$

A "variety" of regular languages (in the sense of Eilenberg [E73]) is a strain of regular languages that is closed under taking left and right "quotients" by words. If $L \subseteq \mathbf{E}_n$ is a regular language and $w \in \mathbf{E}_n$ is a word, then the *left quotient* of L by w is the regular language

$$w^{-1}L = \{x \in \mathbf{E}_n : wx \in L\},$$

and the *right quotient* is the regular language

$$Lw^{-1} = \{x \in \mathbf{E}_n : xw \in L\}.$$

Thus a *variety* of regular languages is a strain $\mathbf{K}$ of regular languages that satisfies the following condition.

Var. For every $L \in \mathbf{K}_n$ and $w \in \mathbf{E}_n$, $L \in \mathbf{Q}_n$ implies $w^{-1}L \in \mathbf{K}_n$ and $Lw^{-1} \in \mathbf{K}_n$.

Our final goal is to characterize varieties of regular languages among global subalgebras by additional closure conditions for their global equivalences. The following condition requires that each $\equiv_n$ be a congruence.

VI. For every $w, x, y, z \in \mathbf{I}_n$, $w \equiv_n y$ and $x \equiv_n z$ imply $wx \equiv_n yz$.

We then have the following.

Theorem 4.3: A strain $\mathbf{K}$ is a variety if and only if the corresponding global equivalence consists of congruences.

Proof: (if) Suppose that $\equiv_n$ is a congruence. Suppose further that $L \in \mathbf{K}_n$ and that $w \in \mathbf{E}_n$. We shall show that $w^{-1}L \in \mathbf{K}_n$; the proof that $Lw^{-1} \in \mathbf{K}_n$ is similar. Define $g : \mathbf{E}_n \to \mathbf{E}_n$ by $g(x) = wx$. We observe that if a congruence $\equiv$ separates $g(x)$ and $g(y)$, then it also separates x and y. Thus we have $d\big(g(x), g(y)\big) \le d(x, y)$, so g is continuous. Let $f : \mathbf{I}_n \to \mathbf{I}_n$ be the continuous extension of g. Since multiplication is continuous, we have $f(x) = wx$.

Since $L \in \mathbf{K}_n$, $\mathrm{cl}(L)$ is a crisp union of $\equiv_n$-classes. Since f is continuous, $f^{-1}\big(\mathrm{cl}(L)\big)$ is crisp. Suppose, for the sake of contradiction, that it is not a union of $\equiv_n$-classes. Then there exist $x \equiv_n y$ such that $f^{-1}\big(\mathrm{cl}(L)\big)$ separates x and y. Since $\equiv_n$ is a congruence and $x \equiv_n y$, we have $wx \equiv_n wy$. But since $f^{-1}\big(\mathrm{cl}(L)\big)$ separates x and y, we have that $\mathrm{cl}(L)$ separates $f(x) = wx$ and $f(y) = wy$. Since $\mathrm{cl}(L)$ is a union of $\equiv_n$-classes, this implies that $wx \not\equiv_n wy$. This contradiction establishes that the crisp set $f^{-1}\big(\mathrm{cl}(L)\big)$ is a union of $\equiv_n$-classes. By Lemma 4.2, $f^{-1}\big(\mathrm{cl}(L)\big) = \mathrm{cl}\big(g^{-1}(L)\big)$, so $\mathrm{cl}\big(g^{-1}(L)\big)$ is a crisp union of $\equiv_n$-classes. Thus $g^{-1}(L) \in \mathbf{K}_n$. Since $w^{-1}L = g^{-1}(L)$, this completes the proof of the "if" part.

(only if) Suppose that $\mathbf{K}_n$ is closed under left quotients by words. We shall show that $x \equiv_n y$ implies $wx \equiv_n wy$. A similar argument based on closure under right quotients shows that $x \equiv_n y$ implies $xz \equiv_n yz$. These conditions together establish that $\equiv_n$ is a congruence, since then $w \equiv_n y$ and $x \equiv_n z$ imply $wx \equiv_n yx \equiv_n yz$, so that $wx \equiv yz$.

Suppose, for the sake of contradiction, that $x \equiv_n y$ but $wx \not\equiv_n wy$. Let X be a crisp union of $\equiv_n$-classes that separates wx and wy Take $L = X \cap \mathbf{E}_n$. Then $\mathrm{cl}(L) = X$ is crisp, by Lemma 2.4. Thus $L \in \mathbf{K}_n$. Since $\mathbf{K}_n$ is closed under left quotients by words, $w^{-1}L \in \mathbf{K}_n$. Taking g and f as in the "if" part, we have $g^{-1}(L) = w^{-1}L$, so $g^{-1}(L) \in \mathbf{K}_n$. Thus $\mathrm{cl}\big(g^{-1}(L)\big)$ is a crisp union of $\equiv_n$-classes. By Lemma 4.2, $f^{-1}\big(\mathrm{cl}(L)\big) = \mathrm{cl}\big(g^{-1}(L)\big)$, so $f^{-1}\big(\mathrm{cl}(L)\big)$ is a crisp union of $\equiv_n$-classes. Furthermore, is must separate x and y, since $\mathrm{cl}(L)$ separates $f(x) = wx$ and $f(y) = wy$. This contradicts that assumption that $x \equiv_n y$, and completes the proof of the "only if" part. $\triangle$

5. Conclusion

Several directions for further work present themselves. Perhaps the most inviting is to try to obtain for strains an analog of Eilenberg's characterization of varieties. Presumably some enrichment of the notion of "finite semigroup" (to keep track of the allowable sets of initial and final states) would be involved. Another problem, which at present appears hard to even formulate in precise mathematical terms, is the following. A large number of varieties of regular languages have been characterized in terms of implicit identities in the literature (a sampling is given by Almeida [A88]). In almost all cases, the identities involve only ordinary terms and the one implicit term x^ω (introduced in Section 2). Why should this one implicit term (among uncountably many) play such an important role? (The situation is reminiscent of the completion of the rational numbers to the real numbers: of the uncountably many real numbers, a few (such as π and e) play much more important roles than most.)

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