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On The CONSISTENCY and COMPLETENESS of an EXTENDED NaDSet

by

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ABSTRACT

NaDSet in its extended form has been defined in several previous papers describing its applications. It is a Natural Deduction based Set theory and logic. In this paper the logic is shown to enjoy a form of ω-consistency from which simple consistency follows. The proof uses transfinite induction over the ordinals up to ε_0, in the style of Gentzen's consistency proof for arithmetic. A completeness proof in the style of Henkin is also given. Finally the cut rule of deduction is shown to be redundant.

INDEX TO SECTIONS & MAJOR SUBSECTIONS

1. INTRODUCTION

The NaDSet of this paper is a Natural Deduction based Set theory and logic that is an extension of the theory of the same name described in [Gilmore86]. It is the theory described in [Gilmore89] and in [Gilmore&Tsiknis90a,90b,91] and in [Tsiknis91]. NaDSet is presented here as a sequent calculus, although as demonstrated in [Gilmore&Tsiknis91], it can be presented in other natural deduction formats as well.

To keep the paper self-contained, the elementary and logical syntax of NaDSet, as described in [Gilmore89], is repeated in §§ 2, 3.1 and 3.2 in a somewhat abbreviated form. To demonstrate the significance of the abstraction rules of NaDSet, an elementary proof of the consistency of NaDSet*, which is NaDSet without abstraction rules, is given in § 3.3. NaDSet* is barely stronger than first order logic. In § 3.4 the concept of a global substitution is introduced along with a generalized form of the cut rule that depends upon a global substitution.

The consistency proof for NaDSet begins in § 4. It is a simplified and clarified version of the proof offered in [Gilmore90]. Terminology for derivations is introduced in §4.1 and the definition of the degree of an occurrence of a formula in a derivation in § 4.2. This definition is critical for the proof of consistency which is an adaptation of Gentzen's second proof of the consistency of arithmetic [Gentzen38] [Szabo69]. The proof uses transfinite induction up to ε_0 to prove that no derivation of the empty sequent exists. The proof proceeds by defining in § 5.1 five transformations of such derivations and proving in § 5.2 that at least one of them can always be applied. In § 6 the proof of consistency is completed by assigning in § 6.1 an ordinal less than ε_0 to each derivation of NaDSet, and proving that each of the five transformations of a derivation of the empty sequent transforms the derivation into one with a smaller ordinal. The consistency portion of the paper ends in § 6.3 where the form of ω-consistency proved for NaDSet is compared with the form introduced in [Gödel31]

In § 7 a semantics for NaDSet is described which is a direct extension of the semantics defined for NaDSet* in § 3.3. In § 8, a proof of completeness is given that is modeled on the completeness proofs offered in [Henkin49,50]. A consequence of the completeness theorem is that cut is a derivable rule of deduction.

The main theme of [Gilmore89] was that the general form of Cantor's diagonal argument cannot be justified in NaDSet because it involves what amounts to an abuse of use and mention. The general form of the argument is needed to prove what was called Cantor's lemma in [Gilmore89], namely that for each enumeration P of sequences of 0's and 1's, there is a sequence not enumerated by P. In § 9, Cantor's lemma is shown to be not derivable by using the methods developed in § 8 to construct an interpretation of NaDSet that does not satisfy the sequent expressing Cantor's lemma.

It is a pleasure to repeat the acknowledgements of [Gilmore90] which was written while I was visiting the University of Amsterdam: I am grateful to Johan van Benthem for the invitation to spend a term in the Department of Mathematics and Computer Science of the University of Amsterdam and for the material and technical help I received. Conversations with my colleagues Kees Doets, Dick de Jongh, and Anne Troelstra have been particularly helpful. In addition thanks are due to my student and colleague George Tsiknis whose critical reading of an earlier version of this paper helped in the writing of this version. Also comments by a referee of the papers [Gilmore89,90] are acknowledged with thanks; questions and concerns regarding those papers have suggested revisions and improvements.

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2. ELEMENTARY SYNTAX

Five kinds of strings form the basis for the elementary syntax. They are:
- variables that may be bound by quantifiers or abstraction terms,
- first and second order constants, and
- first and second order parameters.

The particular notation used for these objects is unimportant. It is only necessary to assume that there are five distinct notations that admit denumerably many objects of each kind.

To simplify the description of NaDSet and reduce the number of cases that must be considered in subsequent proofs, only a single logical connective '$\downarrow$' and only a universal quantifier '$\forall$' are taken to be primitive. The connective '$\downarrow$' is joint denial, so that $(\mathbf{F}\downarrow\mathbf{G})$ has the same truth table as $(\sim\mathbf{F} \wedge \sim\mathbf{G})$. However, other logical connectives and the existential quantifier will be freely used when convenient.

In the following definition, as throughout the paper, bold letters or pairs of letters represent metavariables over particular sets of strings.

2.1. Definition of Elementary Syntax

2.1.1. Elementary Terms

- A variable is a term. The single occurrence of the variable in the term is a free occurrence in the term.
- Any parameter or constant is a term. No variable has a free occurrence in the term.

2.1.2. Formulas

- If **r** and **s** are any terms, then **r:s** is a formula. A free occurrence of a variable in **r** or in **s**, is a free occurrence of the variable in the formula.
- If **G** and **H** are formulas then **(G↓H)** is a formula. A free occurrence of a variable in **G** or in **H** is a free occurrence in **(G↓H)**.
- If **F** is a formula and **v** a variable, then **∀vF** is a formula. A free occurrence of a variable other than **v** in **F**, is a free occurrence in **∀vF**; no occurrence of **v** is free in **∀vF**.

2.1.3. Abstraction Terms

Let **t** be any term in which there is at least one free occurrence of a variable and no occurrence of a parameter. Let **F** be any formula. Then **{t | F}** is an abstraction term and a term. A free occurrence of a variable in **F** which does not also have a free occurrence in **t**, is a free occurrence in **{t | F}**. A variable with a free occurrence in **t** has no free occurrence in **{t | F}**.

2.1.4. First & Second Order Terms, Atomic & Closed Formulas

- A term is first order if no second order parameter occurs in it; otherwise it is second order.
- A formula **r:s** is atomic if **r** is first order, and **s** is a second order parameter or constant.
- A term or formula in which no variable has a free occurrence is said to be closed.

Clause 2.1.3 of this definition introduces the syntax for set abstraction. It generalizes the conventional syntax **{v | F}** in which **t** may only be a single variable **v**. For example, when **t** is the term <x,y> representing the ordered pair of x and y as defined in 4.1 below, **{<x,y> | F}** is the set of ordered pairs <x,y> for which **F** is satisfied. The more general form of the abstraction term is essential for many of its applications, including those for algebra and category theory.

2.1.5 Closed Formulas

It is important to understand what are free and bound occurrences of variables in a term $\{\mathbf{t} \mid \mathbf{F}\}$. For example, let '[x,B,y / u,v,w]' be a substitution operator that replaces free occurrences of 'u', 'v' and 'w', respectively, by 'x', 'B' and 'y'. Then

$$[x,B,y\ /\ u,v,w](\langle u,v\rangle:\{\langle u,v\rangle \mid u:v \wedge \langle v,w\rangle:B\})$$

is the formula

$$\langle x,B\rangle:\{\langle u,v\rangle \mid u:v \wedge \langle v,y\rangle:B\}$$

since the occurrence of 'w' in

$$\langle u,v\rangle:\{\langle u,v\rangle \mid u:v \wedge \langle v,w\rangle:B\}$$

is free, while only the first occurrence of 'u' and 'v' is free. The variables 'x' and 'y' are the only variables with free occurrences in $\langle x,B\rangle:\{\langle u,v\rangle \mid u:v \wedge \langle v,y\rangle:B\}$.

A closed formula must take one of the following three forms:

a) $(\mathbf{G}\downarrow\mathbf{H})$, where **G** and **H** are closed formulas.

b) $\forall\mathbf{v}\mathbf{F}$, where **F** is a formula in which at most the variable **v** has a free occurrence.

c) **r**:**s**, where both **r** and **s** are closed terms.

The only subforms of the latter are the following three:

i) **s** is $\{\mathbf{t} \mid \mathbf{F}\}$, where the variables with a free occurrence in **F** have a free occurrence in **t**.

ii) **r**:**s** is atomic; that is, **r** is first order and **s** is a second order parameter or constant.

iii) **r** is second order or **s** is a first order parameter or constant.

3. LOGICAL SYNTAX

A sequent takes the form

$$\Gamma \rightarrow \Theta,$$

where Γ and Θ are finite, possibly empty, sequences of closed formulas. The formulas Γ form the antecedent of the sequent, and the formulas of Θ the succedent. A sequent can be interpreted as asserting that one of the formulas of its antecedent is false, or one of the formulas of its succedent is true.

3.1. Definition of Logical Syntax

In the following, **G** and **H** are any closed formulas unless otherwise stated.

3.1.1. Axioms

$$\mathbf{A \to A}$$

where **A** is a closed atomic formula

3.1.2. Logical Rules

Propositional

$$\frac{\Gamma, \mathbf{G} \to \Theta \qquad \Delta, \mathbf{H} \to \Lambda}{\Gamma, \Delta \to (\mathbf{G}\downarrow\mathbf{H}), \Theta, \Lambda} \qquad \frac{\Gamma \to \mathbf{G}, \Theta}{\Gamma, (\mathbf{G}\downarrow\mathbf{H}) \to \Theta} \qquad \frac{\Gamma \to \mathbf{H}, \Theta}{\Gamma, (\mathbf{G}\downarrow\mathbf{H}) \to \Theta}$$

Quantification

$$\frac{\Gamma \to [\mathbf{p}/\mathbf{u}]\mathbf{G}, \Theta}{\Gamma \to \forall\mathbf{u}\mathbf{G}, \Theta} \qquad \frac{\Gamma, [\mathbf{r}/\mathbf{u}]\mathbf{G} \to \Theta}{\Gamma, \forall\mathbf{u}\mathbf{G} \to \Theta}$$

In the first rule, **p** is a parameter that does not occur in **G**, or in any formula of Γ or Θ. In the second rule, **r** is any closed term.

Abstraction

$$\frac{\Gamma \to [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G}, \Theta}{\Gamma \to [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}:\{\mathbf{t} \mid \mathbf{G}\}, \Theta} \qquad \frac{\Gamma, [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G} \to \Theta}{\Gamma, [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}:\{\mathbf{t} \mid \mathbf{G}\} \to \Theta}$$

$\underline{\mathbf{u}}$ is a sequence of the distinct variables with free occurrences in the term **t**.
G is a formula in which only the variables $\underline{\mathbf{u}}$ have free occurrences.
$\underline{\mathbf{r}}$ is a sequence of closed terms, one for each variable in $\underline{\mathbf{u}}$.
$[\underline{\mathbf{r}}/\underline{\mathbf{u}}]$ is a substitution operator that replaces each occurrence of the variables $\underline{\mathbf{u}}$, respectively, with the corresponding terms $\underline{\mathbf{r}}$.

3.1.3. Structural Rules

The structural rules consist of contraction, interchange, and thinning rules. The effect of the contraction and interchange rules is to treat the antecedent and succedent of a sequent as finite sets of formulas.

Contraction

$$\frac{\Gamma \to G, G, \Theta}{\Gamma \to G, \Theta} \qquad \frac{\Gamma, G, G \to \Theta}{\Gamma, G \to \Theta}$$

Interchange

$$\frac{\Gamma \to \Lambda, G, H, \Theta}{\Gamma \to \Lambda, H, G, \Theta} \qquad \frac{\Gamma, G, H, \Delta \to \Theta}{\Gamma, H, G, \Delta \to \Theta}$$

Thinning

$$\frac{\Gamma \to \Theta}{\Gamma \to G, \Theta} \qquad \frac{\Gamma \to \Theta}{\Gamma, G \to \Theta}$$

3.1.4. Cut Rule

$$\frac{\Gamma \to \Theta, G, \Theta' \qquad \Delta, G, \Delta' \to \Lambda}{\Gamma, \Delta, \Delta' \to \Theta, \Theta', \Lambda}$$

End of Definition 3.1

3.2. Notations and Observations

The propositional, quantification and abstraction rules will be denoted respectively by:

$\to\downarrow$, $\downarrow\to$, $\to\forall$, $\forall\to$, $\to\{\}$ and $\{\}\to$.

The thinning and cut rules will be referred to by name. When it is necessary to distinguish between the two $\downarrow\to$ rules, they will be denoted by $L\downarrow\to$ and $R\downarrow\to$ respectively.

All the usual logical connectives $\sim$, $\wedge$, $\vee$, $\supset$ and $\equiv$ and the existential quantifier $\exists$ can be defined using $\downarrow$ and $\forall$. Corresponding rules of deduction can be derived and when necessary will be denoted respectively by:

$\sim\to$, $\to\sim$, $\to\wedge$, $\wedge\to$, $\to\vee$, $\vee\to$, $\to\supset$, $\supset\to$, $\to\equiv$, $\equiv\to$, $\to\exists$ and $\exists\to$.

The cut rule takes a slightly different form than the usual; this form permits one to ignore where the cut formula **G** appears in the succedent of the first premiss or in the antecedent of the second premiss. Because of the interchange rule, this form of the rule is equivalent to the usual forms of it. Also because of interchange it is possible to be careless about the order of the formulas in the premiss and conclusion of the rules of deduction.

The parameter **p** used in an application of $\rightarrow\forall$ is called the eigenparameter (briefly e-par) of the application, and the term **t** used in an application of $\forall\rightarrow$ is called the eigenterm (briefly e-term) of the application.

Because the axioms are restricted to being sequents of closed formulas and the thinning rules may only introduce closed formulas, only sequents of closed formulas are derivable in NaDSet.

In § 8 it is proved that cut is a redundant rule; that is, if the premisses of an application of the rule are derivable, then so is the conclusion.

3.2.1. Failure of Parameter by Term Replacement

Gentzen's proof of the consistency of arithmetic made use of a simple property of the axioms of his theory: Any parameter occurring in a logical axiom $\mathbf{F} \rightarrow \mathbf{F}$ can be replaced in all of its occurrences by a term, and the resulting sequent will still be an axiom. As a consequence, the replacement of all occurrences of a parameter in a derivation by a term results in a derivation. This is not true in general of NaDSet: A second order parameter cannot in general be replaced by a term in a derivation, nor can a first order parameter ever be replaced by a second order term. This failure of parameter by term replacement is the main source of complications in the proof of consistency of NaDSet given below. However, three usual properties of logics can be verified for NaDSet.

3.2.2. Lemma:

All occurrences of a first order parameter in a derivation may be replaced by a first order term, provided the parameter is not an e-par for an application of $\rightarrow\forall$, and no parameter appearing in the term is the e-par of an application of $\rightarrow\forall$.

3.2.3. Lemma:

The e-par of an application of the $\rightarrow\forall$ rule in a given derivation can be changed to any other parameter of the same order provided the parameter does not occur in the given derivation.

3.2.4. Lemma:

Let $\Gamma \rightarrow \Theta$ be a sequent in which a first order parameter **p** occurs and let **c** be a first order constant which is distinct from any appearing in $\Gamma \rightarrow \Theta$. Let $\Gamma' \rightarrow \Theta'$ be obtained from $\Gamma \rightarrow \Theta$ by replacing each occurrence of **p** by **c**. Then $\Gamma \rightarrow \Theta$ is derivable if and only if $\Gamma' \rightarrow \Theta'$ is.

3.3. The Consistency of NaDSet Without Abstraction Rules

Consider the logic NaDSet* that has the same elementary syntax as NaDSet, but has a logical syntax differing from NaDSet simply in the absence of the two abstraction rules. NaDSet* differs very little from first order logic, and a consistency proof for it is no harder to construct.

Define $\mathbb{d}$ to be the set of all first order terms in which no parameter occurs. An interpretation of NaDSet* is then defined:

3.3.1. Definition: Interpretation

An interpretation consists of a set $\mathbb{D}$ of subsets of $\mathbb{d}$, called the domain of the interpretation, and a function Φ called the assignment of the interpretation, satisfying:

- For each first order parameter **p**, $\Phi[\mathbf{p}] \in \mathbb{d}$;
- For each second order parameter or constant **PC**, $\Phi[\mathbf{PC}] \in \mathbb{D}$.

End of Definition

For a given closed first order term **r**, $\Phi[\mathbf{r}]$ is defined to be the term obtained from **r** by replacing every occurrence of a parameter **p** in **r** by $\Phi[\mathbf{p}]$; note that since **r** is first order, **p** is necessarily first order, and that constants occurring in **r** are unaffected. For a given closed first order term **r**, $\Phi[\mathbf{r}]$ is necessarily a member of $\mathbb{d}$.

An assignment Φ of an interpretation assigns one and only one of the truth values true or false to each closed atomic formula **r**:**PC**: It is assigned true if $\Phi[\mathbf{r}]$ is a member of $\Phi[\mathbf{PC}]$, otherwise it is false. Truth values are assigned to other closed formulas indirectly through sets $\Omega[\Phi, \mathbb{D}]$ of signed formulas defined by finite induction. A signed formula is a closed formula prefixed with a + or - sign. A formula **F** is assigned true if $+\mathbf{F} \in \Omega[\Phi, \mathbb{D}]$, and is assigned false if $-\mathbf{F} \in \Omega[\Phi, \mathbb{D}]$.

3.3.2. Definition: The Set $\Omega[\Phi, \mathbb{D}]$ of Signed Formulas

Given the base $\mathbb{D}$ and assignment Φ of an interpretation, $\Omega[\Phi, \mathbb{D}]$ is the set

$$\cup\{\Omega_n[\Phi, \mathbb{D}] \mid n \geq 0\},$$

where $\Omega_n[\Phi, \mathbb{D}]$, abbreviated to Ω_n, are sets of signed formulas defined inductively for integers n, $n \geq 0$, as follows:

1. If $\Phi[\mathbf{r}] \in \Phi[\mathbf{PC}]$, then $+\mathbf{r}:\mathbf{PC} \in \Omega_0$, otherwise $-\mathbf{r}:\mathbf{PC} \in \Omega_0$.
2. Assuming Ω_n is defined for an integer n, Ω_{n+1} is the least set satifying:

$\cup$ $\Omega_{n+1} \supseteq \Omega_n$

$+\downarrow$ $-G \in \Omega_n$ and $-H \in \Omega_n \Rightarrow +(G\downarrow H) \in \Omega_{n+1}$

$-\downarrow$ $+G \in \Omega_n$ or $+H \in \Omega_n \Rightarrow -(G\downarrow H) \in \Omega_{n+1}$

Let **G** be a formula in which **u** is the only variable with a free occurrence.

$+\forall$ Let **p** be a first or second order parameter not occurring in the formula **G**, and let $+[p/u]G \in \Omega_n[\Phi', \mathbb{D}]$ for every assignment Φ' that differs from Φ only in the value of $\Phi[p]$. Then $+\forall uG \in \Omega_{n+1}$.

$-\forall$ For any closed term **r**, $-[r/u]G \in \Omega_n \Rightarrow -\forall uF \in \Omega_{n+1}$.

End of Definition

By simple induction on n it is possible to prove that for no closed formula **F** is both **+F** and **-F** a member of $\Omega_n[\Phi, \mathbb{D}]$, for any domain $\mathbb{D}$ and assignment Φ.

The degree of a formula can be defined in the usual way as the count of the number of occurrences of $\downarrow$ and of $\forall$ in the formula. By simple induction on degrees it is possible to prove that for every formula **F**, one of **+F** and **-F** is a member of $\Omega[\Phi, \mathbb{D}]$.

A sequent $\Gamma \rightarrow \Theta$ is satisfied by an interpretation with base $\mathbb{D}$ and assignment Φ, if for some formula **F**,

$$F \in \Gamma \text{ and } -F \in \Omega[\Phi, \mathbb{D}], \text{ or } F \in \Theta \text{ and } +F \in \Omega[\Phi, \mathbb{D}];$$

that is, one of the premisses in Γ is false, or one of the conclusions in Θ is true.

By simple induction on the number of applications of logical rules appearing in a derivation it is possible to prove that every sequent derivable in NaDSet* is satisfied by every interpretation. Since the empty sequent is satisfied by no interpretation, it follows that the empty sequent is not derivable in NaDSet* and therefore that the logic is consistent.

These results for the logic NaDSet* illuminates the special role that the abstraction rules play in NaDSet; that role will be further illuminated by the consistency proof offered below for NaDSet.

3.4. Derivations with Global Substitutions

Because of the failure of parameter by term replacement it is necessary to generalize the cut rule of derivation to what is called a σ-cut, where σ is a global substitution defined in 3.4.1. A σ-derivation is then a derivation in which applications of σ-cuts may appear.

3.4.1. Global Substitutions

Each application of $\rightarrow\forall$ has an e-par and each application of $\forall\rightarrow$ has an e-term. During the reduction process to be described for a derivation of the empty sequent, a one-to-one mapping of applications of $\rightarrow\forall$ onto applications of $\forall\rightarrow$ is constructed which leads to a mapping of e-pars onto e-terms. In Gentzen's proof of consistency, each e-par is replaced by its corresponding e-term. But in NaDSet this is not possible, so a record must be kept of the substitutions that would be made if they could be made. The record is kept as a global substitution consisting of zero or more components of the form $[\mathbf{r}_i / \mathbf{p}_i]$, where $\mathbf{r}_i$ is a closed term and $\mathbf{p}_i$ is a parameter, satisfying the condition:

A parameter $\mathbf{p}_i$ occurs in a term $\mathbf{r}_j$ only if $i < j$.

The order of the components is therefore significant; it is determined by the order in which the parameters $\mathbf{p}_i$ are encountered in the reduction process as e-pars for applications of $\rightarrow\forall$, with $\mathbf{r}_i$ being the e-term of the corresponding application of $\forall\rightarrow$.

Let a global substitution σ have components $[\mathbf{r}_1 / \mathbf{p}_1], [\mathbf{r}_2 / \mathbf{p}_2], \ldots, [\mathbf{r}_k / \mathbf{p}_k]$. Each component $[\mathbf{r}_i / \mathbf{p}_i]$ of σ has the effect of replacing every occurrence of $\mathbf{p}_i$ in a term or formula to which it is applied by $\mathbf{r}_i$. The result σ(**F**) of applying σ to a term or formula **F** is the result of successively applying the components in the reverse order: first $[\mathbf{r}_k / \mathbf{p}_k]$, then $[\mathbf{r}_{k-1} / \mathbf{p}_{k-1}], \ldots$, and then finally $[\mathbf{r}_1 / \mathbf{p}_1]$.

Because of lemma 3.2.2, it may be assumed of a component $[\mathbf{r}_i / \mathbf{p}_i]$ of a global substitution that at least one of $\mathbf{r}_i$ and $\mathbf{p}_i$ is second order.

3.4.2. σ-Cuts

Given a global substitution σ, the σ-cut rule takes the form

$$\frac{\Gamma \rightarrow \Theta, \mathbf{G1}, \Theta' \qquad \Delta, \mathbf{G2}, \Delta' \rightarrow \Lambda}{\Gamma, \Delta, \Delta' \rightarrow \Theta, \Theta', \Lambda}$$

where **G1** and **G2** are closed formulas for which σ(**G1**) is σ(**G2**).

G1 and **G2** are called respectively the succedent and antecedent cut formulas of the application.

A σ-derivation is a derivation in which any number of applications of the σ-cut rule may appear, provided they are all applications for the same global substitution σ. Since an application of the cut rule is an application of the σ-cut rule when σ is the empty substitution, it follow that a derivation is a σ-derivation for the empty substitution σ.

Henceforth, unless otherwise noted, by a derivation is meant a σ-derivation for a given σ, and by a cut is meant a σ-cut.

3.4.3. Classification of Cuts

Let **G1** and **G2** be respectively the succedent and antecedent formulas of an application of a cut in a derivation. Since σ(**G1**) is σ(**G2**) for the given σ of the derivation, both **G1** and **G2** together must be of one of the three forms (a), (b), or (c) described in 2.1.5. In the first two cases the cut is said to be respectively a ↓ cut or a ∀ cut. In the third case the edge is called a {} cut if both **G1** and **G2** have the form (ci), an atomic cut if at least one of them has the form (cii) and the other does not have the form (ciii), and a thinned cut if either has the form (ciii). The latter name is used because a formula of the form (ciii) can only be introduced into a derivation by an application of thinning.

4. TERMINOLOGY & DEGREES

Although the consistency proof is an adaptation of Gentzen's second proof of consistency of elementary number theory [Gentzen], the terminolgy used differs at times from that used in the translation of that paper offered in chapter 8 of [Szabo69]; the differences are largely all noted in § 4.1. The terminology introduced in § 4.1 is necessary for the definition of the degree of an occurrence of a formula in a derivation given in § 4.2.

4.1 Terminology

4.1.1. Endsequent and Branch of a Derivation

A σ–derivation, henceforth called simply a derivation, takes the form of a tree with leaves that are axioms, and with a single sequent at the root of the tree called the endsequent of the derivation. Each sequent in the tree, other than an axiom at a leaf, is the conclusion of an application of a rule of deduction with the premiss or premisses of the application immediately above the conclusion in the tree. (In [Szabo69], an axiom is called a basic sequence, a rule of deduction an inference figure schemata, and an application of a rule of deduction an inference figure.) When there is no risk of confusion, an application of a rule of deduction in a derivation will be referred to simply as a rule of deduction.

A branch of a given derivation is a sequence $Sq_1, \ldots, Sq_n$ of sequents Sq_i, $i \geq 1$, where Sq_{i+1} is a premiss of a rule of deduction with conclusion Sq_i. Thus the order of sequents in a branch is upwards in the tree. One sequent Sq_j is said to be above (respectively below) another sequent Sq_i, if there is a branch of the tree with Sq_i and Sq_j as members in which the first precedes (succedes) the second in the branch.

4.1.2. Principal and Corresponding Formulas

In an application of a rule of deduction, specific formulas must replace the metasystem variables printed in bold in the description of the rule, and specific sequences of formulas must replace the sequences denoted by uppercase Greek letters. The specific formulas replacing the metasystem variables are called the principal formulas of the application. An application of any logical rule, or of any structural rule other than interchange, has a single principal formula in its conclusion, while an application of interchange has two principal formulas in its conclusion. The premiss of an application of contraction has two principal formulas. Each premiss of an application of a logical rule has a single principal formula. Each application of a thinning rule has no principal formula in its premiss, while each application of cut has no principal formula in its conclusion. (In [Szabo69] only an application of a logical rule has a principal formula, and it is the principal formula of the conclusion)

Each principal formula in the conclusion of an application of a rule other than contraction has a single corresponding principal formula in each premiss of the rule. The principal formula in the conclusion of an application of contraction has two corresponding principal formulas in its premiss. Each formula that is not a principal formula of the conclusion of an application has a single corresponding identical formula in one premiss of the rule. The Gentzen rules can be derived from the rules used here by zero or more applications of contraction and interchange.

Note that these results are a consequence of the slightly different formulation of the rules of deduction of NaDSet from those of [Gentzen38]; the formulation used here is similar to that of [Kleene52].

4.1.3. Predecessor, Logical Predecessor, and (Top) Identical Predecessor

Consider an occurrence of a formula in a derivation. A *predecessor* of the occurrence is the occurrence itself, or a formula in a premiss of an application of a rule of deduction corresponding to a predecessor in the conclusion. The principal formula of a premiss of an application of a logical rule is a *logical predecessor* of the principal formula of its conclusion.

A predecessor **H** of an occurrence **F** is called an *identical predecessor*, if **H**, and every predecessor **G** of **F** of which **H** is a predecessor, is an occurrence of the same formula as **F**. A *top* identical predecessor is an identical predecessor that is the antecedent or succedent of an axiom, or the principal formula in the conclusion of an application of thinning or of a logical rule.

4.1.4. Succesor and Last Successor

An occurrence **G** is a *successor* of an occurrence **F**, if **F** is a predecessor of **G**. It is the *last successor* of **F** if it is a cut formula in an application of cut.

4.1.5. Blocked Applications of $\forall\rightarrow$

The principal formula of the premiss of an application of the $\rightarrow\forall$ rule takes the form [**p/u**]**F**, where **p** is the e-par of the application. The requirement that the e-par of an application cannot occur in any formula in the conclusion is referred to briefly as the *e-par restriction*. The principal formula of the premiss of an application of the $\forall\rightarrow$ rule takes the form [**r/u**]**F**, where **r** is the e-term of the application.

There is no e-term restriction similar to the e-par restriction. However, complications arise from interactions between applications of $\forall\rightarrow$ and $\rightarrow\forall$. Consider the following example derivation in which the horizontal bars between premiss and conclusion have been omitted:

$c{:}P \rightarrow c{:}P$	axiom
$\forall x\ c{:}x \rightarrow c{:}P$	$\forall\rightarrow$
$\forall x\ c{:}x \rightarrow \forall x\ c{:}x$	$\rightarrow\forall$
$\forall x\ c{:}x,\ (\forall x\ c{:}x \downarrow \forall x\ c{:}x) \rightarrow$	$\downarrow\rightarrow$
$\forall x\ c{:}x,\ c{:}\{u \mid (\forall x\ u{:}x \downarrow \forall x\ u{:}x)\} \rightarrow$	$\{\}\rightarrow$
$\forall x\ c{:}x,\ \forall x\ c{:}x \rightarrow$	$\forall\rightarrow$
$\forall x\ c{:}x \rightarrow$	contraction

The first rule applied is $\forall\rightarrow$; its e-term P is the e-par of the application of $\rightarrow\forall$ that follows it. Thus the $\forall\rightarrow$ rule removes an occurrence of the parameter P in the antecedent of the premiss of the $\rightarrow\forall$ rule, permitting that application to satisfy the e-par restriction. The removal of the application of $\forall\rightarrow$ would prevent the application of $\rightarrow\forall$ from being made; the application of $\forall\rightarrow$ is said to be blocked by the application of $\rightarrow\forall$. The second application of $\forall\rightarrow$ is however not blocked by any application of $\rightarrow\forall$.

The full definition of blocking follows: Consider a branch of a derivation ending in the premiss

$\Gamma \rightarrow$ **[p /u]F**, Θ

of an application of $\rightarrow\forall$, and consider an application of $\forall\rightarrow$ occuring in the branch. The application of $\forall\rightarrow$ is said to be blocked by the application of $\rightarrow\forall$ if the e-par **p** occurs in the e-term of the application of $\forall\rightarrow$. An application of $\forall\rightarrow$ is said to be blocked in a derivation if it is blocked by some application of $\rightarrow\forall$ in the derivation.

4.2. Definition of Degree & Height

The definition of the degree of a formula given in § 3.3 as the count of the number of occurrences of $\downarrow$ and of $\forall$ in the formula is no longer useful in NaDSet because of the interaction between the { } and $\forall\rightarrow$ rules. For example, the last application of $\forall\rightarrow$ in the derivation appearing in 4.1.5 removes a term that has been introduced by a { } rule. The definition defined ultimately in 4.2.1 is that of the degree of an occurrence of a formula in a derivation.

4.2.1. Definition: Degree Paths and Degrees of Occurrences of Formulas

A degree path dp in a derivation Derv is a sequence $\mathbf{F}_1, \dots, \mathbf{F}_m$, $m \geq 1$, of distinct occurrences of formulas in Derv for which $\mathbf{F}_i$ and $\mathbf{F}_{i+1}$ satisfy one of the following conditions, for $i < m$:

1. $\mathbf{F}_{i+1}$ is a logical predecessor of $\mathbf{F}_i$;
2. $\mathbf{F}_{i+1}$ is a distinct immediate predecessor of $\mathbf{F}_i$;
3. $\mathbf{F}_i$ and $\mathbf{F}_{i+1}$ are the cut formulas of an application of cut; or
4. One of $\mathbf{F}_i$ and $\mathbf{F}_{i+1}$ is the antecedent and the other is the succedent formula of an axiom.

The degree deg(dp, Derv) of a degree path dp with elements $\mathbf{F}_1, \dots, \mathbf{F}_m$ is defined recursively as follows:

1. If dp has a single element $\mathbf{F}_1$, then deg(dp, Derv) is 0;
2. Let dp have elements $\mathbf{F}_1, \dots, \mathbf{F}_i$, and dp' elements $\mathbf{F}_1, \dots, \mathbf{F}_i, \mathbf{F}_{i+1}$. Then

   deg(dp', Derv) = deg(dp, Derv) + 1 if $\mathbf{F}_i, \mathbf{F}_{i+1}$ satisfy clause (1) in the definition of degree path; otherwise

   deg(dp', Derv) = deg(dp, Derv).

The degree deg(**F**, Derv) of an occurrence **F** in a derivation Derv is the maximum of the degrees of

degree paths with **F** as first element.
End of Definition 4.2.1

Consider, for example, the derivation appearing in 4.1.5. There is one degree path of degree 1 beginning with the occurrence of ∀x c:x in the endsequent of the derivation and ending in the antecedent formula c:P of the axiom; although this path can be extended to the succedent formula of the axiom, no increase in degree results. A second degree path of degree 4 has the occurrence of c:P in the succedent of the axiom as last element. The degree of the occurrence of ∀x c:x is therefore 4. Note, however, that the degrees of the two occurrences of this same formula in the penultimate sequent are respectively 1 and 4.

It might be supposed that the definition of degree could be simplified by omitting clauses 3 and 4 from the definition of degree path. But the need for these clauses will become evident when the effect of an axiom-transformation defined in 5.1.5 is examined.

The following lemma states two observations that follow immediately from definition 4.2.1:

4.2.2. Lemma:

1. Let **F** be the principal formula in the conclusion of an application of a logical rule and let **G** be a principal formula in a premiss. Then the deg(**F**, Derv) < deg(**G**, Derv).
2. Let **F** and **G** be the cut formulas in an application of cut in a derivation Derv. Let Derv1 be the derivation of the premiss in which **F** occurs, and let Derv2 be the derivation of the premiss in which **G** occurs. Then

   deg(**F**, Derv) = deg(**G**, Derv) = max{ deg(**F**, Derv1), deg(**G**, Derv2)}

The property (1) of degrees is the same as the property of degrees exploited in Gentzen's consistency proof for arithmetic. That deg(**F**, Derv) = deg(**G**, Derv) for cut formulas was also used. However, note that a significant difference with the traditional definition of degrees is this: The degree of an occurrence of a formula in a derivation can be affected by changes in the derivation.

By (2) of lemma 4.2.2, the cut formulas of an application of cut have identical degrees in any derivation in which the application appears. The degree of an application of cut is the the degree of its cut formulas.

The following definition is adapted from Gentzen:

4.2.3. Definition: Height of a Sequent in a Derivation
The height of a sequent in a derivation is the maximum of the degrees of cuts with conclusion appearing below the sequent.

In [Szabo69] the height of a sequent is called its level, but the definition of level also involves the degree of CJ rules of deduction that are absent in NaDSet.

Thus if h2 is the height of the conclusion of an application of cut, and d is the degree of the application, then the height h1 of the premisses of the application satisfies:

$$h1 = \max\{d, h2\}$$

5. TRANSFORMATIONS OF DERIVATIONS

The proof of consistency proceeds by showing that any derivation Derv of the empty sequent can be reduced to a simpler one Derv*, in a sense defined in § 6.1. Five transformations of derivations are described in 5.1 that will be shown in § 6.2 to produce simpler derivations. As noted in 4.2, the degree of an occurrence of a formula in a derivation can be affected by changes in the derivation; the effect of the transformations on degrees is described along with the transformations in the subsections of 5.1. Given any derivation of the empty sequent, it is proved in 5.2 that at least one of the transformations can be applied.

5.1. The Transformations

Of the five derivations described in this section three correspond directly to transformations described in [Gentzen38], [Szabo69], while one arising from applications of the { } rules is simpler than the Gentzen transformations and a fifth is a new kind made necessary by the character of σ-cuts.

Each of the transformations acts on a given application or applications of cut in a given derivation Derv and produces another derivation Derv*. It is assumed that a global substitution σ is given for Derv of which the following is an application of σ-cut:

$$\frac{\Gamma \rightarrow \mathbf{G1}, \Theta \quad \Delta, \mathbf{G2} \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda}$$

that is, σ(**G1**) is σ(**G2**).

5.1.1. A Thinning-Transformation

Assume that each top identical predecessor of **G1** is the conclusion of an application of thinning. Then all these applications of thinning are dropped from the derivation of $\Gamma \rightarrow \mathbf{G1}, \Theta$ to produce a derivation of $\Gamma \rightarrow \Theta$. A derivation of $\Gamma, \Delta \rightarrow \Theta, \Lambda$ can then be obtained by zero or more thinnings and interchanges.

The derivations Derv and Derv* are illustrated here:

$$\dfrac{\dfrac{\overset{\cdots}{\Gamma \rightarrow \mathbf{G1}, \Theta} \qquad \Delta, \mathbf{G2} \rightarrow \Lambda}{\cdots \quad \Gamma, \Delta \rightarrow \Theta, \Lambda \quad \cdots}}{\rightarrow} \qquad\qquad \dfrac{\dfrac{\overset{\cdots}{\Gamma \rightarrow \Theta}}{\cdots \quad \Gamma, \Delta \rightarrow \Theta, \Lambda \quad \cdots}}{\rightarrow}$$

Here the occurrences of '. . .' represent portions of the derivation not explicitly displayed, while the double dotted lines represent zero or more applications of rules; the first of the double lines in Derv* represent zero or more applications of thinnings and interchanges.

The degrees of occurrences of formulas in Γ and Θ are unaffected by the transformation. The degrees of occurrences in Δ and Λ are reduced to zero since they have been introduced by thinnings. The height of a sequent in Derv* may be less than the corresponding sequent in Derv since a cut has been removed.

A similar transformation is possible if each top identical predecessor of **G2** is the conclusion of an application of thinning.

5.1.2. A ↓-Transformation

Consider an application with cut formulas **(G1↓H1)** and **(G2↓H2)**. Assume that a thinning-transformation cannot be applied. Therefore each of the cut formulas has at least one top identical predecessor that is the principal formula in the conclusion of an application of the →↓ and ↓→ rules respectively. Assume that the latter is an application of the L↓→ rule. Then the derivation Derv takes the following form:

$$
\begin{array}{ccc|c}
\cdots \quad \Gamma', G1 \rightarrow \Theta' \qquad \cdots \quad \Gamma'', H1 \rightarrow \Theta'' & & \cdots \quad \Delta' \rightarrow G2, \Lambda' & \text{Height} \\
\hline
\cdots \quad \Gamma', \Gamma'' \rightarrow (G1{\downarrow}H1), \Theta', \Theta'' \quad \cdots & & \cdots \quad \Delta', (G2{\downarrow}H2) \rightarrow \Lambda' \quad \cdots & \\
\hline\hline
\Gamma \rightarrow (G1{\downarrow}H1), \Theta & & \Delta, (G2{\downarrow}H2) \rightarrow \Lambda & h1 \\
\hline
& \cdots \quad \Gamma, \Delta \rightarrow \Theta, \Lambda \quad \cdots & & h1 \\
\hline\hline
& \Sigma' \rightarrow \Pi' \qquad \Sigma'' \rightarrow \Pi'' & & h1 \\
\hline
& \cdots \quad \Sigma \rightarrow \Pi \quad \cdots & & h < h1 \\
\hline\hline
& \rightarrow & & 0
\end{array}
$$

The sequent $\Sigma \rightarrow \Pi$, which plays a critical role in the definition of Derv*, is the first sequent below the premisses of the first displayed cut with a height h smaller than the height h1 of the premisses. There must exist such a sequent since the endsequent of the derivation has height h. Necessarily the sequent is the conclusion of a cut with premisses of height h1. It is displayed as the conclusion of a cut with premisses $\Sigma' \rightarrow \Pi'$ and $\Sigma'' \rightarrow \Pi''$. Necessarily $h1 \geq d$, where d is the degree of the cut. The sequent $\Sigma \rightarrow \Pi$ may be identical with either $\Gamma, \Delta \rightarrow \Theta, \Lambda$ or with $\rightarrow$, or with both, or be distinct from both. The illustration is best understood as an illustration of the latter case.

Denote by Derv1 the derivation of the first premiss of the displayed cut, and by Derv2, the derivation of the second premiss. Let Derv1' and Derv2' be the following derivations:

$$
\begin{array}{ccc}
\text{Derv1'} & & \text{Derv2'} \\
\cdots & & \cdots \\
\Gamma', G1 \rightarrow \Theta' & & \Delta' \rightarrow G2, \Lambda' \\
\hline\hline
\cdots \quad \Gamma', \Gamma'', G1 \rightarrow (G1{\downarrow}H1), \Theta', \Theta'' \quad \cdots & & \cdots \quad \Delta', (G2{\downarrow}H2) \rightarrow G2, \Lambda' \quad \cdots \\
\hline\hline
\Gamma, G1 \rightarrow (G1{\downarrow}H1), \Theta & & \Delta, (G2{\downarrow}H2) \rightarrow G2, \Lambda
\end{array}
$$

Derv1' has been obtained from Derv1 by dropping the application of $\rightarrow\downarrow$, and replacing it with one or more thinnings and possibly interchanges represented by the first of the double lines; one of the thinnings has **(G1↓H1)** as the principal formula of its conclusion. Derv2' has been obtained from Derv2 by dropping the application of $L{\downarrow}\rightarrow$ and replacing it with an application of thinning with (G2↓H2) as the principal formula of its conclusion. Then the transformed derivation Derv* takes the following form:

Derv1'	Derv2	Derv1	Derv2'	Height
Γ, **G1** $\rightarrow$ **(G1↓H1)**, Θ	Δ, **(G2↓H2)** $\rightarrow \Lambda$	$\Gamma \rightarrow$ **(G1↓H1)**, Θ	Δ, **(G2↓H2)** $\rightarrow$ **G2**, Λ	h1
... Γ, **G1**, $\Delta \rightarrow \Theta, \Lambda$ ...		... $\Gamma, \Delta \rightarrow \Theta$, **G2**, Λ ...		h1
Σ', **G1** $\rightarrow \Pi'$	$\Sigma'' \rightarrow \Pi''$	$\Sigma' \rightarrow \Pi'$, **G2**	$\Sigma'' \rightarrow \Pi''$	h1
Σ, **G1** $\rightarrow \Pi$		$\Sigma \rightarrow$ **G2**, Π		h*
... $\Sigma, \Sigma \rightarrow \Pi, \Pi$ ...				h
$\rightarrow$				0

In this illustration, the endsequents of the derivations Derv1, Derv2, Derv1' and Derv2' have been explicitly displayed.

Note that the third cut with cut formulas **G1** and **G2** has been deferred until the sequent $\Sigma \rightarrow \Pi$ identified in the original derivation above.

Let dL and dR be the degrees of the top left and top right cuts respectively in Derv*, and d be the degree of the cut in Derv. Then from lemma 4.2.2 follows

dL = max{ deg(**(G1↓H1)**, Derv1'), deg(**(G2↓H2)**, Derv2))

dR = max{ deg(**(G1↓H1)**, Derv1), deg(**(G2↓H2)**, Derv2'))

d = max{ deg(**(G1↓H1)**, Derv1), deg(**(G2↓H2)**, Derv2))

Further,

deg(**(G1↓H1)**, Derv1') $\leq$ deg(**(G1↓H1)**, Derv1)

deg(**(G2↓H2)**, Derv2') $\leq$ deg(**(G2↓H2)**, Derv2)

so that

(2) dL, dR $\leq$ max{dL, dR} = d

Let h1' and h1" be the heights respectively of the premisses of the top left and right cuts in Derv*, and let h2' and h2" be the heights of their conclusions.

Let d* be the degree and h* the height of the third cut of Derv*, then

(3) d* < d

(4) h1 > h* $\geq$ h

since h* = max{d*, h} < max{d, h1} = h1.

5.1.3. A ∀-Transformation

Consider an application of cut with cut formulas ∀v**G1** and ∀v**G2**. Let no thinning transformation be applicable. Then the succedent cut formula has a top identical predecessor which is the principal formula in the conclusion of an application of →∀ with e-par **p** satisfying the e-par restriction; therefore in particular, **p** does not occur in **G1**. Similarly let the antecedent cut formula have a top identical formula which is the principal formula in the conclusion of an application of ∀→ that is not blocked, and that has e-term **r**. The derivation Derv may therefore be assumed to take the form:

$$
\dfrac{\dfrac{\cdots \quad \dfrac{\dfrac{\dfrac{\cdots}{\Gamma' \rightarrow [\mathbf{p}/\mathbf{v}]\mathbf{G1}, \Theta'}}{\Gamma' \rightarrow \forall \mathbf{v}\mathbf{G1}, \Theta'} \quad \cdots}{\Gamma \rightarrow \forall \mathbf{v}\mathbf{G1}, \Theta} \qquad \dfrac{\cdots \quad \dfrac{\dfrac{\cdots}{\Delta', [\mathbf{r}/\mathbf{v}]\mathbf{G2} \rightarrow \Lambda'}}{\Delta', \forall \mathbf{v}\mathbf{G2} \rightarrow \Lambda'} \quad \cdots}{\Delta, \forall \mathbf{v}\mathbf{G2} \rightarrow \Lambda}}{\cdots \quad \Gamma, \Delta \rightarrow \Theta, \Lambda \quad \cdots}}{\dfrac{\Sigma \rightarrow \Pi}{\rightarrow}}
$$

As observed in 3.2, it may be assumed that **p** is distinct from any other e-par of the derivation, from any parameter occurring in the derivation of the right premiss of cut, as well as from any parameter occurring in the components of σ. In particular, it may be assumed that neither **p** nor any other e-par of the derivation of the left premiss of cut occurs in **r** or in **G2**.

For those with access to [Szabo69], Derv corresponds to the derivation illustrated on page 271. The transformed derivation Derv* illustrated below corresponds to the one illustrated on page 273.

The sequent Σ → Π that is diplayed has been chosen in the same manner as the similarly named sequent in the description of the ↓-transformation of the previous subsection. As with that transformation, the ∀-transformation replaces the single cut displayed in the original derivation by three cuts.

Denote by Derv1 the derivation of the first premiss of the displayed cut, minus that premiss, and by Derv2, the deivation of the second premiss, minus that premiss. Let Derv1' be the derivation obtained from Derv1 by dropping the application of →∀ and replacing it with an application of succedent thinning with ∀v**G1** its principal formula. Let Derv2' be obtained from Derv2 in a similar manner by dropping the application of ∀→ and replacing it with an application of thinning. Thus the following sequents may be added as endsequents of Derv1' and Derv2' respectively:

$$\Gamma \rightarrow [\mathbf{p}/\mathbf{v}]\mathbf{G1}, \forall\mathbf{v}\mathbf{G1}, \Theta \text{ and } \Delta, [\mathbf{r}/\mathbf{v}]\mathbf{G2}, \forall\mathbf{v}\mathbf{G2} \rightarrow \Lambda$$

The new derivation Derv* is the following:

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}\text{Derv1'} & \text{Derv2} \\ \Gamma \rightarrow [\mathbf{p}/\mathbf{v}]\mathbf{G1}, \forall\mathbf{v}\mathbf{G1}, \Theta & \Delta, \forall\mathbf{v}\mathbf{G2} \rightarrow \Lambda\end{array}}{\ldots \quad \Gamma, \Delta \rightarrow [\mathbf{p}/\mathbf{v}]\mathbf{G1}, \Theta, \Lambda \quad \ldots}}{\Sigma \rightarrow [\mathbf{p}/\mathbf{v}]\mathbf{G1}, \Pi} \qquad \dfrac{\dfrac{\begin{array}{cc}\text{Derv1} & \text{Derv2'} \\ \Gamma \rightarrow \forall\mathbf{v}\mathbf{G1}, \Theta & \Delta, [\mathbf{r}/\mathbf{v}]\mathbf{G2}, \forall\mathbf{v}\mathbf{G2} \rightarrow \Lambda\end{array}}{\ldots \quad \Gamma, \Delta, [\mathbf{r}/\mathbf{v}]\mathbf{G2} \rightarrow \Theta, \Lambda \quad \ldots}}{\Sigma, [\mathbf{r}/\mathbf{v}]\mathbf{G2} \rightarrow \Pi}}{\dfrac{\ldots \quad \Sigma, \Sigma \rightarrow \Pi, \Pi \quad \ldots}{\rightarrow}}$$

The global substitution σ' for the new derivation is defined to be $\sigma[\mathbf{r}/\mathbf{p}]$. Since **p** does not occur in **G1**, **r**, or in **G2**, $\sigma([\mathbf{r}/\mathbf{p}]([\mathbf{p}/\mathbf{v}]\mathbf{G1})$ is $\sigma([\mathbf{r}/\mathbf{v}]\mathbf{G1})$ is $[\sigma(\mathbf{r})/\mathbf{v}]\sigma(\mathbf{G1})$. But since $\sigma(\forall\mathbf{v}\mathbf{G1})$ is $\sigma(\forall\mathbf{v}\mathbf{G2})$, $\sigma(\mathbf{G1})$ is $\sigma(\mathbf{G2})$. Therefore, $[\sigma(\mathbf{r})/\mathbf{v}]\sigma(\mathbf{G1})$ is $[\sigma(\mathbf{r})/\mathbf{v}]\sigma(\mathbf{G2})$ is $\sigma([\mathbf{r}/\mathbf{v}]\mathbf{G2})$ is $\sigma([\mathbf{r}/\mathbf{p}]([\mathbf{r}/\mathbf{v}]\mathbf{G2})$, and the new derivation is a σ'-derivation.

The definitions of degrees and heights given in 5.1.2 can be adapted here, and the equalties and inequalties (1, 2, 3, 4) derived in the same mannter.

5.1.4. A {}-Transformation

Consider an application of cut with cut formulas **r1**:{ **t1** | **G1**} and **r2**:{ **t2** | **G2**}. Since by 2.1.3 no parameter may occur in **t1** or **t2**, it may be assumed that these terms are identical so that the cut formulas may be assumed to be **r1**:{**t** | **G1**} and **r2**:{**t** | **G2**} It may further be assumed that $\sigma(\mathbf{G1})$ is $\sigma(\mathbf{G2})$, and that $\sigma(\mathbf{r1})$ is $\sigma(\mathbf{r2})$. Assume that the thinning-transformation cannot be applied. It may be assumed therefore that each of the cut formulas has a top identical predecessor that is the principal formula in the conclusion of an application of a { } rule.

Consider such an application for the cut formula **r1**:{**t** | **G1**}. Necessarily **r1** must be of the form $[\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{t}$, and the principal formula in the premiss of the application has the form $[\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{G1}$, where $\underline{\mathbf{u}}$ is a sequence of all the variables with free occurrences in **t**, and $\underline{\mathbf{r1}}$ is a sequence of terms of the same length. Similarly it follows that **r2** must be of the form $[\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{t}$, and that $[\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{G2}$ is the principal formula in the premiss of an application of { }→ with **r2**:{**t** | **G2**} as principal formula of its conclusion; as before $\underline{\mathbf{r2}}$ is a sequence of terms of the same length as $\underline{\mathbf{u}}$.

The following illustrates the original derivation Derv with only one application of $\to\{\}$ and of $\{\}\to$ displayed:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\cdots \\
\Gamma' \to [\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{G1}, \Theta' \\ \hline
\cdots \quad \Gamma' \to [\underline{\mathbf{r1}}/\underline{\mathbf{u}}]:\{\mathbf{t} \mid \mathbf{G1}\}, \Theta' \quad \cdots \\ \hline\hline
\Gamma \to [\underline{\mathbf{r1}}/\underline{\mathbf{u}}]:\{\mathbf{t} \mid \mathbf{G1}\}, \Theta
\end{array}
&
\begin{array}{c}
\cdots \\
\Delta', [\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{G2} \to \Lambda' \\ \hline
\cdots \quad \Delta', [\underline{\mathbf{r2}}/\underline{\mathbf{u}}]:\{\mathbf{t} \mid \mathbf{G2}\} \to \Lambda' \quad \cdots \\ \hline\hline
\Delta, [\underline{\mathbf{r2}}/\underline{\mathbf{u}}]:\{\mathbf{t} \mid \mathbf{G2}\} \to \Lambda
\end{array}
\end{array} \\ \hline
\cdots \qquad \Gamma, \Delta \to \Theta, \Lambda \qquad \cdots \\ \hline\hline
\Sigma \to \Pi \\ \hline\hline
\to
\end{array}
$$

The transformed derivation is obtained from the given derivation by dropping every application of thinning and of the { } rules for which a top identical predecessor of one of the cut formulas is the principal formula of its conclusion. In this case thinnings do not replace the removed logical rules. The derivation Derv* resulting from the transformation is illustrated next:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\cdots \\
\cdots \quad \Gamma' \to [\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{G1}, \Theta' \quad \cdots \\ \hline\hline
\Gamma \to [\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{G1}, \Theta
\end{array}
&
\begin{array}{c}
\cdots \\
\cdots \quad \Delta', [\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{G2} \to \Lambda' \quad \cdots \\ \hline\hline
\Delta, [\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{G2} \to \Lambda
\end{array}
\end{array} \\ \hline
\cdots \qquad \Gamma, \Delta \to \Theta, \Lambda \qquad \cdots \\ \hline\hline
\Sigma \to \Pi \\ \hline\hline
\to
\end{array}
$$

Since any parameter occurring in $[\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{G1}$ (respectively $[\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{G2}$) must occur in $[\underline{\mathbf{r1}}/\underline{\mathbf{u}}]\mathbf{t}:\{\mathbf{t} \mid \mathbf{G1}\}$ (respectively $[\underline{\mathbf{r2}}/\underline{\mathbf{u}}]\mathbf{t}:\{\mathbf{t} \mid \mathbf{G2}\}$), the eigen parameter restrictions of the original derivation are respected in the transformed derivation.

Let d be the degree of the cut in Derv and d* of the cut in Derv*. Since the { } rules are both single premiss rules, it follows from the definition of degree in 4.2.1 that

$$d^* = d - 1$$

Let h be the height of the conclusion of the application of cut in both Derv and Derv*, and h1 and h1* be the heights of the premisses in Derv and Derv* respectively. Then

$$h1 = \max\{d, h\} \geq \max\{d^*, h\} = h1^*$$

5.1.5. An Axiom-Transformation

Let both the antecedent and succedent formulas of an axiom $\mathbf{A \to A}$ be the cut formulas of two cuts one with succedent formula **G1** and the other with antecedent formula **G2**. Thus

$$\sigma(\mathbf{G1}) \text{ is } \sigma(\mathbf{A}) \text{ is } \sigma(\mathbf{G2})$$

Assume that the cut in which **G1** is the succedent formula appears above the cut in which **G2** is the antecedent formula. The derivation Derv may therefore be assumed to take the following form:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\begin{array}{cc} \begin{array}{c} \cdots \\ \Gamma' \to G1, \Theta' \end{array} & A \to A \end{array} \\ \hline
\cdots \quad \Gamma' \to A, \Theta' \quad \cdots \\ \hline\hline
\Gamma \to A, \Theta
\end{array}
&
\begin{array}{c} \cdots \\ \Delta, G2 \to \Lambda \end{array}
\end{array} \\ \hline
\cdots \quad \Gamma, \Delta \to \Theta, \Lambda \quad \cdots \\ \hline\hline
\to
\end{array}
$$

The new derivation Derv* is obtained by dropping the first application of cut as well as the axiom, and replacing the second cut with one in which **G1** and **G2** are the cut formulas:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\cdots \quad \begin{array}{c} \cdots \\ \Gamma' \to G1, \Theta' \end{array} \quad \cdots \\ \hline\hline
\Gamma \to G1, \Theta
\end{array}
&
\begin{array}{c} \cdots \\ \Delta, G2 \to \Lambda \end{array}
\end{array} \\ \hline
\cdots \quad \Gamma, \Delta \to \Theta, \Lambda \quad \cdots \\ \hline\hline
\to
\end{array}
$$

Consider Derv and the following degree path in it:

$$\mathbf{G2, A, \ldots, A, A, G1}$$

where **A, ... , A** is the sequence of identical predecessors of the succedent formula of the axiom beginning with the cut formula of the second cut, and where the **A** following this sequence and preceding **G1** is the antecedent formula of the axiom. This degree path can be followed by any degree path beginning with **G1**, minus of course **G1**. It follows therefore that

$$\mathbf{deg(G1, Derv) \leq deg(G2, Derv)}$$

Note that this conclusion depends upon clauses (3) and (4) in the definition 4.2.1 of degree path. The conclusion is essential to the proof given in 6.2.5 that an axiom-transformation reduces the ordinal of a derivation. Should the cut in which **G2** is the antecedent formula appear above the cut in which **G1** is the succedent formula, this inequality is reversed.

Let d1 be the degree of the first cut in Derv and d2 the degree of the second. Then from lemma 4.2.2 and the above inequality follows,

d1 = deg(**G1**, Derv)

d2 = max{ deg(**G1**, Derv), deg(**G2**, Derv) = deg(**G2**, Derv)

Therefore the degree of the cut in Derv* is d2.

The height h2 of the conclusion of the second cut in Derv is also the height of the conclusion of the only cut in Derv*.

5.2. A Transformation Can Always Be Applied

Throughout this section it is assumed that a derivation Derv, that is a σ-derivation of the empty sequent, has been given for some global substitution σ. It is further assumed that no thinning-transformation can be applied, which means that no thinning cut can appear in the derivation, as described in 3.4.3. Therefore every cut is atomic or a $\downarrow$, $\forall$, or { }-cut. If there exist a $\downarrow$-cut, then a $\downarrow$-transformation can be applied, and if there exists a { }-cut, a { }-transformation can be applied. It is therefore only necessary to show that if no thinning, $\downarrow$, or { }-transformation can be applied, then an atomic or a $\forall$-transformation can be applied.

Recall now the definition in 4.1.5 of blocked applications of $\forall\!\rightarrow$. It is not enough to have a $\forall$-cut in order to have a $\forall$-transformation that can be applied; an application of $\forall\!\rightarrow$ with the antecedent cut formula as principal conclusion, must not be blocked. Call such a cut unblocked.

5.2.1. Lemma:

Let there be no $\downarrow$ or { }-cut in the derivation. Then there is either an unblocked $\forall$-cut or an atomic cut

Proof:

Since it is assumed that no thinning-transformation can be applied, each cut is either atomic or a $\forall$-cut. Therefore the pairs of cut formulas, listed in order of succedent cut formula followed by antecedent cut formula are of three kinds:

< ∀vG1, ∀vG2 >, < A, G >, and **< G, A >**

Assume that no atomic-transformation nor $\forall$-transformation can be applied; in particlular, therefore, every $\forall$-cut in the derivation is blocked. Under these assumptions, each pair of cut formulas is necessarily linked to another distinct pair of the same or of a different kind in the following sense:

1) The cut formula **∀vG2** of a pair **< ∀vG1, ∀vG2 >** is blocked by an application of $\rightarrow\!\forall$. Let **H** be the last successor of the principal formula in the conclusion of this application. **H** is one of

the cut formulas of a cut and is therefore either the first or second element of a cut pair. It is to this cut pair of which **H** is an element that **< ∀vG1, ∀vG2 >** is said to be linked.

2) The atomic cut formula **A** of a pair **< A, G >** or **< G, A >** is the last successor of either the antecedent or succedent formula of an axiom **A → A**. Let **H** be the last successor of the other formula of the axiom. **H** cannot be **A**, for then an axiom-transformation could be applied. **H** is one of the cut formulas of a cut and is therefore either the first or second element of a cut pair. It is to this cut pair of which **H** is an element that **< A, G >** or **< G, A >** is said to be linked.

Let $CtPr_1, \ldots, CtPr_n$ be a sequence of cut pairs of the derivation for which $CtPr_i$ is linked to $CtPr_{i+1}$, for $1 \leq i < n$. Necessarily there can be no repetitions in this sequence. Further, for every n, $CtPr_n$ must have a successor in the sequence; that is a cut pair to which it is linked. But this is impossible for a finite derivation, so it must be possible to either apply an atomic-transformation or find a ∀-cut for which the antecedent cut formula is not blocked.

End of Proof of Lemma 5.2.1

6. CONSISTENCY PROOF

A proof of the following theorem will be provided in the subsections of this section:

6.0.1. Theorem: For no global substitution σ, is there a σ-derivation of the empty sequent.

The proof proceeds by contradiction. In § 6.1 an ordinal less than ε_0 is assigned to derivations. In § 6.2 it is shown that each of the transformations defined in § 5.3 reduces the ordinal of a derivation when it is applied. In § 5.3 it was shown that at least one of the derivations can always be applied to a derivation of the empty sequent. Thus there cannot be a derivation of the empty sequent in NaDSet since the empty sequent is not an axiom.

The consistency proved in theorem 6.0.1 is a form of ω-consistency from which the simple consistency of NaDSet follows. The connection with ω-consistency, as defined in [Gödel31], is explained in § 6.3.

6.1. Ordinals of Derivations

The method employed here for assigning an ordinal number to a σ-derivation is an adaptation of the method Gentzen used to assign an ordinal number to a derivation of the empty sequent as described in [Gentzen38] and [Szabo69]. An adaptation is necessary since NaDSet has no explicit induction rule of deduction CJ, and no basic mathematical sequents.

6.1.1. Properties of Ordinals

By an <u>ordinal</u> is meant any ordinal number less than ε_0. Collected in this subsection are all the properties of ordinals required for the consistency proof.

For an ordinal μ and an integer k, $k \geq 0$, $\omega_k(\mu)$ is defined recursively: $\omega_0(\mu) = \mu$; $\omega_1(\mu) = \omega^\mu$; and $\omega_{k+1}(\mu) = \omega_1(\ \omega_k(\mu)\ )$.

Each ordinal μ, $\mu > 0$, has a unique representation in the following normal form:

$$\mu = \omega_1(\delta_1) + \omega_1(\delta_2) + \ldots + \omega_1(\delta_m),$$

where $\mu > \delta_1 \geq \delta_2 \geq \ldots \geq \delta_m$, $m \geq 1$.

Let μ and ν be two ordinal numbers with normal forms as follows:

$$\mu = \omega_1(\delta_1) + \omega_1(\delta_2) + \ldots + \omega_1(\delta_m)$$

$$\nu = \omega_1(\gamma_1) + \omega_1(\gamma_2) + \ldots + \omega_1(\gamma_n)$$

The <u>natural sum</u> of the two ordinals μ and ν is

$$\mu \# \nu = \omega_1(\lambda_1) + \omega_1(\lambda_2) + \ldots + \omega_1(\lambda_{m+n}),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{m+n}$ is the sequence obtained by merging the sequences $\delta_1, \delta_2, \ldots, \delta_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ with duplicates maintained, and then reordering the resulting sequence so that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{m+n}.$$

Properties of # and ω_1 that will be used in the proof of consistency are for ordinals $\gamma, \mu, \nu \geq 1$:

1. # is commutative and associative
2. $\mu < \mu \# \nu$
3. $\mu < \nu \Rightarrow \mu \# \gamma < \mu \# \gamma$
4. $\mu < \omega_1(\mu)$
5. $\mu < \nu \Rightarrow \omega_1(\mu) < \omega_1(\nu)$
6. $\mu, \nu < \gamma \Rightarrow \omega_1(\mu) \# \omega_1(\nu) < \omega_1(\gamma)$

6.1.2. Definition: The Ordinal of a Derivation

The ordinal number assigned to a derivation is the ordinal number assigned to its endsequent by the following process:

1. The ordinal 1 is assigned to each axiom used in the derivation.

2. The ordinal number of the conclusion of an application of a structural rule is the ordinal of its premiss.
3. The ordinal number of the conclusion of an application of a single premiss logical rule is $\mu \# 1$, where μ is the ordinal of the premiss.
4. The ordinal number of the conclusion of an application of the two premiss logical rule $\rightarrow\downarrow$ is $\mu \# \nu$, where μ and ν are the ordinals of the premisses.
5. The ordinal number of the conclusion of an application of cut is $\omega_{h1-h2}(\mu \# \nu)$, where μ and ν are the ordinals of the premisses, and h1 and h2 are respectively the height of the premisses and of the conclusion of the application.

End of Definition

This definition differs in two unimportant respects from the comparable definition of Gentzen. First, it assigns ordinals only to sequents, not as well to a "line of inference" in the words of [Szabo69] page 279. Second, clauses (3) and (4) of this definition differ slightly from Gentzen in the use of # instead of + in (3), and the use of $\mu \# \nu$ instead of $\max\{\mu, \nu\} + 1$ in (4). These changes have no important effect upon the proof of Gentzen, but do assist in the proof of lemma 6.1.4 below.

6.1.3. Monotone Functions

A function ϕ of a single ordinal argument is said to be <u>monotone</u> if $\mu < \nu \Rightarrow \phi(\mu) < \phi(\nu)$. A function of more than one ordinal is said to be <u>monotone</u> if the function is monotone in each of its arguments.

For each single premiss rule of deduction there is a monotone function ϕ such that $\phi(\mu)$ is the ordinal assigned to its conclusion if μ is assigned to its premiss. Similarly for each two premiss rule, namely either $\rightarrow\downarrow$ or cut, then there is a monotone function ϕ such that $\phi(\mu 1, \mu 2)$ is the ordinal assigned to the conclusion if $\mu 1$ and $\mu 2$ are assigned to its premisses. In the case of cut, the function is dependent upon the difference h1 - h2 in the height h1 of the premisses of the application and the height h2 of the conclusion.

Let Seq be a sequent in a derivation Derv, and let $Seq_1, \ldots, Seq_k$ be sequents standing above Seq in Derv that satisfy the following condition: Every branch of Derv that begins with Seq has exactly one of the sequents $Seq_1, \ldots, Seq_k$ as an element. Then there is a monotone function ϕ of k arguments such that if $\mu_1, \ldots, \mu_k$ are the ordinals assigned to $Seq_1, \ldots, Seq_k$, then $\phi(\mu_1, \ldots, \mu_k)$ is the ordinal assigned to Seq. That there is such a function follows from the fact that the composition of monotone functions is necessarily monotone in each of its arguments.

The next lemma deals with a change that can take place in a derivation Derv when it is transformed into a derivation Derv*. Consider a cut of degree d in Derv that is transformed into a cut d* in Derv* for which d > d*. Call the cut in Derv simply cut A. Let h2 be the height of the premisses of cut A and h3 that of the conclusion where h2 > h3. If h2 = d, then reducing d to d* will reduce h2 to max{d*, h3}. This reduction in the height of the premisses of cut A can increase the ordinals assigned to its premisses. For let there be a second cut, cut B, standing above cut A with h2 the height of its conclusion and h1 the height of its premisses. If h1 is the degree of cut B, then decreasing h2 can increase the ordinal assigned to the conclusion of cut B and therefore increase the ordinal of a premiss of cut A. Nevertheless the next lemma demonstrates that the ordinal of the conclusion of cut A is decreased by the decrease in h2.

A cut for which the height of its premisses is the degree of the cut is said to have a height determined by its degree. Note that no ordinal of a premiss of a cut with height determined by its degree can be affected by a decrease in the height of the conclusion of the cut.

6.1.4. Lemma:

Let Derv be any derivation. Let Seq be a sequent of height h3 in Derv that is the conclusion of an application of cut with premisses of height h2 for which h2 > h3. Let $Seq_1, \dots, Seq_k$ be sequents standing above Seq in Derv that satisfy the following conditions:

1. Every branch of Derv that begins with Seq has exactly one of the sequents $Seq_1, \dots, Seq_k$ as an element.
2. Each Seq_i has height h2.
3. If there is a branch that has an element that is the conclusion of an application of cut with height determined by its degree, then the Seq_i that is an element of the branch is the first sequent above Seq in the branch that is the conclusion of such a cut.

Let $\mu_1, \dots, \mu_k$ be the ordinals assigned to $Seq_1, \dots, Seq_k$ and $\phi(\mu_1, \dots, \mu_k)$ be the ordinal assigned to Seq. Let the height h2 be reduced to h2-1, and let $\mu_1^*, \dots, \mu_k^*$ be the ordinals assigned to $Seq_1, \dots, Seq_k$ resulting from this change. Then

$$\phi(\mu_1^*, \dots, \mu_k^*) < \phi(\mu_1, \dots, \mu_k)$$

Proof:

Let the premisses of the cut of which Seq is the conclusion be Seq1 and Seq2. Then there are monotone functions $\phi1$ and $\phi2$ and a partition of the set $\{\mu_1, \dots, \mu_k\}$ of ordinals into disjoint subsets $\{\mu_1, \dots, \mu_m\}$ and $\{\nu_1, \dots, \nu_n\}$ for which

$$\phi(\mu_1, \dots, \mu_k) = \omega_a(\phi1(\mu_1, \dots, \mu_m) \# \phi2(\nu_1, \dots, \nu_n))$$

where a = h2- h3.

Consider now the form of the functions $\phi 1$ and $\phi 2$. All sequents between either of Seq1 and Seq2 and any Seq_i are of height h2. Therefore for some ordinals $\delta 1$ and $\delta 2$,

$$\phi 1(\mu_1, \dots, \mu_m) = \mu_1 \# \dots \# \mu_m \# \delta 1$$

$$\phi 2(\nu_1, \dots, \nu_n) = \nu_1 \# \dots \# \nu_n \# \delta 2$$

Consider the relationship between μ_i^* and μ_i. If Seq_i is the conclusion of an application of cut with height determined by its degree, then $\mu_i^* = \omega_1(\mu_i)$; otherwise $\mu_i^* = \mu_i$. It may therefore be assumed that the former is the case for each Seq_i since ϕ can be assumed to incorporate the other cases. Therefore

$$\phi 1(\mu_1^*, \dots, \mu_m^*) = \mu_1^* \# \dots \# \mu_m^* \# \delta 1 = \omega_1(\mu_1) \# \dots \# \omega_1(\mu_m) \# \delta 1$$

$$\phi 2(\nu_1^*, \dots, \nu_n^*) = \nu_1^* \# \dots \# \nu_n^* \# \delta 2 = \omega_1(\nu_1) \# \dots \# \omega_1(\nu_n) \# \delta 2$$

The new ordinal $\phi(\mu_1^*, \dots, \mu_k^*)$ assigned to Seq after the reduction in height is therefore

$$\omega_{a-1}(\omega_1(\mu_1) \# \dots \# \omega_1(\mu_m) \# \delta 1 \# \omega_1(\nu_1) \# \dots \# \omega_1(\nu_n) \# \delta 2)$$

$$\leq \omega_{a-1}(\omega_1(\mu_1) \# \dots \# \omega_1(\mu_m) \# \omega_1(\delta 1) \# \omega_1(\nu_1) \# \dots \# \omega_1(\nu_n) \# \omega_1(\delta 1))$$

$$< \omega_{a-1}(\omega_1(\mu_1 \# \dots \# \mu_m \# \nu_1 \# \dots \# \nu_n \# \delta 1 \# \delta 2))$$

$$= \omega_a(\mu_1 \# \dots \# \mu_m \# \nu_1 \# \dots \# \nu_n \# \delta 1 \# \delta 2)$$

$$= \phi(\mu_1, \dots, \mu_k)$$

The first non strict inequality allows for the possibility that $\delta 1$ and $\delta 2$ are absent from the natural sum defining the functions $\phi 1$ and $\phi 2$. If they are not absent, the inequality becomes strict because of property (4) and other properties of 6.1.1. The first strict inequality follows in particular from repeated applications of (6) when γ is $\mu \# \nu$.

End of Proof of Lemma 6.1.4

If the decrease in the height h2 is greater than 1, the same conclusion may be drawn from repeated applications of the lemma.

6.2. Transformations Reduce Ordinals

The following lemma will be proved in this section:

6.2. Lemma:

Let Derv be a derivation of the empty sequent and Derv* a derivation that is obtained from it by applying one of the transformations described in 5.1. Then the ordinal of Derv* is less than the ordinal of Derv.

Proof:

The lemma will be proved for each transformation in turn. The numbers (1) - (6) refer to the properties of ordinals listed in 6.1.2.

6.2.1. Thinning-Transformation

Consider the cut to be removed from the derivation Derv. Let the premisses of the cut have ordinals μ and ν, where μ is assumed to be the ordinal of the premiss not removed. The ordinal of the conclusion of the cut is $\omega_{h1-h2}(\mu \# \nu)$, where h1 is the height of the premisses and h2 the height of the conclusion.

If h1 = h2, then the removal of this cut cannot affect the ordinal μ of the remaining premiss in Derv*. In this case the ordinal of the sequent in Derv* that replaces the conclusion of the cut in Derv is μ.

If h1 > h2, then the removal of the cut can affect the ordinal μ. But by lemma 6.1.4 the effect can only be to decrease μ to say μ^*.

The ordinal of the endsequent of Derv is $\phi(\omega_{h1-h2}(\mu \# \nu))$ for some monotone function ϕ. But by properties (2) and (5) of 6.1.1 it follows that

$$\phi(\mu^*) < \phi(\mu) < \phi(\omega_{h1-h2}(\mu \# \nu))$$

6.2.2. $\downarrow$-Transformation

A proof for this case can be adapted from the following case of a $\forall$-transformation.

6.2.3. $\forall$-Transformation

This is the case treated in detail by Gentzen. One complication not dealt with by Gentzen arises here from the possibility of the degree of an occurrence of a formula changing under a transformation. However, the examination of the consequences of this complication will be deferred until after lemma 6.2 has been established in this case under the assumption of unchanging degrees.

The notation introduced in 5.1.3 is the following:
h1 is the height of the premisses of the cut in Derv, and d its degree
h is the height of the first sequent below the premisses of the cut for which $h1 > h$
h* is the height of the premisses of the last cut in Derv* and d* is its degree
dL and dR are the degrees of the top left and top right cuts respectively in Derv*

The following conclusions were drawn in 5.1.3:
(1) $dL, dR \leq \max\{dL, dR\} = d$
(2) $d^* < d$
(3) $h1 > h^* \geq h$

Under the assumption of unchanging degrees, $dL = dR = d$. Let α and β be the ordinals of the premiss of the $\rightarrow\forall$ and $\forall\rightarrow$ rules respectively, and let δ be the ordinal assigned to $\Sigma'' \rightarrow \Pi''$. Then the following tree of ordinals indicates the ordinals assigned to other sequents in Derv.

$$
\begin{array}{llll}
 & \cdots & \cdots & \text{Height} \\
 & \alpha & \beta & \\
 \cline{2-2}\cline{3-3}
 & \alpha \# 1 & \beta \# 1 & \\
 \cline{2-2}\cline{3-3}\cline{2-2}\cline{3-3}
 & \phi1(\alpha \# 1) & \phi2(\beta \# 1) & h1 \\
 \cline{2-3}
 \gamma = & \multicolumn{2}{l}{\phi1(\alpha \# 1) \# \phi2(\beta \# 1)} & h1 \\
 \cline{2-3}\cline{2-3}
 & \phi3(\gamma) \quad \delta & & h1 \\
 \cline{2-2}
 & \multicolumn{2}{l}{\omega_{h1-h}(\phi3(\gamma) \# \delta)} & h \\
 \cline{2-2}\cline{2-2}
 & \multicolumn{2}{l}{\phi4(\omega_{h1-h}(\phi3(\gamma) \# \delta))} & 0
\end{array}
$$

where $\phi1$, $\phi2$, $\phi3$ and $\phi4$ are monotone functions.

Under the assumption of unchanging degrees, the height of the premisses and conclusions of the top left and top right cuts in Derv* are the same as those of the cut in Derv. Therefore this tree of ordinals is transformed in Derv* into the following:

$$
\begin{array}{llll|l}
 & \text{Derv1'} \quad \text{Derv2} & & \text{Derv1} \quad \text{Derv2'} & \text{Height} \\
 & \phi1(\alpha) \quad \phi2(\beta \# 1) & & \phi1(\alpha \# 1) \quad \phi2(\beta) & h1 \\
\hline
\gamma L = & \phi1(\alpha) \# \phi2(\beta \# 1) & \gamma R = & \phi1(\alpha \# 1) \# \phi2(\beta) & h1 \\
\hline\hline
 & \phi3(\gamma L) \quad \delta & & \phi3(\gamma R) \quad \delta & h1 \\
\hline
\gamma L' = & \omega_{h1-h^*}(\phi3(\gamma L) \# \delta) & \gamma R' = & \omega_{h1-h^*}(\phi3(\gamma R) \# \delta) & h^* \\
\hline
 & & \omega_{h^*-h}(\gamma L' \# \gamma R') & & h \\
\hline\hline
 & & \phi4(\omega_{h^*-h}(\gamma L' \# \gamma R')) & & 0
\end{array}
$$

It is required to prove therefore that

$$\phi4(\omega_{h^*-h}(\gamma L' \# \gamma R')) < \phi4(\omega_{h1-h}(\phi3(\gamma) \# \delta))$$

Since $\phi4$ is monotonic, and since

$$\omega_{h1-h}(\phi3(\gamma) \# \delta) = \omega_{h^*-h}(\omega_{h1-h^*}(\phi3(\gamma) \# \delta))$$

by property (5) it is sufficient to prove

$$\gamma L' \# \gamma R' < \omega_{h1-h^*}(\phi3(\gamma) \# \delta)$$

where

$$\gamma L' = \omega_{h1-h^*}(\phi3(\gamma L) \# \delta)$$

$$\gamma R' = \omega_{h1-h^*}(\phi3(\gamma R) \# \delta)$$

$$\gamma L = \phi1(\alpha) \# \phi2(\beta \# 1)$$

$$\gamma R = \phi1(\alpha \# 1) \# \phi2(\beta)$$

Note that

$$\gamma L, \gamma R < \gamma$$

so that

$$\phi3(\gamma L), \phi3(\gamma R) < \phi3(\gamma)$$

and therefore

$$\phi3(\gamma L) \# \delta, \phi3(\gamma R) \# \delta < \phi3(\gamma) \# \delta$$

Since $h1 - h^* \geq 1$, there follows from property (6) that

$$\omega_{h1-h^*}(\phi3(\gamma) \# \delta) \# \omega_{h1-h^*}(\phi3(\gamma R) \# \delta) < \omega_{h1-h^*}(\phi3(\gamma) \# \delta)$$

as required.

Now consider the effect on this inequality that may result from the changing of degrees from Derv to Derv*, with these changes reflected in changing heights. It is possible that now $dL < d$ or $dR < d$, but not both since $d = \max\{dL, dR\}$. A decrease in the degree of the top left cut from d to dL, can by lemma 6.1.4 however only decrease the ordinal assigned to its conclusion, with such a decrease reflected in a decrease of the ordinal of the endsequent.

6.2.4. {}-Transformation

In 5.1.4 it was concluded that

$d^* = d - 1$

$h1^* \leq h1,$

where d is the degree of the cut in Derv and d* of the cut in Derv*, and h1 and h1* the heights of the premisses of the cuts in Derv and Derv* respectively. The height of the conclusions of the cuts is h in both derivations.

Let μ and ν be the ordinals of the premisses of the cut in Derv and γ the ordinal of its conclusion. Let μ^* and ν^* be the ordinals of the premisses of the cut in Derv* when h1* is fixed at h1, and let γ^* be the ordinal of its conclusion under the same assumption. Then necessarily

$\mu^* < \mu$ and $\nu^* < \nu$

since applications of $\rightarrow\{\}$ and $\{\}\rightarrow$ have been omitted. Therefore

$\gamma^* < \gamma$

Now consider these ordinals when $h1^* < h1$. Let $\gamma^{*\prime}$ be the ordinal of the conclusion of the cut when h1* is reduced from h1. By lemma 6.1.4, $\gamma^{*\prime} < \gamma^*$, so that $\gamma^{*\prime} < \gamma$. But the ordinal of the endsequent of Derv is $\phi(\gamma)$ for some monotone function ϕ. Therefore $\phi(\gamma^{*\prime}) < \phi(\gamma)$.

6.2.5. Axiom-Transformation

In 5.1.5 it was concluded that $d1 \leq d2$, where $d1 = deg(\mathbf{G1}, Derv)$ and $d2 = deg(\mathbf{G2}, Derv)$, and that therefore the degree of the only cut in Derv* is d2. It was also concluded that the height h2 of the conclusion of the second cut in Derv is the height of the conclusion of the only cut in Derv*. Let h1 be the height of the conclusion of the premisses of the first cut in Derv. Define

$a1 = \max\{d1, h1\} - h1$

$a2 = \max\{d2, h2\} - h2$

Let μ be the ordinal of the first premiss of the first cut in Derv, and ν the ordinal of the second premiss of the second cut. of this cut. The following ordinal tree indicates the ordinals assigned to other sequents in Derv and Derv*.

$$
\dfrac{\cdots \quad \dfrac{\dfrac{\cdots \quad \dfrac{\begin{matrix}\cdots\\ \mu\end{matrix} \qquad 1}{\omega_{\alpha 1}(\mu \# 1)} \quad \cdots}{\phi(\omega_{\alpha 1}(\mu \# 1))} \qquad \begin{matrix}\cdots\\ \nu\end{matrix}}{\omega_{\alpha 2}(\phi(\omega_{\alpha 1}(\mu \# 1)) \# \nu)} \quad \cdots}{\psi(\omega_{\alpha 2}(\phi(\omega_{\alpha 1}(\mu \# 1)) \# \nu))}
$$

$$
\dfrac{\cdots \quad \dfrac{\dfrac{\cdots \quad \begin{matrix}\cdots\\ \mu\end{matrix} \quad \cdots}{\phi(\mu)} \qquad \begin{matrix}\cdots\\ \nu\end{matrix}}{\omega_{\alpha 2}(\phi(\mu) \# \nu)} \quad \cdots}{\psi(\omega_{\alpha 2}(\phi(\mu) \# \nu))}
$$

Here ϕ and ψ are monotone functions.

It is necessary to prove that

$$\psi(\omega_{a2}(\phi(\mu) \# \nu)) < \psi(\omega_{a2}(\phi(\omega_{a1}(\mu \# 1)) \# \nu))$$

Since ψ is monotone, it is sufficient to prove

$$\omega_{a2}(\phi(\mu) \# \nu) < \omega_{a2}(\phi(\omega_{a1}(\mu \# 1)) \# \nu)$$

By (5) it is sufficient to prove therefore that

$$\phi(\mu) \# \nu < \phi(\omega_{a1}(\mu \# 1)) \# \nu$$

But this follows from (2), the monotonicity of ϕ, and (4).

End of Proof of Lemma 6.2

Given a derivation of the empty sequent, by lemma 5.2 one of the transformations defined in 5.1 can always be applied. By lemma 6.2, any one of the transformations will transform the derivation into another derivation of the empty sequent, but one with a smaller ordinal number. Since the empty sequent is not an axiom of NaDSet, there cannot exist a derivation of the empty sequent.

End of Proof of Theorem 6.0.1

6.3. Omega Consistency

The proof of theorem 6.0.1 demonstrates that there cannot exist a σ-derivation for any global substitution σ. In particular therefore, there cannot exist formulas **G1** and **G2** and a global substitution σ for which σ(**G1**) is σ(**G2**) and both $\rightarrow$ **G1** and **G2** $\rightarrow$ have σ-derivations. That this is a form of ω-consistency as defined in [Gödel31] is evident from the following consequent: For no parameter **p** and term **r** are both $\rightarrow$ **[p /v] F** and **[r /v] F** $\rightarrow$ derivable in NaDSet.

7. SEMANTICS

The traditional semantics for classical logics is described in [Tarski36]. It is reductionist in the following sense: An interpretation assigns one and only one truth value to each closed atomic formula; a closed formula that is not atomic receives a truth value dependant upon the truth values assigned to simpler closed formulas. A natural deduction presentation of a logic is reductionist in the same sense, as is evident from the logical syntax for NaDSet; the presentation is called "natural" because the rules of deduction correspond faithfully to the semantics of the three fundamental logical concepts, namely, truth functions, quantifiers and abstraction terms.

Here a traditional semantics will be provided for NaDSet. However, finite induction no longer suffices, and not all closed formulas receive a truth value; for example, the paradoxical formula {u | ~u:u }:{u | ~u:u } of Russell does not.

7.1. Interpreting Atomic Closed Formulas

The provision of a Tarskian semantics for NaDSet requires first an interpretation of the atomic closed formulas. These formulas take the form

r:**PC**

where **r** is a closed first order term and **PC** is a second order parameter or constant. Interpretations of these formulas are conventional: Given a domain of discourse $\mathbb{d}$, an interpretation assigns an object in $\mathbb{d}$ to **r** and a subset of $\mathbb{d}$ to **PC**. The formula is true in the interpretation, if the object assigned to **r** is a member of the subset assigned to **PC**, and is otherwise false.

All interpretations of NaDSet have the same first order domain $\mathbb{d}$, consisting of all closed terms in which no parameter, first or second order, has an occurrence. The domain $\mathbb{D}$ for a given interpretation, on the other hand, consists of just some, but not necessarily all, subsets of $\mathbb{d}$. Thus an interpetation of NaDSet may be a nonstandard model in the sense of [Henkin50].

7.1.1. Definition: Interpretation

An interpretation consists of a set $\mathbb{D}$ of subsets of $\mathbb{d}$, called the domain of the interpretation, and a function Φ, called the assignment of the interpretation, satisfing:

- For each first order parameter **p**, Φ[**p**] $\in \mathbb{d}$;
- For each second order parameter or constant **PC**, Φ[**PC**] $\in \mathbb{D}$.

End of Definition

For a given closed first order term **r**, **Φ[r]** is defined to be the term obtained from **r** by replacing every occurrence of a parameter **p** in **r** by **Φ[p]**; note that since **r** is first order, **p** is necessarily first order, and that constants occurring in **r** are unaffected. For a given closed first order term **r**, **Φ[r]** is necessarily a member of **d**.

An assignment Φ of an interpretation assigns one and only one of the truth values true or false to each closed atomic formula **r:PC**: It is assigned true if **Φ[r]** is a member of **Φ[PC]**, otherwise it is false. For example, let C be a second order constant and p a first order parameter, and consider the following closed atomic formulas: C:C, p:C, and {u | ~u=u }:C. Let Φ[C] be the set {C, {u |~u=u}} and let Φ[p] be {u |~u=u}. Then Φ assigns true to each of the atomic formulas since Φ['C'] is C and C ∈ Φ[C]; Φ[p] is {u | ~u=u } and {u | ~u=u } ∈ Φ[C]; and Φ[{u | ~u=u }] is {u | ~u=u } and {u | ~u=u } ∈ Φ[C]. Note that in the atomic formula C:C, the occurence of 'C' to the left of ':' is being mentioned, since the occurrence is a name for itself, while the occurrence to the right of ':' is being used as a name for the set {C, {u |~u=u}}.

7.2. Assigning Truth Values to Nonatomic Formulas

Here the definitions given in 3.3 for NaDSet* are updated to include the { } rules of NaDSet. As with conventional Tarski semantics, truth values are assigned to other closed formulas by induction on a measure of the complexity of the formulas. For first order logic the measure is related to the degree of the formula, as is evident in definition 3.3.2. But abstraction terms make the conventional definition of degree unsuitable for NaDSet. Instead truth values are assigned indirectly through sets Ω[Φ, 𝔻] of signed formulas defined by transfinite induction, rather than finite induction. A signed formula is a closed formula prefixed with a + or - sign. A formula **F** is assigned true if **+F** ∈ Ω[Φ, 𝔻], and is assigned false if **-F** ∈ Ω[Φ, 𝔻]; it is assigned no truth value if **+F** ∉ Ω[Φ, 𝔻] and **+F** ∉ Ω[Φ, 𝔻].

7.2.1. Definition: The Set Ω[Φ, 𝔻] of Signed Formulas

Given the domain 𝔻 and assignment Φ of an interpretation, Ω[Φ, 𝔻] is the set

$$\cup\{\Omega_\mu[\Phi, \mathbb{D}] \mid \mu \geq 0\},$$

where $\Omega_\mu[\Phi, \mathbb{D}]$, abbreviated to Ω_μ, are sets of signed formulas defined inductively for ordinal numbers μ, μ ≥ 0, as follows:

1. If **Φ[r]** ∈ **Φ[PC]**, then **+r:PC** ∈ Ω_0, otherwise **-r:PC** ∈ Ω_0.

2. Assuming Ω_μ is defined for an ordinal number μ, $\Omega_{\mu+1}$ is the least set satifying:

$\cup$ $\Omega_{\mu+1} \supseteq \Omega_\mu$

$+\downarrow$ $-\mathbf{G} \in \Omega_\mu$ and $-\mathbf{H} \in \Omega_\mu \Rightarrow +(\mathbf{G}\downarrow\mathbf{H}) \in \Omega_{\mu+1}$

$-\downarrow$ $+\mathbf{G} \in \Omega_\mu$ or $+\mathbf{H} \in \Omega_\mu \Rightarrow -(\mathbf{G}\downarrow\mathbf{H}) \in \Omega_{\mu+1}$

Let **G** be a formula in which **u** is the only variable with a free occurrence.

$+\forall$ Let **p** be a first or second order parameter not occurring in the formula **G**, and let $+[\mathbf{p}/\mathbf{u}]\mathbf{G} \in \Omega_\mu[\Phi', \mathbb{D}]$ for every assignment Φ' that differs from Φ only in the value of $\Phi[\mathbf{p}]$. Then $+\forall\mathbf{uG} \in \Omega_{\mu+1}$.

$-\forall$ For any closed term **r**, $-[\mathbf{r}/\mathbf{u}]\mathbf{G} \in \Omega_\mu \Rightarrow -\forall\mathbf{uF} \in \Omega_{\mu+1}$.

Let **G** be a formula in which only the variables in the sequence $\underline{\mathbf{u}}$ of distinct variables has a free occurrence, and let $\underline{\mathbf{r}}$ be a sequence of closed terms of the same length as $\underline{\mathbf{u}}$.

$\pm\{\}$ $\pm[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G} \in \Omega_\mu \Rightarrow \pm[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{G}\} \in \Omega_{\mu+1}$.

3. For a limit ordinal ν, Ω_ν is $\cup\{\Omega_\mu \mid \nu > \mu \geq 0\}$

End of Definition 7.2.1

Since there are only denumerably many signed formulas, necessarily $\Omega_{\nu+1}[\Phi, \mathbb{D}]$ is $\Omega_\nu[\Phi, \mathbb{D}]$ when ν is ε_0.

Without the clauses $\pm\{\}$, the definition could be given in terms of finite induction, as in definition 3.3.2, and the resulting set $\Omega[\Phi, \mathbb{D}]$ would be consistent in the sense that for no closed formula **F** is both **+F** and **-F** members of $\Omega[\Phi, \mathbb{D}]$. With these clauses, however, not every set $\Omega[\Phi, \mathbb{D}]$ need be consistent.

7.2.2. **Definition:** Consistent Interpretation

An interpretation with domain $\mathbb{D}$ and assignment Φ is said to be <u>consistent</u> if for no closed formula F is both +F and -F members of $\Omega[\Phi, \mathbb{D}]$.

There are interpretations that are not consistent. For example, let $\mathbb{D}$ have $\mathbb{d}$ and {c} as its only members, where c is a first order constant, and let Φ be any assignment. Then both $+\forall$y c:y and $-\forall$y c:y are in $\Omega[\Phi, \mathbb{D}]$. For let P be a second order parameter. Necessarily +c:P $\in \Omega[\Phi, \mathbb{D}]$ and

+c:P ∈ Ω[Φ', 𝔻] for every assignment Φ' differing from Φ only in the value of Φ[P]; therefore the premiss of clause +∀ is satisfied and +∀y c:y ∈ Ω[Φ, 𝔻]. But +c=c ∈ Ω[Φ, 𝔻], so that --c=c ∈ Ω[Φ, 𝔻] and therefore -c:{x | ~x=x} ∈ Ω[Φ, 𝔻] and hence -∀y c:y ∈ Ω[Φ, 𝔻].

The source of the contradiction here is the incompleteness of 𝔻; it does not have sufficiently many subsets of 𝕕 as members. Using the terminology of [Henkin50], the inconsistent interpretation is a frame that is not a general model. By adapting the methods described in [Henkin49] and using the result of § 6 that the empty sequent is not derivable in NaDSet, consistent interpretations of NaDSet are constructed in § 8.

There are inconsistent interpretations for first order logic, namely those in which an atomic formula is assigned both true and false, but they, like the inconsistent interpretations of NaDSet are not of interest.

7.3. Satisfiability and Validity

A sequent Γ → Θ is satisfied by an interpretation with domain 𝔻 and assignment Φ, if for some formula **F**,

F ∈ Γ and -**F** ∈ Ω[Φ, 𝔻], or **F** ∈ Θ and +**F** ∈ Ω[Φ, 𝔻];

that is, one of the premisses in Γ is false, or one of the conclusions in Θ is true.

Because there exist inconsistent interpretations, a sequent cannot be said to be valid if it is satisfied by every interpretation; rather, it is said to be weakly valid. It is said to be valid only if it is satisfied by every consistent interpretation.

It may be suprising that the following theorem can be proved, given the existence of inconsistent interpretations. But the corresponding theorem for first order logic is also provable.

7.3.1. Theorem: A sequent derivable without the use of cut is weakly valid.

Proof:

Since each axiom of NaDSet is weakly valid, it is sufficient to prove of each rule of deduction that if there is an interpretation in which the conclusion of the rule is not satisfied, then there is one in which at least one of its premisses is not satisfied. For each rule 𝔻 will be assumed to be the domain and Φ the assignment of an interpretation which does not satisfy the conclusion of the rule.

$\rightarrow\downarrow$ Let the conclusion $\Gamma, \Delta \rightarrow (\mathbf{G}\downarrow\mathbf{H}), \Theta, \Lambda$ of this rule be not satisfied in the interpretation. Then for every $\mathbf{F}$

$$\mathbf{F} \in \Gamma, \Delta \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in (\mathbf{G}\downarrow\mathbf{H}), \Theta, \Lambda \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

But

$$+(\mathbf{G}\downarrow\mathbf{H}) \notin \Omega[\Phi, \mathbb{D}] \Rightarrow -\mathbf{G} \notin \Omega[\Phi, \mathbb{D}] \text{ or } -\mathbf{H} \notin \Omega[\Phi, \mathbb{D}]$$

Hence, either for every $\mathbf{F}$

$$\mathbf{F} \in \Gamma, \mathbf{G} \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

or for every $\mathbf{F}$

$$\mathbf{F} \in \Delta, \mathbf{H} \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in \Lambda \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

Therefore one of the premisses $\Gamma, \mathbf{G} \rightarrow \Theta$ or $\Gamma \rightarrow \neg\mathbf{G}, \Theta$ of the rule is not satisfied by the interpretation.

$\downarrow\rightarrow$ A similar argument suffices in this case for each of the rules of deduction.

$\rightarrow\forall$ Let the conclusion $\Gamma \rightarrow \forall\mathbf{u}\mathbf{G}, \Theta$ of this rule be not satisfied in the interpretation. Then for every $\mathbf{F}$

$$\mathbf{F} \in \Gamma \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in \forall\mathbf{u}\mathbf{G}, \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

But

$$+\forall\mathbf{u}\mathbf{G} \notin \Omega[\Phi, \mathbb{D}] \Rightarrow +[\mathbf{p}/\mathbf{u}]\mathbf{G} \notin \Omega[\Phi', \mathbb{D}]$$

where Φ' is some assignment that differs from Φ only in the value assigned to $\Phi[\mathbf{p}]$. Since $\mathbf{p}$ does not occur in any formula of Γ or Θ, it follows that for every $\mathbf{F}$

$$\mathbf{F} \in \Gamma \Rightarrow -\mathbf{F} \notin \Omega[\Phi', \mathbb{D}] \text{ and } \mathbf{F} \in [\mathbf{p}/\mathbf{u}]\mathbf{G}, \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi', \mathbb{D}]$$

Hence the premiss of the rule is not satisfied by Φ'.

$\forall\rightarrow$ Let conclusion $\Gamma, \forall\mathbf{u}\mathbf{G} \rightarrow \Theta$ of this rule be not satisfied in the interpetation. Then for every $\mathbf{F}$

$$\mathbf{F} \in \Gamma, \forall\mathbf{u}\mathbf{G} \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

But

$$-\forall\mathbf{u}\mathbf{G} \notin \Omega[\Phi, \mathbb{D}] \Rightarrow -[\mathbf{r}/\mathbf{u}]\mathbf{G} \notin \Omega[\Phi, \mathbb{D}]$$

Hence for every $\mathbf{F}$

$$\mathbf{F} \in \Gamma, [\mathbf{r}/\mathbf{u}]\mathbf{G} \Rightarrow -\mathbf{F} \notin \Omega[\Phi', \mathbb{D}] \text{ and } \mathbf{F} \in \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi', \mathbb{D}]$$

and the premiss of the rule is not satisfied by Φ

$\rightarrow\{\}$ Let the conclusion $\Gamma \rightarrow [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{G}\}, \Theta$ of this rule be not satisfied in the interpetation. Then for every **F**

$$\mathbf{F} \in \Gamma \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{G}\}, \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

But

$$+[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{G}\} \notin \Omega[\Phi, \mathbb{D}] \Rightarrow +[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G} \notin \Omega[\Phi, \mathbb{D}]$$

Hence for every **F**

$$\mathbf{F} \in \Gamma \Rightarrow -\mathbf{F} \notin \Omega[\Phi', \mathbb{D}] \text{ and } \mathbf{F} \in [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G}, \Theta \Rightarrow +\mathbf{F} \notin \Omega[\Phi', \mathbb{D}]$$

and the premiss of the rule is not satisfied by Φ

$\{\}\rightarrow$ This case is similar to the previous.

End of Proof

Cut must be excluded from the derivations of theorem 7.3.1 because inconsistent interpretations exist. But the following theorem is easily proved:

7.3.2. Theorem: Every sequent derivable in NaDSet with cut is valid.

Proof:

It is sufficient to prove that cut preserves validity. Let $\mathbb{D}$ be the domain and Φ the assignment of a consistent interpretation which does not satisfy the conclusion $\Gamma, \Delta \rightarrow \Theta, \Lambda$ of this rule. Then for every **F**

$$\mathbf{F} \in \Gamma, \Delta \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}] \text{ and } \mathbf{F} \in \Theta, \Lambda \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}]$$

But if both of the premisses are valid, then necessarily

$$-\mathbf{G} \in \Omega[\Phi, \mathbb{D}] \text{ and } +\mathbf{G} \in \Omega[\Phi, \mathbb{D}]$$

which is not possible for a consistent interpretation.

End of Proof

The converse of 7.3.2, the completeness theorem for NaDSet, is proved in the following section.

8. COMPLETENESS

Since by theorem 6.0.1 the empty sequent is not derivable in NaDSet with cut, there are sequents that are not derivable. Indeed, for every pair of sequents $\rightarrow \mathbf{G}$ and $\mathbf{G} \rightarrow$, at least one of them is necessarily not derivable in NaDSet with cut.

Given a sequent that is not derivable, a method will be described for constructing a consistent interpretation which does not satisfy it. But not only will the completeness theorem for NaDSet be proved, but also the redundancy of the cut rule of deduction. For this reason attention will be focused first on NaDSet without cut. Unless otherwise stated, in this section a sequent is said to be derivable, or not derivable, if it has, or has not, a derivation within NaDSet in which no application of cut is made.

8.1. Consistent Sets

In this subsection only a restricted set of closed formulas will be considered, namely those in which no first order parameter occurs; here they will be referred to simply as restricted formulas. Similarly by a restricted term is meant a closed term in which no first order parameter occurs; note that a restricted first order term is a member of **d**. Focusing on restricted formulas and terms assists in the construction of consistent interpretations.

Given a finite set Σ of signed restricted formulas, $+\Sigma$ is a sequence of all the formulas **F** for which $+$**F** is in Σ, while $-\Sigma$ is a corresponding sequence of all the formulas **F** for which $-$**F** is in Σ. By fixing on an order for the sequences $+\Sigma$ and $-\Sigma$, a sequent $+\Sigma \rightarrow -\Sigma$ is determined by Σ.

8.1.1. Definition: Consistent Sets of Signed Restricted Formulas
A finite set of signed restricted formulas Σ is consistent if the sequent $+\Sigma \rightarrow -\Sigma$ is not derivable, and is cut-consistent if it is not derivable in NaDSet with cut. An infinite set of signed restricted formulas is (cut-) consistent if every finite subset of it is (cut-)consistent.

The following definition has been adapted from [Henkin49]

8.1.2. Definition: A (cut-)consistency completion Σ^* ($^c\Sigma^*$) of Σ
Let Σ be a finite consistent set of signed restricted formulas. A (cut-)consistency completion Σ^* ($^c\Sigma^*$) of Σ is any set

$$\cup\{\Sigma_i \mid i \geq 0\}$$

where the sets Σ_i are defined, together with sets Δ_i, as follows:

1. Σ_0 is Σ, and

   Δ_0 is the empty set.

2. Let Σ_i and Δ_i be defined.

   a) If there is no formula of the form $-\forall$**vG** in $\Sigma_i - \Delta_i$, then Σ'_{i+1} is Σ_i.

   Otherwise, let $-\forall$**vG** be the first formula in $\Sigma_i - \Delta_i$ in a given ordering of the formulas of the form $-\forall \mathbf{v}\mathbf{G}$. Then

Δ_{i+1} is $\Delta_i \cup \{-\forall \mathbf{v}\mathbf{G}\}$ and

Σ'_{i+1} is $\Sigma_i \cup \{-[\mathbf{c}/\mathbf{v}]\mathbf{G}, -[\mathbf{P}/\mathbf{v}]\mathbf{G}\}$,

where **c** is a first order constant and **P** a second order parameter not occurring in Σ_i.

b) Let **sF** be a signed restricted formula for which $\mathbf{sF} \notin \Sigma'_{i+1}$ and $\Sigma'_{i+1} \cup \{\mathbf{sF}\}$ is (cut-)consistent. Then Σ_{i+1} is $\Sigma'_{i+1} \cup \{\mathbf{sF}\}$.

End of Definition 8.1.2.

Necessarily Σ^* and ${}^c\Sigma^*$ are sets of signed restricted formulas since Σ is such a set and only signed restricted formulas are added to Σ in the formation of Σ^* or ${}^c\Sigma^*$.

8.1.3. Theorem:

Let Σ be a finite (cut-)consistent set of signed restricted formulas, and Σ^* a (cut-)consistency completion of it. Then Σ^* has the following properties:

0.1. Σ^* is (cut-)consistent

0.2. For no atomic formula **A** is $+\mathbf{A} \in \Sigma^*$ and $-\mathbf{A} \in \Sigma^*$

1.1. $+(\mathbf{G}\downarrow\mathbf{H}) \in \Sigma^* \Rightarrow -\mathbf{G} \in \Sigma^*$ and $-\mathbf{H} \in \Sigma^*$

1.2. $-(\mathbf{G}\downarrow\mathbf{H}) \in \Sigma^* \Rightarrow +\mathbf{G} \in \Sigma^*$ or $+\mathbf{H} \in \Sigma^*$

2.1. $+\forall \mathbf{v}\mathbf{G} \in \Sigma^* \Rightarrow +[\mathbf{r}/\mathbf{v}]\mathbf{G} \in \Sigma^*$ for every restricted term **r**.

2.2. $-\forall \mathbf{v}\mathbf{G} \in \Sigma^* \Rightarrow -[\mathbf{c}/\mathbf{v}]\mathbf{G} \in \Sigma^*$ and $-[\mathbf{P}/\mathbf{v}]\mathbf{G} \in \Sigma^*$ for some first order constant **c**, and some second order parameter **P**.

3.1. $+[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{G}\} \in \Sigma^* \Rightarrow +[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G} \in \Sigma^*$

3.2. $-[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{G}\} \in \Sigma^* \Rightarrow -[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{G} \in \Sigma^*$

Proof:

0.1. It is sufficient to consider the case of consistent sets, since the case of cut-consistent sets is similar.

If Σ^* is not consistent, then for some i, $i \geq 0$, Σ_{i+1} is not consistent, since Σ is consistent. Let i be the smallest index for which Σ_{i+1} is not consistent. If Σ'_{i+1} is Σ_i then Σ'_{i+1} is consistent. Assume, therefore, that Σ'_{i+1} is not Σ_i. Let $-\forall \mathbf{v}\mathbf{G}$ be the first formula in $\Sigma_i - \Delta_i$, so that Σ'_{i+1} is $\Sigma_i \cup \{ -[\mathbf{c}/\mathbf{v}]\mathbf{G}, -[\mathbf{P}/\mathbf{v}]\mathbf{G}\}$, where **c** is a first order constant and **P** a second order parameter not occurring in Σ_i. Necessarily Σ'_{i+1} is consistent since otherwise $+\Sigma_i \rightarrow -\Sigma_i, [\mathbf{c}/\mathbf{v}]\mathbf{G}, [\mathbf{P}/\mathbf{v}]\mathbf{G}$ would be derivable, and therefore by lemma 3.2.4 also $+\Sigma_i \rightarrow -\Sigma_i, [\mathbf{p}/\mathbf{v}]\mathbf{G}, [\mathbf{P}/\mathbf{v}]\mathbf{G}$, where **p** is a first order parameter not occurring anywhere in the derivation of $+\Sigma_i \rightarrow -\Sigma_i$, [c/v]G, [P/v]G. Two applications of $\rightarrow\forall$ and applications of contraction and possibly interchange would result in $+\Sigma_i \rightarrow -\Sigma_i$ being derivabable, contradicting the consistency of Σ_i. But if Σ'_{i+1} is consistent, so also is Σ_{i+1} by definition.

0.2. Since $A \rightarrow A$ is an axiom, $+A \in \Sigma^*$ and $-A \in \Sigma^*$ would contradict 0.1.

1.1. If $-G \notin \Sigma^*$, then for some i, $+\Sigma_i \rightarrow -\Sigma_i$, G is derivable so that one application of $\downarrow\rightarrow$ yields a derivation of $+\Sigma_i$, $(G\downarrow H) \rightarrow -\Sigma_i$. Therefore $+(G\downarrow H) \notin \Sigma^*$. A similar argument applies if $-H \notin \Sigma^*$.

1.2. If $+G \notin \Sigma^*$ and $+H \notin \Sigma^*$, then for some i, both $+\Sigma_i$, $G \rightarrow -\Sigma_i$ and $+\Sigma_i$, $H \rightarrow -\Sigma_i$ are derivable so that one application of $\rightarrow\downarrow$ yields a derivation of $+\Sigma_i \rightarrow -\Sigma_i$, $(G\downarrow H)$. Therefore $-(G\downarrow H) \notin \Sigma^*$.

2.1. +If $[r/v]G \notin \Sigma^*$ for some restricted term **r**, then for some i, $+\Sigma_i$, $[r/v]G \rightarrow -\Sigma_i$ is derivable so that one application of $\forall\rightarrow$ yields a derivation of $+\Sigma_i$, $\forall vG \rightarrow -\Sigma_i$. Therefore $+\forall vG \notin \Sigma^*$.

2.2. Let $-\forall vG \in \Sigma^*$, and let i be the smallest index for which $-\forall vG \in \Sigma_i$. Then for some j, $i \leq j$, $-\forall vG$ is the first formula of that form that is in $\Sigma_j - \Delta_j$. Hence, Σ'_{j+1} is $\Sigma_j \cup \{ -[c/v]G, -[P/v]G\}$, for some first order constant **c**, and some second order parameter **P**.

3.1. +If $[\underline{r} / \underline{u}]G \notin \Sigma^*$, then for some i, $+\Sigma_i$, $[\underline{r} / \underline{u}]G \rightarrow -\Sigma_i$ is derivable so that one application of $\{\}\rightarrow$ yields a derivation of $+\Sigma_i$, $[\underline{r} / \underline{u}]t{:}\{t \mid G\} \rightarrow -\Sigma_i$. Therefore $+[\underline{r} / \underline{u}]t{:}\{t \mid G\} \notin \Sigma^*$.

3.2. If $-[\underline{r} / \underline{u}]G \notin \Sigma^*$, then for some i, $+\Sigma_i \rightarrow -\Sigma_i$, $[\underline{r} / \underline{u}]G$ is derivable so that one application of $\rightarrow\{\}$ yields a derivation of $+\Sigma_i \rightarrow -\Sigma_i$, $[\underline{r} / \underline{u}]t{:}\{t \mid G\}$. Therefore $-[\underline{r} / \underline{u}]t{:}\{t \mid G\} \notin \Sigma^*$.

End of Proof of Theorem 8.1.3

Corollary:

Let Σ be a finite cut-consistent set of signed restricted formulas, and ${}^c\Sigma^*$ a cut-consistency completion of it. Then for every restricted formula **F**, at least one of +F or -F is in ${}^c\Sigma^*$.

Proof: Let $+F \notin {}^c\Sigma^*$ and $-F \notin {}^c\Sigma^*$. Then for some i, both $+\Sigma_i, F \rightarrow -\Sigma_i$ and $+\Sigma_i \rightarrow -\Sigma_i, F$ are derivable in NaDSet with cut. From cut, interchange and contraction it follows that $+\Sigma_i \rightarrow -\Sigma_i$ is derivable with cut, contradicting the cut-consistency of Σ_i.

End of Proof of Corollary

This property of ${}^c\Sigma^*$ depends upon the fact that cut-consistency completion is defined for NaDSet with cut. From the next theorem it will be possible to conclude that it is also a property of consistency completions defined for NaDSet without cut.

8.1.4. Theorem:

Let Σ be a finite cut-consistent set and let Σ^* be a consistency completion of it. Then there is a cut-consistency completion ${}^c\Sigma^*$ of Σ for which $\Sigma^* \supseteq {}^c\Sigma^*$.

Proof:

Define ${}^c\Sigma_0$ to be Σ_0 to be Σ, and ${}^c\Delta_0$ to be Δ_0 to be the empty set. Σ^* is $\cup\{\Sigma_i \mid i \geq 0\}$, where sets Σ_i and Δ_i, $i > 0$, have been chosen to satisfy part (2) of definition 8.1.2. It is sufficient to find sets ${}^c\Sigma_i$ for which $\cup\{{}^c\Sigma_i \mid i \geq 0\}$ is a cut-consistency completion of Σ and for which $\Sigma_i \supseteq {}^c\Sigma_i$, for $i \geq 0$. While defining such sets, sets ${}^c\Delta_i$, $i > 0$, will also be defined for which $\Delta_i \supseteq {}^c\Delta_i$.

Consider now Σ_{i+1}, ${}^c\Sigma_{i+1}$, Δ_{i+1}, and ${}^c\Delta_{i+1}$.

a) If Σ'_{i+1} is Σ_i, or if Σ'_{i+1} is not Σ_i and $-\forall v\mathbf{G}$ is not in ${}^c\Sigma_i - {}^c\Delta_i$, then define ${}^c\Sigma'_{i+1}$ to be ${}^c\Sigma_i$ and ${}^c\Delta_{i+1}$ to be ${}^c\Delta_i$. Otherwise, if $-\forall v\mathbf{G}$ is in ${}^c\Sigma_i - {}^c\Delta_i$, define ${}^c\Sigma'_{i+1}$ to be ${}^c\Sigma_i \cup \{-[\mathbf{c}/\mathbf{v}]\mathbf{G}, -[\mathbf{P}/\mathbf{v}]\mathbf{G}\}$ and ${}^c\Delta_{i+1}$ to be ${}^c\Delta_i \cup \{-\forall \mathbf{v}\mathbf{G}\}$.

b) If Σ_{i+1} is $\Sigma'_{i+1} \cup \{\mathbf{sF}\}$ and ${}^c\Sigma'_{i+1} \cup \{\mathbf{sF}\}$ is cut-consistent, then ${}^c\Sigma_{i+1}$ is ${}^c\Sigma'_{i+1} \cup \{\mathbf{sF}\}$; otherwise ${}^c\Sigma_{i+1}$ is ${}^c\Sigma'_{i+1}$.

It follows from these definitions that $\Sigma_{i+1} \supseteq {}^c\Sigma_{i+1}$ and $\Delta_{i+1} \supseteq {}^c\Delta_{i+1}$.

It is now necessary to prove that $\cup\{{}^c\Sigma_i \mid i \geq 0\}$ is a cut-consistency completion of Σ. The sequence of sets ${}^c\Sigma_i$ is transformed as follows:

i) If ${}^c\Sigma_{i+1}$ is ${}^c\Sigma_i$, then ${}^c\Sigma_{i+1}$ is dropped.

ii) Consider the remaining sets and consider the first set ${}^c\Sigma_{i+1}$ for which ${}^c\Sigma'_{i+1}$ is ${}^c\Sigma_i$ but for which $-\forall v\mathbf{G}$ is in ${}^c\Sigma_i - {}^c\Delta_i$ for some formula of that form. For some j, $j \geq i+1$, ${}^c\Sigma'_{j+1}$ is ${}^c\Sigma_j \cup \{-[\mathbf{c}/\mathbf{v}]\mathbf{G}, -[\mathbf{P}/\mathbf{v}]\mathbf{G}\}$, where **c** and **P** do not occur in ${}^c\Sigma_j$ or in **G** and ${}^c\Delta_{j+1}$ is

${}^{c}\Delta_j \cup \{-\forall v G\}$. Redefine ${}^{c}\Sigma'_{i+1}$ to be ${}^{c}\Sigma_i \cup \{-[c/v]G, -[P/v]G\}$, ${}^{c}\Delta_{i+1}$ to be ${}^{c}\Delta_i \cup \{-\forall v G\}$and ${}^{c}\Delta_{j+1}$ to be ${}^{c}\Delta_j$. If as a consequence ${}^{c}\Sigma_{j+1}$ is ${}^{c}\Sigma_j$, then drop ${}^{c}\Sigma_{j+1}$.

iii) Consider the remaining sets and consider the first set ${}^{c}\Sigma_{i+1}$ for which ${}^{c}\Sigma'_{i+1}$ is not ${}^{c}\Sigma_i$, but for which ${}^{c}\Sigma_{i+1}$ is ${}^{c}\Sigma'_{i+1}$. For some j, $j \geq i+1$, ${}^{c}\Sigma_{j+1}$ is ${}^{c}\Sigma'_{j+1} \cup \{sF\}$, where ${}^{c}\Sigma'_{j+1} \cup \{sF\}$ is cut-consistent, and $sF \notin {}^{c}\Sigma_i$. Redefine ${}^{c}\Sigma'_{i+1}$ to be ${}^{c}\Sigma_i \cup \{sF\}$, and ${}^{c}\Sigma_{j+1}$ to be ${}^{c}\Sigma_j$. If as a consequence ${}^{c}\Sigma_{j+1}$ is ${}^{c}\Sigma_j$, then drop ${}^{c}\Sigma_{j+1}$.

The resulting sequence of sets satisfies the conditions of definition 8.1.2.
End of Proof of Theorem 8.1.4

The following corollary follows immediately from the corollary to theorem 8.1.3.

Corollary:
Let Σ be a finite cut-consistent set of signed restricted formulas, and Σ^* a consistency completion of it. Then for every restricted formula **F**, at least one of **+F** or **-F** is in Σ^*.

8.2. Consistent Interpretations

Let Σ be a finite consistent set of signed restricted formulas and Σ^* a consistency completion of it. For each second order parameter or constant **PC**, $\beta[\Sigma^*, PC]$ is the subset of $\mathbb{d}$ consisting of those restricted first order terms **r** for which $+r{:}PC \in \Sigma^*$.

8.2.1. Definition: An interpretation determined by Σ^*
An interpetation with domain $\mathbb{D}$ consisting of all the sets $\beta[\Sigma^*, PC]$, and an assignment Φ for which $\Phi[PC]$ is $\beta[\Sigma^*, PC]$, is said to be determined by Σ^*. The value of $\Phi[p]$ for a first order parameter **p** may be any member of $\mathbb{d}$.

The notation $\Phi 1[\pm F]$ is used to denote the signed restricted formula obtained from a signed closed formula $\pm F$ by replacing each occurrence of a first order parameter **p** in **F** by $\Phi[p]$.

8.2.2. Theorem:

Let Σ be a finite consistent set of signed restricted formulas and Σ^* a consistency completion of it. Let $\mathbb{D}$ be the domain and Φ the assignment of an interpretation determined by Σ^*. Then for all restricted formulas **F** and all ordinals μ,

$$+\mathbf{F} \in \Omega_\mu[\Phi, \mathbb{D}] \Rightarrow \Phi 1[-\mathbf{F}] \notin \Sigma^*$$

$$-\mathbf{F} \in \Omega_\mu[\Phi, \mathbb{D}] \Rightarrow \Phi 1[+\mathbf{F}] \notin \Sigma^*$$

Proof:

Throughout the proof, $\Omega_\mu[\Phi, \mathbb{D}]$ will be abbreviated by Ω_μ. The proof will be by induction on μ. The case μ=0 follows immediately from the definition of Ω_0. Assume therefore that the conclusion is true for μ, $\mu \geq 0$, and consider the possibilities for **F** for the case μ+1.

F is (**G↓H**).

$+(\mathbf{G}\downarrow\mathbf{H}) \in \Omega_{\mu+1} - \Omega_\mu \Rightarrow -\mathbf{G} \in \Omega_\mu$ and $-\mathbf{H} \in \Omega_\mu$ by definition,

$\Rightarrow \Phi 1[+\mathbf{G}] \notin \Sigma^*$ and $\Phi 1[+\mathbf{H}] \notin \Sigma^*$ by induction.

$\Phi 1[-(\mathbf{G}\downarrow\mathbf{H})] \in \Sigma^* \Rightarrow \Phi 1[+\mathbf{G}] \in \Sigma^*$ or $\Phi 1[+\mathbf{H}] \in \Sigma^*$ by 1.2 of theorem 8.1.3.

Hence $\Phi 1[-(\mathbf{G}\downarrow\mathbf{H})] \notin \Sigma^*$.

$-(\mathbf{G}\downarrow\mathbf{H}) \in \Omega_{\mu+1} - \Omega_\mu \Rightarrow +\mathbf{G} \in \Omega_\mu$ or $+\mathbf{H} \in \Omega_\mu$ by definition.

$\Rightarrow \Phi 1[-\mathbf{G}] \notin \Sigma^*$ or $\Phi 1[-\mathbf{H}] \notin \Sigma^*$ by induction.

$\Phi 1[+(\mathbf{G}\downarrow\mathbf{H})] \in \Sigma^* \Rightarrow \Phi 1[-\mathbf{G}] \in \Sigma^*$ and $\Phi 1[-\mathbf{H}] \in \Sigma^*$ by 1.1 of theorem 8.1.3.

Hence $\Phi 1[+(\mathbf{G}\downarrow\mathbf{H})] \notin \Sigma^*$.

F is $\forall$**uG**.

$+\forall\mathbf{uG} \in \Omega_{\mu+1} - \Omega_\mu \Rightarrow +[\mathbf{q}/\mathbf{u}]\mathbf{G} \in \Omega_\mu[\Phi', \mathbb{D}]$ for every assignment Φ' that differs from Φ only in the assignment to the parameter **q** not occuring in **G**; by definition.

$\Phi 1[-\forall\mathbf{uG}] \in \Sigma^* \Rightarrow \Phi 1[-[\mathbf{c}/\mathbf{u}]\mathbf{G}] \in \Sigma^*$ and $\Phi 1[-[\mathbf{P}/\mathbf{u}]\mathbf{G}] \in \Sigma^*$, for some first order constant **c** and second order parameter **P**, by 2.2 of theorem 8.1.3.

Let $\Phi'[\mathbf{q}]$ be **c**, if **q** is first order, and let it be $\Phi[\mathbf{P}]$, if **q** is second order.

Then $+[\mathbf{q}/\mathbf{u}]\mathbf{G} \in \Omega_\mu[\Phi', \mathbb{D}] \Rightarrow +[\mathbf{c}/\mathbf{u}]\mathbf{G} \in \Omega_\mu[\Phi', \mathbb{D}]$, if **q** is first order and

$\Rightarrow +[\mathbf{P}/\mathbf{u}]\mathbf{G} \in \Omega_\mu[\Phi', \mathbb{D}]$, if **q** is second order.

By the induction assumption, therefore, either

$\Phi' 1[-[\mathbf{c}/\mathbf{u}]\mathbf{G}] \notin \Sigma^*$ or $\Phi' 1[-[\mathbf{P}/\mathbf{u}]\mathbf{G}] \notin \Sigma^*$. But since Φ' differs from Φ only in the assignment to the parameter **q** not occuring in **G**, Φ 1[-[**c**/**u**]**G**] is $\Phi 1[-[\mathbf{c}/\mathbf{u}]\mathbf{G}]$ and

$\Phi'1[-[P/u]G]$ is $\Phi 1[-[P/u]G]$. Hence $\Phi 1[-\forall uG] \notin \Sigma^*$.

$-\forall uG \in \Omega_{\mu+1} - \Omega_\mu \quad \Rightarrow -[r/u]G \in \Omega_\mu$, for some closed term r, by definition

$\Rightarrow \Phi 1[+[r/u]G] \notin \Sigma^*$ by induction.

$\Phi 1[+\forall uG] \in \Sigma^* \quad \Rightarrow [+\Phi[r]/u]\Phi 1[G] \in \Sigma^*$, by 2.1 of theorem 8.1.3.

$\Rightarrow \Phi 1[+[r/u]G] \in \Sigma^*$

Hence $\Phi 1[+\forall uG] \notin \Sigma^*$.

F is $[\underline{r}/\underline{u}]t{:}\{t \mid G\}$

$+[\underline{r}/\underline{u}]t{:}\{t \mid G\} \in \Omega_{\mu+1} - \Omega_\mu \Rightarrow +[\underline{r}/\underline{u}]G \in \Omega_\mu$, by definition

$\Rightarrow -[\underline{r}/\underline{u}]G \notin \Sigma^*$ by induction.

$-[\underline{r}/\underline{u}]t{:}\{t \mid G\} \in \Sigma^* \Rightarrow -[\underline{r}/\underline{u}]G \in \Sigma^*$ by 3.2 of theorem 8.1.3.

Hence $-[\underline{r}/\underline{u}]t{:}\{t \mid G\} \notin \Sigma^*$.

$-[\underline{r}/\underline{u}]t{:}\{t \mid G\} \in \Omega_{\mu+1} - \Omega_\mu \Rightarrow -[\underline{r}/\underline{u}]G \in \Omega_\mu$, by definition

$\Rightarrow +[\underline{r}/\underline{u}]G \notin \Sigma^*$ by induction.

$+[\underline{r}/\underline{u}]t{:}\{t \mid G\} \in \Sigma^* \Rightarrow +[\underline{r}/\underline{u}]G \in \Sigma^*$ by 3.1 of theorem 8.1.3.

Hence $+[\underline{r}/\underline{u}]t{:}\{t \mid G\} \notin \Sigma^*$.

End of Proof of Theorem 8.2.2.

The following corollary follows immediately from the corollary of theorem 8.1.4.

Corollary 1:

The interpetation with domain $\mathbb{D}$ and assignment Φ is consistent. In particular, for all closed formulas **F**,

$+F \in \Omega[\Phi, \mathbb{D}] \Rightarrow \Phi 1[+F] \in \Sigma^*$ and $\Phi 1[-F] \notin \Sigma^*$

$-F \in \Omega[\Phi, \mathbb{D}] \Rightarrow \Phi 1[-F] \in \Sigma^*$ and $\Phi 1[+F] \notin \Sigma^*$

Corollary 2: The Completeness Theorem

If $\Gamma \rightarrow \Theta$ is not derivable, then there is a consistent interpretation in which $\Gamma \rightarrow \Theta$ is not satisfied.

Proof:

Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be all the first order parameters with occurrences in $\Gamma \rightarrow \Theta$ and let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be distinct first order constants distinct from any such constants with occurrences in $\Gamma \rightarrow \Theta$. Let Γ', respectively Θ', be obtained from Γ, respectively Θ, by replacing each occurrence of $\mathbf{p}_i$ in their formulas by $\mathbf{c}_i$, for i = 1, ..., n. Let Σ consist of the formulas of Γ' with + signs and the formulas of Θ' with - signs. Since by lemma 3.2.4, $\Gamma' \rightarrow \Theta'$ has no derivation, Σ is consistent. Let Σ^* be a consistency completion of it. Let $\mathbb{D}$ be the domain and Φ the assignment of an interpretation determined by Σ^* satisfying the additional condition that $\Phi[\mathbf{p}_i]$ is $\mathbf{c}_i$, for i = 1, ..., n. Then for all **F**

$$\mathbf{F} \in \Gamma \Rightarrow \Phi 1[+\mathbf{F}] \in \Sigma^* \Rightarrow -\mathbf{F} \notin \Omega[\Phi, \mathbb{D}], \text{ and}$$

$$\mathbf{F} \in \Theta' \Rightarrow \Phi 1[-\mathbf{F}] \in \Sigma^* \Rightarrow +\mathbf{F} \notin \Omega[\Phi, \mathbb{D}].$$

Therefore $\Gamma \rightarrow \Theta$ is not satisfied by the interpretation.

End of Proof of Corollary 2

Corollary 3: Redundancy of Cut

Proof:

Let $\Gamma \rightarrow \Theta$ be the sequent of corollary 2 that is not derivable in NaDSet without cut. Then it cannot be derivable in NaDSet with cut, since every sequent derivable with cut is satisfied by every consistent interpretation, including the interpretation constructed in the proof of corollary 2.

End of Proof of Corollary 3

8.2.3. Standard Interpretations & Gödel Incompleteness

A standard interpretation is one for which $\mathbb{D}$ consists of all the subsets of $\mathbb{d}$. It is natural to ask if standard interpretations are consistent. A positive answer to the question is not difficult to supply. Extend the elementary syntax of NaDSet to admit a distinct second order constant for each subset of $\mathbb{d}$, but restricting these new constants, as the second order parameters are restricted, to being second order terms only so that a term in which one of them occurs is necessarily second order. Let Φ be any assignment that assigns to each of the new constants **C** the subset that prompted its introduction. Let Σ be the set of all signed closed atomic formulas $\pm\mathbf{r}{:}\mathbf{C}$, for which respectively $\mathbf{r} \in \Phi[\mathbf{C}]$ and $\mathbf{r} \notin \Phi[\mathbf{C}]$. Σ is consistent as is therefore the interpretation by corollary 1 of theorem 8.2.2.

NaDSet satisfies the assumptions of Gödel's Incompleteness Theorem [Gödel31] since second order arithmetic can be developed within it [Gilmore89], despite the fact that a completeness theorem has been proved for it. The remarks made by Henkin in [Henkin50] concerning his completeness theorem for the simple theory of types apply here also: The true but unprovable formula constructed in Gödel's proof of incompleteness is true for all standard interpretations of NaDSet , while it may be true or false in an arbitrary consistent interpretation.

9. CANTOR'S LEMMA

In [Gilmore89] Cantor's lemma was taken to be:

For each enumeration P of sequences of 0's and 1's, there is a sequence not enumerated by P. The correctness of the general form of Cantor's diagonal argument used to prove this lemma was denied since the argument involves an abuse of use and mention. The question as to whether Cantor's lemma could nevertheless be proved in NaDSet remained open. Using the method of semantic trees, a variant of the method of semantic tableaux of [Beth55], a consistent interpretation is constructed below which does not satisfy Cantor's lemma.

Using the definitions of [Gilmore89], 0 is defined to be $\{u \mid {\sim}u{=}u\}$, the successor $\mathbf{t}'$ of $\mathbf{t}$ to be $\{u \mid u{=}\mathbf{t}\}$, the set N of nonnegative integers to be $\{u \mid \forall z(0{:}z \wedge (\forall x\,(x{:}z \supset x'{:}z)) \supset u{:}z)$, 1 to be $0'$, N1 to be $\{u' \mid u{:}N\}$, and Bit to be $\{u \mid u{=}0 \vee u{=}1\}$. The set Sq of sequences of 0's and 1's can then be defined:

$$\mathrm{Sq}\ \textbf{for}\ \{z \mid [\forall n{:}N1][\exists u{:}Bit](\langle n,u\rangle{:}z \wedge [\forall v{:}Bit](\langle n,v\rangle{:}z \supset v{=}u))\}.$$

Extensional identity $=_{e2}$ between sequences was defined to be:

$$=_{e2}\ \textbf{for}\ \{\langle x,y\rangle \mid [\forall u{:}N1][\forall v{:}Bit](\langle u,v\rangle{:}x \equiv \langle u,v\rangle{:}y)\},$$

where $\equiv$ expresses material equivalence, or "if and only if". A term **m** that is an enumeration of sequences of 0's and 1's satisfies M[**m**], which is defined:

$$M[\mathbf{m}]\ \textbf{for}\ [\forall n{:}N1][\exists x{:}Sq](\langle n,x\rangle{:}\mathbf{m} \wedge [\forall y{:}Sq](\langle n,y\rangle{:}\mathbf{m} \supset y{=}_{e2}x))$$

Thus the set Map of all such enumerations is defined:

$$\mathrm{Map}\ \textbf{for}\ \{z \mid M[z]\}$$

A proof of Cantor's lemma within NaDSet is therefore a derivation of the sequent:

$$\rightarrow [\forall z{:}Map][\exists x{:}Sq][\forall n{:}N1]\ {\sim}\langle n,x\rangle{:}z$$

If that sequent is not derivable, then the set Σ with the single member

$$-[\forall z{:}Map][\exists x{:}Sq][\forall n{:}N1]\ {\sim}\langle n,x\rangle{:}z$$

is consistent and a consistency completion Σ^* of the set will yield a consistent interpretation which does not satisfy the sequent. The method of semantic trees obtained from [Beth55] essentially attempts to construct a consistency completion with respect to an enumeration that is determined by the signed formulas in the tree. Should the tree fail to close, that is, should the original sequent be not derivable, then an open branch of the tree determines a consistency completion.

Consider the following incomplete semantic tree, where '...' indicates that a branch of the tree can be extended:

$-[\forall z{:}Map][\exists x{:}Sq][\forall n{:}N1] \sim\langle n,x\rangle{:}z)$

$-(M[P] \supset [\exists x{:}Sq][\forall n{:}N1] \sim\langle n,x\rangle{:}P))$ [P new]

$+M[P]$

$-[\exists x{:}Sq][\forall n{:}N1] \sim\langle n,x\rangle{:}P$

$-(C[P]{:}Sq \wedge [\forall n{:}N1] \sim\langle n,C[P]\rangle{:}P)$

$-C[P]{:}Sq$	$-[\forall n{:}N1] \sim\langle n,C[P]\rangle{:}P)$	
...	$-(c{:}N1 \supset \sim\langle c,C[P]\rangle{:}P)$	[c new]
	$+c{:}N1$	
	$-\sim\langle c,C[P]\rangle{:}P$	
	$+\langle c,C[P]\rangle{:}P$	
	...	

The first signed formula, which is the root vertex of the tree, is the single member of Σ. The second is one of the two signed formulas that should be added because of part (2) of definition 8.1.2; the other should be the same formula with the new second order parameter P replaced by a new first order constant c. The latter is not added because $\langle n,\mathbf{t}\rangle{:}c$ cannot be atomic for any term **t**, and it cannot be a member of $\Omega[\Phi, \mathbb{D}]$ for any interpretation, consistent or not. The formulas +M[P] and $-[\exists x{:}Sq][\forall n{:}N1] \sim\langle n,x\rangle{:}P$ are added next because of the formula appearing before them and the meaning of $\supset$: $\mathbf{G}\supset\mathbf{H}$ is false only if **G** is true and **H** is false. The next formula appearing just above the horizontal line is an instantiation of the existential quantifier appearing in the formula above it; the existentially quantified variable has been instantiated with an unspecified term C[P] which may be any closed term by part 2.1 of theorem 8.1.3. The horizontal bar indicates a branching of the tree resulting from the meaning of $\wedge$: $\mathbf{G}\wedge\mathbf{H}$ is false only if one of **G** or **H** is false. The left branch need not be considered further; the right branch is constructed in the same manner as before. Again only one of the formulas required by part (2) of 8.1.2; namely the one introducing a new first order constant c; the one introducing a new second order parameter, say Q, is ignored because again $\langle Q,C[P]\rangle{:}P$ could not be atomic nor a member of $\Omega[\Phi, \mathbb{D}]$ for any interpretation.

Consider now the possibilities for C[P]. A proof of Cantor's lemma using Cantor's general diagonal argument defines C[P] to be $\{\langle n,b\rangle \mid [\forall x{:}Sq](\langle n,x\rangle{:}P \supset [\forall v{:}Bit](\langle n,v\rangle{:}x \supset \sim v{=}b))\}$. But when C[P] is so defined $\langle c,C[P]\rangle{:}P$ is not atomic nor a member of $\Omega[\Phi, \mathbb{D}]$ for any interpretation. For C[P] is not first order, since P occurs in it, and therefore $\langle c,C[P]\rangle$ is not first order.

Is it possible that C[P] could be replaced by some first order term, say t, and have both branches of the tree close ? If that were the case, then a derivation of the sequent $\rightarrow t{:}Sq \wedge [\forall z{:}Mp][\forall n{:}N1] \sim\langle n,t\rangle{:}z)$ could be provided. But that is clearly not the case since a derivation of $\rightarrow$ FB:Map is provided in § 6.3.1 of [Gilmore89] for a particular term FB. FB enumerates the sequences 1, 0, 0, 0, ... ; 1, 1, 0, 0, ... ; 1, 1, 1, 0, ...; ... .But FB can be modified to FB' that enumerates t first before it enumerates the given sequences. It will then be possible to derive → FB':Map and → <1,t >:FB', but also → ~<1,t >:FB'.

Therefore the sequent

→ [∀z:Map][∃x:Sq][∀n:N1] ~<n,x>:z

is not derivable in NaDSet; i.e., Cantor's lemma in its usual interpretation is not provable in NaDSet. However, as noted in [Gilmore89], an intuitionistic or constructive interpretation is derivable: The Turing computable real numbers cannot be Turing enumerated. The wide acceptance of the non-constructive interpretation of Cantor's lemma may be due to the incontrovertible nature of the constructive lemma.

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