Fault Tolerant Planar Communication Networks

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Abstract Planar communication networks facilitate simultaneous communication in VLSI chips. Planar rearrangeable network and planar superconcentrator are two principle planar communication networks of important theoretical values. We construct planar rearrangeable networks and planar superconcentrators with concerns not only of the size (the number of single-pole single throw switches in the network), but also of the degree of fault tolerance in the presence of random switch failures. In this paper, we study a random switch failure model and present optimal size constructions of fault tolerant planar rearrangeable networks and planar superconcentrators. In fact, we construct schemes in which fault tolerant planar rearrangeable n-networks (networks failing to be planar rearrangeable n-networks with probability approaching 0 as $n \to \infty$) contain $O(n^3)$ switches, and fault tolerant planar n-superconcentrators contain $O(n^2)$ switches. It turns out that both constructions are optimal in the sense that the sizes of the resulting networks are within a constant factor of their minimum values.

1. Introduction

A. Planar Communication Networks

The rapid advances of VLSI technology have made massive parallel computer systems practical. Consequently, the research on planar communication networks on VLSI chips becomes attractive. Consider in a VLSI chip, there are n resource (eg. processors and memory devices) called "transmitters" and n other resource called "receivers", planar communication networks (consisting of electrical links and switching elements) provide simultaneous communication between various combinations of transmitters and receivers. (Condition "simultaneous" is crucial for parallel computation.) Our interest in this paper is in fault tolerant planar networks that accomplish the simultaneous communication by means of disjoint paths of links and switches from transmitters to receivers. Among various schemes of such networks, *planar rearrangeable networks* and *planar superconcentrators* are the two of important theoretical values. To formulate the property more precisely, we describe a network in terms of a graph. Transmitters are represented by n distinguished vertices called *inputs*, and receivers by n other distinguished vertices called *outputs*. Electrical links are represented by vertices other than inputs and outputs, and switches (single-pole single throw, connecting two links) by edges between the two corresponding vertices. Such a network is said to be a "*rearrangeable n-network*" if, given any one-to-one correspondence between the inputs and the outputs, there exists a set of n vertex-disjoint paths joining each input to its corresponding output; it is said to be a "*n-superconcentrator*" if, for every $r \leq n$, every set of r inputs, and every set of r outputs, there exists a set of r vertex-disjoint paths from the given inputs to the given outputs. A "*planar rearrangeable n-network*" is a rearrangeable n-network that can be embedded into a plane. Similar property holds for a "*planar n-superconcentrator*". It is obvious that a rearrangeable network is able to transfer data simultaneously from transmitters to receivers in accordance with any permutation. A superconcentrator, for example, can provide an ideal support for the Task Queue scheme [Co] in parallel computing. Imagine that transmitters are n processors and receivers are n memory devices. The task queue of some problem is distributed on k memories and k processors is working in parallel to solve the problem, the superconcentrator will ensure each processor the access to a task. As tasks are independent, it does not matter which accesses which.

As the number of elements integrated in a VLSI chip increases, the probability that some switches in the the planar communication network fail becomes greater too. Fault tolerance of communication networks is therefore important in order to maintain the reliability of the computer system. The measure of fault tolerance we are interested in in this paper is the probability of the network fulfilling the communication task in the presence of random switch failures. In fact, we explore a random switch failure model and construct planar rearrangeable networks and planar superconcentrators with high degree of fault tolerance. Because of the nature of switch failures, the model studied in this paper is different from the random link failure model or the random faulty processor model in [GG], [LL], [KKLMRRTT] and [Ra]. In fact, their models can be treated as special cases of the model in this paper, in the sense that they only consider one of the two type of failures studied in this paper. This model difference, however, does not prevent us from integrating certain techniques of their researches, especially that of [KKLMRRTT] and [Ra], into our proof of some results, in particular of Lamma 1 in Section 2.

B. The Random Switch Failure Model

The conventional measure of complexity applied to planar communication networks is "size", the number of single-pole single-throw switches (edges) in it, as it affects the number of processors and memory cells integrated onto a VLSI chip. An extensive literature exists concerning the design of planar communication networks, minimizing their sizes as functions of the number of inputs and outputs (see [CS], [AKLLW85], [AKLLW91] and [KL]). To planar rearrangeable networks, the most advanced results are the $O(n^3)$ upper bound due to Cutler and Shiloach [CS] and the $\Omega(n^3)$ lower bound by Klawe and Leighton [KL]; to planar superconcentrators, an $O(n^2)$ upper bound (an $n \times n$ grid) is obvious and Lipton and Tarjan [LT] proved the $\Omega(n^2)$ lower bound.

Our concern in this paper is not only the size of the networks, but also their fault tolerance when edges (electrical switches) are subject to probabilistic failures. In fact, we consider two type of failures. The first is that the two vertices of an edge contract to one (the two vertices become identical), called *closing failure*. The second is that the two vertices of an edge are permanently "separated"(the edge ceases to exist) , called *open failure*. The interpretation of these failures is evident. Closing failures correspond to electrical switches (edges) being permanently "*on*" (failing to be "*off*"); open failures correspond to electrical switches being permanently "*off*" (failing to be "*on*"). Our goal of this paper is to design planar rearrangeable networks and planar superconcentrators that use small number of edges (switches) while have high degree of fault tolerance.

We shall assume each edge randomly and independently subject to closing failure and open failure with probability $0 < \epsilon_1 < 1/2$ and $0 < \epsilon_2 < 1/2$ respectively. For the simplicity of notations, we assume that $\epsilon_1 = \epsilon_2 = \epsilon$. Given $0 < \epsilon < 1/2$, consider a planar network

N subject to the above failure model. Let the event space Ω be the set of all graphs obtained from N. The probability measure on each graph is assigned in accordance with the number of failed edges. Given $0 < \delta < 1$, we say N is a *planar (ϵ, δ)-rearrangeable n-network* if the probability measure on all planar rearrangeable n-networks in Ω is greater than $1-\delta$; N is a *planar (ϵ, δ)-n-superconcentrator* if the probability measure on all planar n-superconcentrators in Ω is greater than $1 - \delta$. It is clear that by choosing arbitrarily small δ, a planar (ϵ, δ)-rearrangeable n-network or a (ϵ, δ)*-n-superconcentrator* can fulfill its communication task with arbitrarily high probability.

Results of Moore and Shannon [MS] are useful to build planar (ϵ, δ)-rearrangeable n-networks and planar (ϵ, δ)-n-superconcentrators with small number of edges. They presented a solution to the construction of general fault tolerant networks. They showed that given network Ψ and $0 < \epsilon < 1/2$ and $0 < \delta < 1$, there exists a network of size $cT(logT)^2$, doing the same task as Ψ does with probability at least $1 - \delta$, where T is the size of Ψ and c is a constant independent of Ψ. Consequences of Moore and Shannon's result are that planar (ϵ, δ)-rearrangeable n-networks are of size $O(n^3(logn)^2)$ and planar (ϵ, δ)-n-superconcentrators are of size $O(n^2(logn)^2)$. However, noticeable gaps exist between these upper bounds and the known lower bounds.

Proposition 1 *Given $0 < \epsilon < 1/2$ and $0 < \delta < 1$, any (ϵ, δ)-rearrangeable n-network must have $\Omega(n^3)$ edges, and any (ϵ, δ)-n-superconcentrator must have $\Omega(n^2)$ edges.*

Proof. It is observed that if a planar graph N is not a planar rearrangeable network, then none of its random instances are planar rearrangeable networks either. This, combined with the $\Omega(n^3)$ lower bound of Klawe and Leighton [KT], implies that any planar (ϵ, δ)-rearrangeable n-network must have $\Omega(n^3)$ edges. Similar observation on planar superconcentrators and the $\Omega(n^2)$ lower bound of Lipton and Tarjan (an immediate consequence of Theorem 5 in [LT]) indicate that any planar (ϵ, δ)-n-superconcentrator must have $\Omega(n^2)$ edges. $\triangle$

We present in this paper the first optimal size planar (ϵ, δ)-rearrangeable n-networks and planar (ϵ, δ)-n-superconcentrators, namely planar (ϵ, δ)-rearrangeable n-networks with $O(n^3)$ edges and planar (ϵ, δ)-n-superconcentrators with $O(n^2)$ edges. The construction of the optimal $O(n^3)$ size planar (ϵ, δ)-rearrangeable n-networks is presented in Section 2, and that of optimal $O(n^2)$ size planar (ϵ, δ)-n-superconcentrators in Section 3. The result on planar rearrangeable networks also applies to the optimal construction of fault tolerant planar permutation networks (see [KL] for the definition of permutation network, and [CS], [AKLLW85] and [AKLLW91] for its applications).

2. Planar Rearrangeable Networks

In this section, we shall construct and analyze an $O(n^3)$ size planar (ϵ, δ)-rearrangeable n-network. We first review the result due to Cutler and Shiloach [CS] on planar rearrangeable networks of reliable edges.

In a $(2n^2+n) \times (2n+1)$ grid, n inputs and n outputs are placed on the middle vertical line at intervals of length n beginning at the top line, with inputs being placed first. Cutler and Shiloach [CS] showed that such a grid is an planar rearrangeable n-network. It obviously has size $O(n^3)$.

Unfortunately this approach is not resilient to edge failures, since the inputs and outputs are of bounded degrees. With a probability of constant value (say, ϵ^4), the four edges adjacent to an input may suffer open failures simultaneously and the input is isolated from the rest of the network. However, a modification of the grid will give a highly fault tolerant planar rearrangeable network with optimal size.

We first construct an $18n^2 \times (12n+1)$ grid. Along the middle vertical line, we shrink the first $3n$ vertices of every $9n$ vertices to one vertex. Let the first n of the $2n$ "shrunk" vertices be inputs and the rest be outputs. A vertex is adjacent to an input (resp. output) if it was a neighbor of one of the $3n$ vertices from which the input (resp. output) is obtained. We call this network N.

To discover the fault tolerance of network N, we shall need the following lemmas.

Lemma 1 (Pippenger) *In a $6r \times t$ grid of which edges are subject to the above failure model, the probability that there are less than $r+1$ vertex-disjoint paths from the left hand boundary to the right hand boundary is at most $d_1 t(7^6 \cdot 56\epsilon)^r$ provided $7^6 \cdot 56\epsilon < 1$, where $d_1 = 1/(1-(7^6 \cdot 56\epsilon)^{1/6})$.*

Proof. Consider any minimal set of edges S which separates the left side boundary from the right side. Condition "minimal" means that any proper subset of S does not separate the left side boundary from the right side. Suppose $|S| = l$. For each edge e in S, call the subnetwork comprising itself and the six edges adjacent to it $C(e)$. We say $C(e)$ is *dead* if one of its edges suffers an open failure or a closing failure, otherwise $C(e)$ is *alive*. Thus the probability of $C(e)$ being dead is at most 14ϵ. It is observed that if the distance of two edges e_1 and e_2 in S is 3, $C(e_1)$ and $C(e_2)$ are disjoint. Thus there are $l/3$ $C(e)$'s in $\{C(e) | e \in S\}$ which are disjoint to each other. Let $C(S)$ be the subset of $\{C(e) | e \in S\}$, which contains $l/3$ such $C(e)$'s. If there are less than $r+1$ vertex-disjoint paths from the left hand side boundary to the right hand side boundary, by Menger's Theorem (see

Chapter 5 of [CL]), there is a set of edges S which separates the left side boundary from the right side, such that there are fewer than $r+1$ $C(e)$'s in $C(S)$ alive. It is clear that events "$C(e_i)$ being alive" (for all $C(e_i) \in C(S)$) are independent, since $C(e_i)$'s are disjoint. Thus the probability that there are fewer than $r+1$ $C(e)$'s in $C(S)$ being alive is at most $\sum_{w<r+1} \binom{l/3}{w} (14\epsilon)^{l/3-w} \le 2^{l/3}(14\epsilon)^{l/3-r}$. For any given l, the number of such S is at most $t \cdot 7^l$, since there are t places to start S and at most 7 ways to continue at each step. On the other hand, $l \ge 6r$, since any edge set separating the left boundary from the right boundary contains at least $6r$ edges. Therefore, the probability that there are less than $r+1$ vertex-disjoint paths from the left hand boundary to the right hand boundary is at most $\sum_{l \ge 6r} t7^l 2^{l/3}(14\epsilon)^{l/3-r}$. Thus Lemma 1 follows. $\triangle$

Similarly,

Lemma 2 *In a $s \times t$ grid, shrink the s vertices at the left hand side boundary to one vertex and named this vertex "new". The probability that there is no path from "new" to the opposite side boundary is at most $d_2 t(7^3 \cdot 14\epsilon)^{s/3}$ provided $7^3 \cdot 14\epsilon < 1$, where $d_2 = 1/(1-(7^3 \cdot 14\epsilon)^{1/3})$.*

To show the fault tolerance of N, imagine horizontally partitioning N into $4n$ layers. Each of the $2n$ odd layers contains vertices of the first $3n$ rows of every $9n$ rows; each of the $2n$ even layers contains vertices of the rest $6n$ rows of every $9n$ rows. It is clear that the inputs and outputs are in the $2n$ odd layers, with each in one layer. It is observed that each odd layer is a $3n \times (12n+1)$ grid, except the $3n$ vertices in the middle vertical line shrink to an input or an output. By Union Law and Lemma 2, the probability that there exists an odd layer in which there is no path from left side boundary to right side boundary (thus through the input or the output in this layer) is at most $24d_2n^2(7^3 \cdot 14\epsilon)^n$. Similarly, each even layer is a $6n \times (12n+1)$ grid. By Union Law and Lemma 1, the probability that there exists an even layer in which there are less than $n+1$ vertex-disjoint paths from the left side boundary to the right side boundary is at most $24d_1n^2(7^6 \cdot 56\epsilon)^n$.

Imagine vertically partitioning N into three sections. The left section consists of vertices in the left hand side of the inputs and outputs. The middle section consists of $2n$ inputs and outputs. The rest comprises the right section. It is clear that the left section and the right section are $6n \times 18n^2$ grids. By Union Law and Lemma 1, the probability that there are less than $n+1$ vertex-disjoint paths running from the top boundary to the bottom boundary in the left section or there are less than $n+1$ paths running from the top boundary to the bottom boundary in the right section is at most $36d_1n^2(7^6 \cdot 56\epsilon)^n$.

We call these vertex-disjoint paths in each (horizontal) layer and (vertical) section *tracks*. Let the subnetwork induced by these tracks be $frame(N)$. We observe that in any two consecutive layers, at most two horizontal tracks have a vertex in common (with one track in each layer). Similarly, of the left section and the right section, at most two vertical tracks have a vertex in common. Therefore, with probability at least

$$1 - 24d_2n^2(7^3 \cdot 14\epsilon)^n - 24d_1n^2(7^6 \cdot 56\epsilon)^n - 36d_1n^2(7^6 \cdot 56\epsilon)^n > 1 - 84d_1n^2(7^6 \cdot 56\epsilon)^n,$$

(notice $d_1 > d_2$ when $7^6 \cdot 56\epsilon < 1$,) the random instance of N contains a $frame(N)$ which has at least $2n^2 + 2n$ horizontal vertex-disjoint tracks, with inputs and outputs on $2n$ horizontal tracks and at least n horizontal tracks between two consecutive input or output vertices, and at least $2n$ vertical vertex-disjoint tracks with at least n tracks on each side of the inputs and outputs. Thus by virtuely the same argument as that in [CS], this $frame(N)$ is shown to be a planar rearrangeable n-network, i.e. the random instance of N is a planar rearrangeable n-network.

In order to see the fault tolerance of N, we observe that $1 - 84d_1n^2(7^6 \cdot 56\epsilon)^n$ is approaching 1 as $n \to \infty$, when $\epsilon < 1/7^6 \cdot 56$. Thus it will be greater than any $1 - \delta$. In order to extend the result to any $0 < \epsilon < 1/2$, we shall need the following lemma due to Moore and Shannon [MS].

We say a *two-terminal network* is a network with two distinguished vertices called *input* and *output*. Given a two-terminal network Φ in which edges are independently subject to random closing and open failures. We say Φ suffers a *closing failure* if the input and output of Φ contract into one vertex; Φ suffers an *open failure* if the input and the output are not in the same connected component.

Lemma 3 (Theorem 6 in [MS]) *Given $0 < \epsilon < 1/2$ and $0 < \epsilon' < \epsilon$. There exists a two-terminal planar network Φ of which the probabilities of closing failure and open failure are both less than ϵ', and Φ contains $c_\epsilon(log\epsilon')^2$ edges, where ϵ is the probability of closing failure and open failure of each edge in Φ, and c_ϵ is a constant depending on ϵ.*

Now we are ready to present the optimal size planar (ϵ, δ)-rearrangeable n-network.

Theorem 1 *For any $0 < \epsilon < 1/2$ and $0 < \delta < 1$, there is an explicit construction of a planar (ϵ, δ)-rearrangeable n-network of size $O(n^3)$.*

Proof. Given any $0 < \epsilon < 1/2$ and $0 < \delta < 1$. Consider in network N, we substitute each edge by a two-terminal planar network Φ of Lemma 3, in which we choose ϵ' such that $7^6 \cdot 56\epsilon' < 1$ and $84d_1n^2(7^6 \cdot 56\epsilon')^n < \delta$, for all $n \geq 1$. It is clear that the resulting network is planar and of size $O(n^3)$. Moreover, it is an (ϵ, δ)-rearrangeable n-network since $1 - 84d_1n^2(7^6 \cdot 56\epsilon')^n > 1 - \delta$. $\triangle$

3. Planar Superconcentrators

Without loss of generality, we assume in this section that $n^{1/2}$ is an integer. The construction of planar (ϵ, δ)-n-superconcentrators is described in terms of stage. The n inputs consist of the first stage. The second stage comprises $2n-1$ disjoint sets of vertices called *group*(j), $1 \leq j \leq 2n-1$. *group*$(2i-1)$ consists of $6n^{1/2}+5$ vertices, $1 \leq i \leq n$, and *group*$(2i)$ of $6n^{1/2}-1$ vertices, $1 \leq i \leq n-1$. Each input i in the first stage, $1 \leq i \leq n$, is adjoining to $6n^{1/2}+5$ edges which lead to the $6n^{1/2}+5$ vertices of *group*$(2i-1)$. We use $2n(6n^{1/2}+2)-6n^{1/2}$ edges to join the $2n(6n^{1/2}+2)-6n^{1/2}+1$ vertices in the second stage on a path (a thread of edges) with vertices of *group*(j) appearing before that of *group*$(j+1)$, $1 \leq j \leq 2n-2$. For any two adjacent vertices in the second stage, if they are in the same *group*(j), they have edges leading to a common vertex in the third stage. Thus the third stage contains $2n(6n^{1/2}+2)-(6n^{1/2}-1)-(2n-1) = 2n(6n^{1/2}+1)-6n^{1/2}+2$ vertices, with each vertex being adjacent to two vertices in the previous stage. Similarly, vertices in the third stage are joined by a path of $2n(6n^{1/2}+1)-6n^{1/2}+1$ edges running from the top to the bottom. In similar ways, we construct the fourth stage, and so on. The number of vertices in each stage decreases by $2n-1$ as one more stage is constructed. Stop the construction after $(6n^{1/2}+1)$-st stage (the $(6n^{1/2}+1)$-st stage contains $6n$ vertices). Now, taking these $6n$ vertices as the left hand boundary, construct a $6n \times 6n$ grid. With the bottom boundary of the grid, associate a $6n^{1/2}+1$ stage subnetwork described above, but replace the n inputs by n outputs. We call this network M. The network is clearly planar and has $O(n^2)$ edges.

In order to show the fault tolerance of this network, we shall need a result on planar superconcentrators with reliable edges. And we shall prove a general result on disjoint paths of planar graphs (Proposition 2).

Consider in an $n \times n$ grid, n inputs are adjoining to the n vertices on the left hand boundary of the grid in any one-to-one correspondence and n outputs are adjoining to the n vertices on the bottom boundary in any one-to-one correspondence. It is obviously a planar n-superconcentrator and has $O(n^2)$ edges.

Proposition 2 *Given a planar graph G, let $v_0, v_1, \cdots, v_n$ be a set of vertices occurring anti-clockwise around the boundary of the exterior face of G. There are n paths $v_i \rightarrow$*

v_0 in G (for $i = 1, \cdots, n$), such that they are vertex-disjoint except at v_0 iff for every interval $\{v_i, v_{i+1}, \cdots, v_{i+r-1}\}$ ($1 \leq i \leq n$ and $1 \leq r \leq n-i+1$), there are r paths from $\{v_i, v_{i+1}, \cdots, v_{i+r-1}\}$ to v_0, vertex-disjoint except for their ends.

Proof. Necessity of the condition is obvious. Before we prove the sufficiency, we need some definitions. Given a set of paths $\{L_1, \cdots, L_p\}$ in G, vertex-disjoint except at their ends, we define a full anti-symmetric order relation *lies above* among it. We say L_i lies above L_{i+1}, for $i = 1, \cdots, p-1$, if $Upperboundary, L_1, \cdots, L_p$ form an anti-clockwise order, where $Upperboundary$ is the segment between v_1 and v_0 of the boundary of the exterior face that contains no $v_i, i = 2, \cdots, n$. Given two simple paths A and B in G, let $cap(A, B)$ be a new path by starting from A's initial vertex and, between each two consecutive intersecting vertices of A and B, choosing the segment that lies above. Similarly we define $cup(A, B)$ except we choose the segment that lies below. To prove sufficiency, we do induction on n. The case when $n = 1$ is trivially true. We assume the proposition is true when $n = k$. We consider the case $n = k+1$. We have $k+1$ paths $R_i, i = 1, \cdots, k+1$, from $\{v_1, v_2, \cdots, v_{k+1}\}$ to v_0, vertex-disjoint except at their ends. Without loss of generality, we assume that R_i lies above R_{i+1}. By induction hypothesis, we have k paths P_i: $v_i \rightarrow v_0$ for $i = 1, \cdots, k$, vertex-disjoint except at v_0, and k paths Q_i: $v_i \rightarrow v_0$ for $i = 2, \cdots, k+1$, vertex-disjoint except at v_0. Without loss of generality, we assume all P_i's, Q_i's and R_i's are simple. Let $height(i)$ be the smallest j such that R_i and Q_j have a common vertex other than v_0. It is clear that $2 \leq height(1) \leq height(2) \leq \cdots \leq height(k+1) \leq k+1$. Thus $\exists i$, such that $height(j) \leq j$, for all $j \geq i$. Let i_0 be the smallest such i. Let $T_j = cup(Q_j, R_j)$, for $j = i_0, \cdots, k+1$ and, $T_j = P_j$ for $j = 1, \cdots, i_0 - 1$, if R_{i_0-1} and P_{i_0-1} are vertex-disjoint except at v_0; $T_j = cap(P_j, Q_{j+1})$, for $j = 1, \cdots, i_0 - 2$, and $T_{i_0-1} = cap(P_{i_0-1}, R_{i_0-1})$, if R_{i_0-1} and P_{i_0-1} have a common vertex other than v_0. It is observed that in either case, T_j is a simple path from v_j to v_0, for $j = 1, \cdots, k+1$, and $T_1, \cdots, T_{k+1}$ are vertex-disjoint except at v_0. This completes the induction. $\triangle$

An application of Proposition 2 on network M immediately suggests Corollary 1.

Corollary 1 *If given any r consecutive inputs i, $i+1$, $\cdots$, $i+r-1$ in the first stage, for any $1 \leq i \leq n$ and $1 \leq r \leq n-i+1$, there are r paths from a subset of the r inputs to the right side boundary of the grid, disjoint except initial vertices, then there are n vertex-disjoint paths from the n inputs to the right side boundary of the grid.*

Corollary 1 shows that in order to ensure the existence of n vertex-disjoint paths from n inputs to the right side boundary of the grid, besides the distinctness of n inputs (no two inputs contract to one vertex, due to the simultaneous closing failure of edges linking the two inputs), there are at most $n + (n-1) + \cdots + 1 = n(n+1)/2 \le n^2$ "conditions" need to be satisfied. By a "condition", we mean "there are r paths from a subset of r consecutive inputs to the right side boundary of the grid, disjoint except initial vertices".

Lemma 4 *Given r consecutive inputs i, $i+1$, $\cdots$, $i+r-1$, the probability that there do not exist r paths from a subset of the r inputs to the right side boundary of the grid, disjoint except initial vertices, is at most $24d_3n(9^6 \cdot 88\epsilon)^{n^{1/2}}$ provided $9^6 \cdot 88\epsilon < 1$, where $d_3 = 1/(1-(9^6 \cdot 88\epsilon)^{1/6})$.*

Proof. We observe two facts. Firstly, any set of edges which separates inputs $i, i+1, \cdots, i+r-1$ from the right side boundary of the grid must contain at least $l_s = min\{6n, 6r + 6n^{1/2}\}$ edges. Secondly, each edge in the network is adjacent to at most ten other edges. The same argument as that in Lemma 1 produces this probability value, noticing that $l_s \ge 6r$ and $l_s \ge 6n^{1/2}$. $\triangle$

Lemma 4 indicates that the probability of such a "condition" not being satisfied is at most $24d_3n(9^6 \cdot 88\epsilon)^{n^{1/2}}$, when $9^6 \cdot 88\epsilon < 1$.

Lemma 5 *With probability at most $16d_4n^3(6\epsilon)^{6n^{1/2}}$, $d_4 = 1/1 - 6\epsilon$, there exist input(s) or output(s) i_1 and i_2, $1 \le i_1 \ne i_2 \le n$, such that i_1 and i_2 contract to one vertex.*

Proof. Consider any such pair i_1 and i_2. Any simple path from i_1 to i_2 must contains at least $6n^{1/2}$ edges. And for any $l \ge 6n^{1/2}$, there are at most $(6n^{1/2}+5)^2 6^{l-2} < (n^{1/2}+1)^2 6^l < 4n6^l$ such paths of length l, as the degree of inputs and outputs is $6n^{1/2} + 5$, and that of the other vertices is at most 6. Thus the probability that there exist i_1 and i_2, contracting to one vertex due to the simultaneous closing failure of edges along a path from i_1 to i_2, is at most $4n^2 \sum_{l \ge 6n^{1/2}} 4n6^l\epsilon^l < 16d_4n^3(6\epsilon)^{6n^{1/2}}$. $\triangle$

It is clear now when $9^6 \cdot 88\epsilon < 1$, with probability at most $24d_3n^3(9^6 \cdot 88\epsilon)^{n^{1/2}} + 16d_4n^3(6\epsilon)^{6n^{1/2}} < 40d_3n^3(9^6 \cdot 88\epsilon)^{n^{1/2}}$, there do not exist n vertex-disjoint paths from n inputs to the right side boundary of the grid. Similar properties hold for the vertex-disjoint paths from n outputs to the top boundary of the grid.

Recall $frame(M)$ defined in the preceding section. With probability at least $1 - 40d_3n^3(9^6 \cdot 88\epsilon)^{n^{1/2}} - 40d_3n^3(9^6 \cdot 88\epsilon)^{n^{1/2}} = 1 - 80d_3n^3(9^6 \cdot 88\epsilon)^{n^{1/2}}$, the random instance of M contains a $frame(M)$ which has n horizontal vertex-disjoint paths from n inputs to the right side boundary of the grid and n vertical vertex-disjoint paths from n outputs to the top boundary of the grid. Thus by virtually the same argument as that applied to the $n \times n$ grid of reliable edges, this $frame(M)$ is shown to be a planar n-superconcentrator.

Theorem 2 *For any $0 < \epsilon < 1/2$ and $0 < \delta < 1$, there is an explicit construction of a planar (ϵ, δ)-n-superconcentrator of size $O(n^2)$.*

Proof. Same arguments as that in Theorem 1, except in which we replace network N by network M, and choose ϵ' such that $9^6 \cdot 88\epsilon' < 1$ and $80 d_3 n^3 (9^6 \cdot 88\epsilon')^{n^{1/2}} < \delta$, for all $n \geq 1$. $\triangle$

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