Starshaped Sets, Distance Functions, and Star Hulls

by

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Technical Report 91-4
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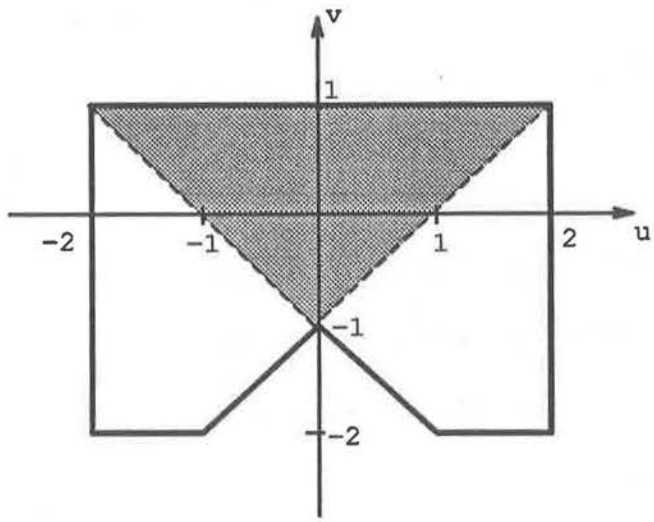

Figure 1: A starshaped object in R^2 with shaded area as its kernel.

1 Introduction

We begin with the definition and an example of starshaped sets.

Definition 1.1 Let $S \subseteq R^d$. For points $x \in S$, $y \in S$, we say x sees y via S (y is *visible* from x via S) if the the line segment $\overline{xy}$ lies in S. Let A and B be subsets of S. We say A sees B provided, for all $x \in A$ and $y \in B$, $\overline{xy} \subset S$.

Definition 1.2 A set $S \subseteq R^d$ is *starshaped* if there exist a point $x \in S$ such that x sees every point of S via S. We say that S is starshaped with respect to x.

Definition 1.3 The (convex) *kernel* of a set $S \subseteq R^d$ is the set of all points $x \in S$ such that every point of S is visible from x via S. Denoted ker S.

Figure 1 gives an example of a starshaped set in R^2 along with its kernel.

The motivation of our study of starshaped sets is that we want to be able to handle non-convex surfaces in solving computational vision problems. We perceive starshapedness as a good intermediate step from convexity to non-convexity.

Early studies of starshaped sets by mathematicians had been concentrated on the kernels of starshaped sets and conditions for a set to be starshaped. Also studied are unions of starshaped sets. Concepts that are more general than starshapedness are related with the study of visibilities, which has been a research topic not only in mathematics but also in computational geometry.

The goal of this paper is to promote studying and applying the concept of starshapedness in solving computational vision problems. It is known that the theory of convex bodies can be applied to solve computational vision problems [17, 20]. We regard starshapedness as an extended notion of convexity rather than a special version of visibility. Basically, we are interested in making the notion of orientation based representations [19] applicable to objects that are not convex. Thus, we are particularly interested in functions and tools intrinsically related to starshaped sets.

Section 2 surveys the results on necessary and sufficient conditions for a set to be starshaped. Section 3 surveys results about the characteristics, in particular, dimension and size, of the kernel of a starshaped set. Section 4 studies the distance functions of starshaped sets. Section 5 defines a notion star hull and compares starshapedness with other notions that are intended to extend the concept of convexity. Section 6 proposes the use of distance function in solving the problem of attitude determination in computational vision.

Throughout this paper, we present our surveys and results from mathematical point of view rather than that of computational geometry, although some results from computational geometry will be mentioned.

2 Conditions for Being Starshaped

By definitions, a set is starshaped if and only if its kernel is not empty. From computational point of view, checking whether a set is starshaped or not is not hard.

Theorem 2.1 (Preparata and Shamos [23], Theorem 7.13, page 298) The kernel of an N-vertex polygon can be computed in optimal $\theta(N)$ time.

Results surveyed in this section state necessary and sufficient conditions which we can check on every point of S or subset of S in order to determine whether S is starshaped or not. We start with the pioneer work of Krasnoselskii.

Theorem 2.2 (Krasnoselskii 1946, see [18] page 53) Let S be a compact subset of E^d that contains at least $d+1$ points. Suppose that for each $d+1$ points of S there is a point from which all $d+1$ are visible. Then S is starshaped.

□

The theorem says that if S is a nonempty compact set in R^d, then S is starshaped if and only if every $d+1$ points of S see a common point of S via S. This common point may depend on the choice of the $d+1$ points. A stronger result may be obtained by replacing, in the statement of the theorem, points of S with boundary points of S (mentioned without proof in [7]). The compactness requirement is very important. In fact the problem involving the existence of Krasnoselskii-type for sets that are neither closed nor bounded was open as of 1985 [11].

The combinatorial flavor of the theorem is very appealing, although the theorem itself does not constitute an effective procedure for determining if a set is starshaped. Mathematicians were inspired to ask the following question: Can the nature of ker S be determined from the behavior of appropriately selected subsets of S? A result by Nick Stavrakas [25] answers this question in terms of *points of local nonconvexity* which was first explored by F. A. Valentine [28].

Definition 2.1 (Valentine [28], page 39) Let $S \subseteq R^d$. A point x in S is called a *point of local convexity* of S if there is a neighborhood N_x of x such that $N_x \cap S$ is convex. Otherwise x is called a *point of local nonconvexity* (lnc point) of S.

Definition 2.2 Let $S \subseteq R^d$. For points $x \in S$, $y \in \text{cl } S$ (the closure of S), we say y is *clearly visible* from x via S if there is some neighborhood N_y of y such that x sees each point of $N_y \cap S$ via S.

Theorem 2.3 (Stavrakas 1972 [25]) Let S be a compact connected subset of R^d. Then $x \in \ker S$ if and only if all lnc points of S are clearly visible from x.

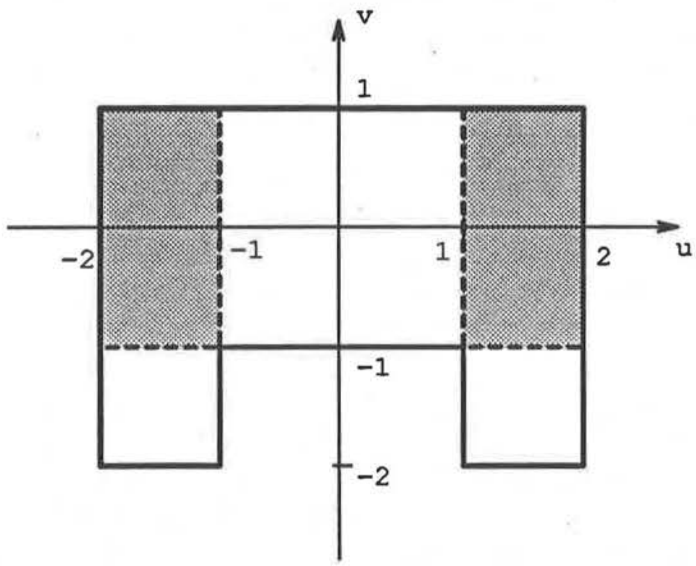

Figure 2: A set with two lnc points which is not starshaped.

This theorem makes it easier to check whether a set is starshaped or not. A set S is starshaped if and only if there is a point in S from which all lnc points of S are clearly visible. The next theorem by Breen says that in R^2, we only need to check every three lnc points of S to see whether a set is starshaped or not.

Theorem 2.4 (Breen 1982 [8]) Let S be a nonempty compact connected set in R^2. Then S is starshaped if and only if every 3 lnc points of S are clearly visible from a common point of S. The number 3 is the best possible.

□

This theorem makes it even easier to check if a set is starshaped. For example, the set depicted in Figure 1 has only one lnc point. Thus we immediately conclude that it is starshaped. The set depicted in Figure 2 has two lnc points, (1, -1) and (-1, -1), the regions of points from which they are clearly visible do not intersect. Thus the set is not starshaped.

Theorem 2.4 can be generalized to open bounded sets in R^2 (Theorem 2.5). If lnc points are replaced by boundary points, the theorem can also be generalized to bounded sets in R^2 (Theorem 2.6). But the problem of whether it has a d-dimensional analogue was unknown.

Theorem 2.5 (Breen 1982 [9]) Let $S \neq \emptyset$ be a bounded set in R^2, and assume that every 3 or fewer lnc points of S are clearly visible from a common point of S. Then for some point p in S, the set $A \equiv \{x : x \in S, \overline{px} \not\subseteq S\}$ is nowhere dense in S. Moreover, $A \subseteq \text{bd } S$ (i.e., if S is open, then S is starshaped).

□

Theorem 2.6 (Breen 1985 [10]) Let S be a nonempty bounded set in R^2. Then S is starshaped if and only if every 3 or fewer boundary points of S are clearly visible via S from a common point of S. The number 3 is best possible.

□

Another set of conditions was given by Goodey [13] in terms of *separating set* and *extreme points*.

Definition 2.3 A set is called a separating set if its complement is not connected.

Definition 2.4 Let $S \subset R^d$. The (d-2)-extreme points of S are those points x of S such that if $D \subset S$ is a (d-1)-dimensional simplex then $x \notin \text{rel int } D$ (relative interior). The set of all extreme points is denoted by $E(S)$.

Theorem 2.7 If $S \subset R^d$ is a nonseparating compact set and

$$\cap_{y \in E(S)} \{x : \overline{xy} \subset S\} \neq \emptyset \,,$$

then S is starshaped.

□

3 Kernels of Starshaped Sets

Most of the results concerning the dimension of the kernel of a starshaped set are aimed in answering the following question: What is the necessary and sufficient conditions for dim ker $S \geq k$, given $S \subseteq R^d$ and $1 \leq k \leq d$?

We will list a few results that answer these questions in terms of points of S, boundary points of S, and lnc points of S, respectively.

Theorem 3.1 (Stavrakas 1972 [25]) Let S be a compact connected subset of R^d. Then dim ker $S \geq k$, $0 \leq k \leq d$, if and only if there exists a flat F, dim $F = k$ and a point $x \in$ rel int $F \cap S$ (relative interior) such that given lnc point y there exists open sets N_y and N_x^y such that $N_x^y \cap S \cap F$ sees $N_y \cap S$.

□

Theorem 3.2 (Breen 1981 [6]) For each k and d, $1 \leq k \leq d$, let $f(d,d) = d+1$ and $f(d,k) = 2d$ if $1 \leq k \leq d-1$. Let S be a compact set in some linear topological space L. Then for a k with $1 \leq k \leq d$, dim ker $S \geq k$ if and only if for some $\epsilon > 0$ and some d-dimensional flat F in L, every $f(d,k)$ points of S see via S a common k-dimensional ϵ-neighborhood in F. If $k = 1$ or $k = d$, the result is best possible.

□

Theorem 3.3 (Breen 1982 [7]) For each k and d, let $f(d,d) = d+1$ and $f(d,k) = 2d$ if $1 \leq k \leq d-1$. Let S be a nonempty compact set in R^d. Then for a k with $1 \leq k \leq d$, dim ker $S \geq k$ if and only if every $f(d,k)$ boundary points of S are clearly visible from a common k-dimensional subset of S. If $k = 1$ or $k = d$, the result is best possible.

□

If S is in R^2, then the boundary of S may be replaced by the lnc points of S.

Theorem 3.4 (Breen 1982 [7]) Let S be a compact, connected, nonconvex set in R^2. Then for $k = 1$ or $k = 2$, dim ker $S \geq k$ if and only if every $g(k) = \max\ \{3, 6-2k\}$ lnc points of S are clearly visible from a common k-dimensional subset of S. The result is best possible.

Quantitative information concerning the size of the kernel is not easy to formulate. Some results give the necessary and sufficient conditions for the kernel of a starshaped set to contain an ε-interval, $\varepsilon > 0$.

Theorem 3.5 (Breen 1980 [4]) Let S be a nonempty compact set in R^2 having n lnc points. The kernel of S contains an interval of radius $\varepsilon > 0$ if and only if every $f(n) = \max\{4, 2n\}$ (or fewer) points of S see via S a common interval of radius ε. The number $f(n)$ is best possible for every $n \geq 1$.

□

Another question that had been asked concerns the characteristics of the convex sets which are admissible as the kernel of some nonconvex starshaped sets: if D is a convex subset of R^d, is there a starshaped set $S \neq D$ in R^d whose kernel is D? It had been well answered by the following theorem.

Theorem 3.6 (Breen 1981 [5]) Let D be a nonempty compact convex set in R^d, $d \geq 2$. Then there is a compact set $S \neq D$ in R^d with $\ker S = D$.

□

4 Distance Functions of Starshaped Sets

Support function plays an important role in the theory of convex bodies. Since support function is defined for any non-empty set, we can talk about the support function of a starshaped set. Since a set starshaped with respect to origin O has the same support function as its convex hull, support function loses its eligibility for being an intrinsic function related to starshaped sets.

Although distance function was defined only for convex sets by Minkowski (see Bonnesen-Fenchel [3]), its definition can actually be used on starshaped sets. Replacing "convex" by "starshaped", we have the following generalized definition.

Definition 4.1 Let S be a compact starshaped set in R^d with the origin O in the interior of its kernel. For any $x \in R^d \setminus \{O\}$, let ξ_x be the (unique) intersection point of the ray $\overrightarrow{Ox}$ with the boundary of S. The distance function $F(x)$, $x \in R^d$, of S is defined as

1. $g(S; O) = 0$, and
2. $g(S; x) = \|x\| / \|\xi_x\|$, $x \in R^d \setminus \{O\}$.

Some basic properties of distance functions of convex bodies still hold for starshaped compact sets.

1. The points that satisfy the inequality $g(S; x) \leq 1$ are precisely the points of S.
2. If two compact sets starshaped with respect to O have the same distance function, the two must be the same.
3. If the distance function of S is $g(S; x)$, that of λS is $\frac{1}{\lambda} g(S; x)$.
4. For compact sets S_1 and S_2 starshaped with respect to O, $g(S_1; x) \geq g(S_2; x), \forall x \in R^d$ if and only $S_1 \subseteq S_2$.

Properties 1 and 2 mean that a starshaped set is uniquely determined by its distance function.

It is known that the distance function of a convex set is convex. In fact, it is proved that a starshaped set is convex if and only if its distance function is convex. Before we state the theorem, we first give a more general definition of distance function used by Valentine [29], Lay [18], and Beer [2], which does not require compactness.

Definition 4.2 (Valentine [29], page 32) Let S be a set in a linear space $\mathcal{L}$, starshaped with respect to the origin O. The generalized distance function of S is the function $g : \mathcal{L} \rightarrow [0, \infty]$ defined by

$$g(S; x) = \inf \{\lambda : \lambda > 0 \text{ and } x \in \lambda S\} . \tag{1}$$

It should be noted that the origin O should be in the interior of S, otherwise the domain of $g(S; x)$ may not be the whole space. Figure 3 gives an example of distance function whose domain does not contain the sector spanned by the two sides of the polygon emitting from the origin.

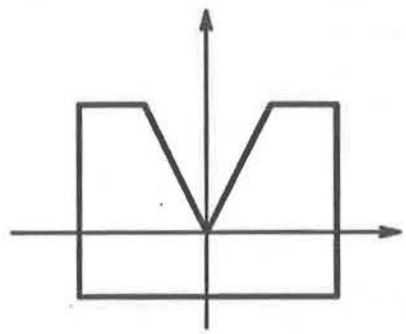

Figure 3: A starshaped set whose distance function is not defined on the whole space.

Theorem 4.1 (Valentine [29], page 32) Suppose $S \subset \mathcal{L}$ is starshaped with respect to O and each line through O intersects S in a relatively closed set. Then S is convex if and only if the distance function g of S is subadditive and positively homogeneous; that is

1. $g(S; v + w) \leq g(S; v) + g(S; w)$ for all $v, w \in \mathcal{L}$,
2. $g(S; \lambda v) = \lambda g(S; v)$ for all $\lambda \geq 0,\ v \in \mathcal{L}$.

□

Figure 4 shows the distance function g of the starshaped set in Figure 1, where the analytical expression in each sector bounded by the dashed lines is the value of g taking on the variables in the corresponding region. We see that $g((-2,-3)) = 1.5$, $g((1,-2)) = 1$, $g((-2,-3)) + g((1,-2)) = 2.5 < g((-2,-3)+(1,-2)) = g((-1,-5)) = 4$. Thus g is not convex.

Gerald Beer [2] established a selection theorem for starshaped sets by considering the distance function of starshaped sets. He called distance function gauge, and he believed that distance function was intrinsically related to starshaped set.

It is stated in his paper that if g is the gauge of a nontrivial closed set starshaped with respect to the origin, then g is a nonnegative extended valued positively homogeneous lower-semicontinuous function, and there exists

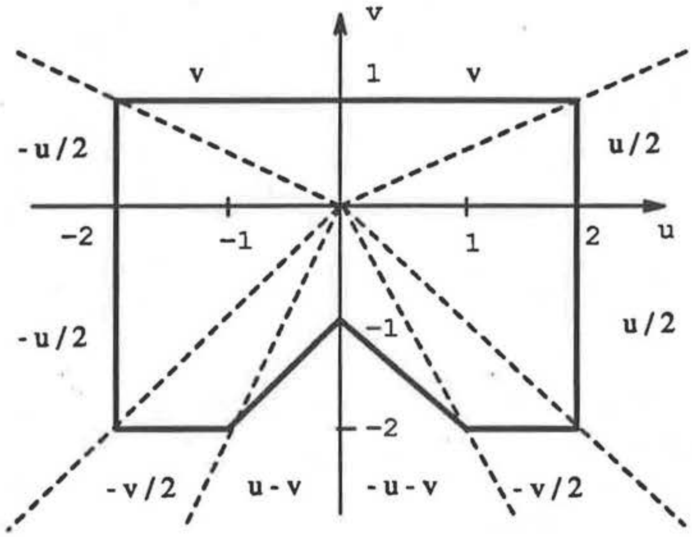

Figure 4: The distance function of the starshaped object in Figure 1.

$x_0 \neq O$ satisfying $g(x_0) \neq \infty$. Conversely, any function f with these properties is the gauge of such a set, namely, $S = \{x : f(x) \leq 1\}$. If $O \in$ int ker S, then the gauge is continuous. Moreover, it is Lipschitz.

Beer studied the gauge of parallel bodies of starshaped sets. We only cite one of his theorems here.

Theorem 4.2 (Beer 1955 [2]) Let $\{S_k\}$ be a sequence of compact starshaped sets each contained in $\{x \in R^d : \|x\| \leq M\}$. Then $\{S_k\}$ has a subsequence convergent in the Hausdorff metric to a compact starshaped set.

□

In the study of classes of starshaped sets in E^3, Melzak asserted that a class of starshaped sets is identifiable with the class of all real valued positive functions on the sphere S^2 which satisfy a Lipschitz condition. Define $\mathcal{H}$ as follows: $S \in \mathcal{H}$ if and only if S is a bounded closed set in E^3 and $O \in$ int ker S. Let f_S be the distance function of S. Then we have the following theorems, where Theorem 4.3 was mentioned similarly in Beer [2].

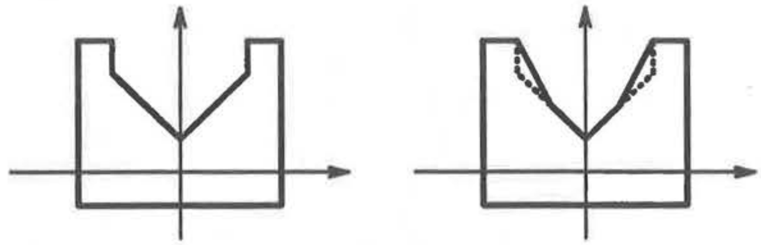

Figure 5: Two polygons with the same distance function.

Theorem 4.3 (Melzak 1959 [22]) If $S \in \mathcal{H}$ then $f_S(u_1) > 0$ and $|f_S(u_1) - f_S(u_2)| < \gamma_S |u_1 u_2|, 0 < \gamma_S < \infty$, where $u_1, u_2 \in S^2$ and $|u_1 u_2|$ is the length of the line segment between u_1 and u_2. Conversely, any such function f defines a set in $\mathcal{H}$.

□

Theorem 4.4 (Melzak 1959 [22]) Given any convex set $K \in \mathcal{H}$, any $\lambda > 0$ such that $K \subset \lambda B^2$, and any $\varepsilon > 0$, there exists $S \in \mathcal{H}$ such that a) $\ker S = K$, b) $K \subset \operatorname{int} S$, c) $\lambda B^2 \subset S \subset (\lambda + \varepsilon) B^2$.

□

5 Star Hulls and Generalized Convexity

We have seen that a starshaped set is uniquely determined by its distance function and is convex if and only if its distance function is convex. We naturally would ask the question: Can distance function be defined for arbitrary set? If yes, what properties does it have? If we look at Definition 4.2, we notice that it is not necessary to require convexity or starshapedness for (1) to be meaningful. Thus we say that Definition 4.2 can be applied on arbitrary sets. The distance function, however, is no longer unique when non-starshaped sets are involved. Figure 5 shows two different sets that have the same distance function if we apply Definition 4.2 directly on them.

It is known that a nonconvex set and its convex hull have the same support function, we want to ask the following question: Can we define a notion of

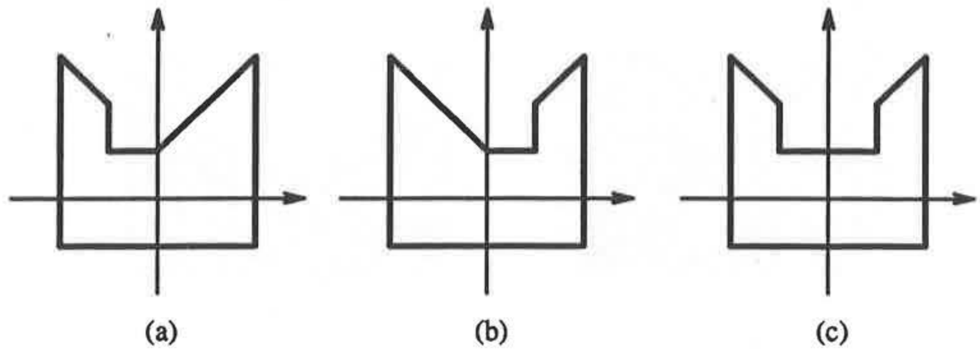

Figure 6: Two starshaped polygons (a) and (b) whose intersection (c) is not starshaped.

star hull so that a set and its star hull have the same distance function? Trying to answer this question, we first have the following observations.

1. Intersection does not preserve starshapedness. Figure 6(a) and Figure 6(b) shows two starshaped sets whose intersection, Figure 6(c), is not starshaped.

2. The smallest, in terms of containment, starshaped set that contains a given set does not generally exist. To see this, using Figure 6 again, let A be the set in Figure 6(c), S_1 and S_2 be the sets in Figure 6(a) and Figure 6(b) respectively. Both S_1 and S_2 are starshaped and both contain A, but $S_1 \cap S_2 = A$. Thus there is no smallest starshaped set that contains A.

Considering the nature of distance function, we give the following definition of star hull.

Definition 5.1 Let S be a set with origin O in its interior. Define the star hull $\mathrm{SH}(S)$ of S as

$$\mathrm{SH}(S) \triangleq \bigcup_{x \in S} \{ \overline{Ox} \} .$$

With star hull such defined, a set and its star hull have the same distance function. But this definition is not very satisfactory because of the following two reasons.

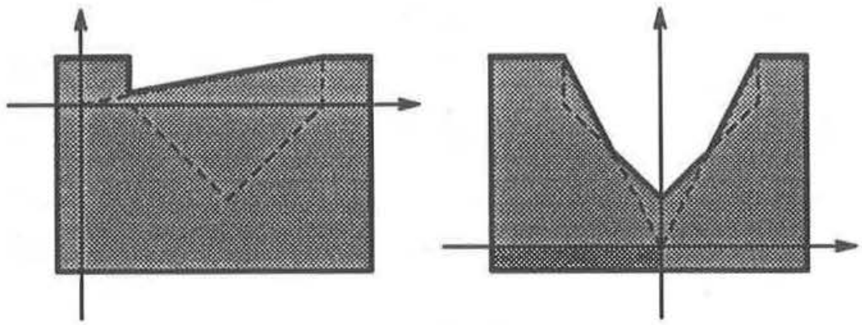

Figure 7: Star hull as defined in Definition 5.1 depends on the choice of the origin.

1. The star hull such defined depends on the choice the origin O (see Figure 7). This could lead to non-interesting star hulls like the one in Figure 7(a).

2. The distance function of a star hull may not be continuous. Because the star hull is defined as the union of all the segments $\overline{Ox}$, $x \in S$, a ray emitting from the origin may intersect the boundary of the star hull in a segment instead of a single point.

To overcome the first disadvantage, it may be desirable for the origin to be so chosen that it clearly sees the biggest number of lnc points of the set. For example, the origin in Figure 7(a) clearly sees only one lnc point, while the origin in Figure 7(b) clearly sees two lnc points which are all the lnc points the set has.

To overcome the second disadvantage, one may define a notion of λ star hull.

Definition 5.2 Let S be a set with origin O in its interior. Define the λ star hull $\mathrm{SH}_\lambda(S)$ of S as

$$\mathrm{SH}_\lambda(S) \triangleq \bigcup_{p \in B(O,\lambda)} \bigcup_{x \in S} \{ \overline{px} \} ,$$

where $B(O, \lambda)$ is the ball of radius λ centered at O.

We perceive starshapedness as one type of generalized convexity. There is a wide range of notions of generalized convexity (see [12] Section 9 for a list). In the following, we briefly introduce a few notions of generalized convexity that are of interest in the context of starshapedness.

Definition 5.3 (Horn and Valentine [15]) A set S in R^d is called an L_n set if each pair of points in S can be joined by a polygonal line in S having at most n segments.

Obviously, a starshaped set is an L_2 set. But an L_2 set is not necessarily a starshaped set. One such example is $B^d \setminus \{O\}$.

Theorem 5.1 (Valentine [27]) Suppose S is a closed connected set in R^d which has at most n points of local nonconvexity. The S is an L_{n+1} set.

Definition 5.4 (Valentine [26]) A set S in R^d is said to possess the three-point convexity property P_3 if for each triple of points x, y, z in S at least one of the closed segments $\overline{xy}$, $\overline{yz}$, $\overline{xz}$ is in S.

A starshaped set does not necessarily have property P_3. Examples of such starshaped sets are sets in Figure 6(a) and Figure 6(b). But a P_3 set is starshaped as stated in the following theorem.

Theorem 5.2 (Valentine [26]) Let S be a closed connected set in R^d which has property P_3. Then either S is convex or S is starshaped with respect to each of its points of local nonconvexity.

Another notion of generalized convexity is called *restricted-oriented convexity.*

Definition 5.5 (Rawlins and Wood [24]) Let $\mathcal{O}$ be a set of orientations. A collection of lines is said to be $\mathcal{O}$-oriented if the set of orientations of the lines is a subset of $\mathcal{O}$. Thus we speak of $\mathcal{O}$-lines. Let P be a subset of R^2. We say that P is $\mathcal{O}$-convex if the intersection of P and any $\mathcal{O}$-lines is either empty or connected.

Definition 5.6 (Rawlins and Wood [24]) The intersection of all $\mathcal{O}$-convex sets containing P is called the $\mathcal{O}$-hull of P.

In computational geometry, starshapedness is regarded as a property of visibility in a set from a fixed point. One extension of this viewpoint is the notion of visibility from an edge.

Definition 5.7 (Avis and Toussaint [1]) Let P be a simple planar polygon.

1. P is said to be completely visible from an edge e if for every $x \in P$ and every $y \in e$, $\overline{xy}$ is in P.
2. P is said to be strongly visible from an edge e if there exists a $y \in e$ such that for every $x \in P$, $\overline{xy}$ is in P.
3. P is said to be weakly visible from an edge e if for each $x \in P$, there exists a $y \in e$ such that, $\overline{xy}$ is in P.

6 Applications in Computational Vision

Generic tasks that robot vision systems perform are [30]: 1) recognition 2) localization and 3) inspection. Localization involves attitude determination. This section proposes the use of distance functions in attitude determination. Throughout this section, we are only concerned with compact sets in R^3. The critical tool we will be using is *dual mixed volume*. The first part of this section will quote theorems about dual mixed volumes which are due mainly to Lutwak [21]. The rest of this section will define our problem of attitude determination, and find a theoretical solution for the problem.

Definition 6.1 (Lutwak [21] page 531.) The *radial function* of a convex body K is defined as

$$\rho(K;\xi) \triangleq \sup\{\lambda > 0 |\ \lambda\xi \in K\}\ ,\ \text{for } \xi \in S^{d-1}\ . \tag{2}$$

Recall Definition 4.2, we know

$$\rho(K;\xi) = \frac{1}{g(K;\xi)}\ ,$$

where $g(K;\xi)$ is the distance function of K. It is not hard to see that Definition 6.1 can be directly applied on any compact starshaped set whose kernel contains the origin.

Definition 6.2 (Lutwak [21] page 532.) The *dual mixed volume* of $K_1, \ldots, K_d$ is defined as

$$\tilde{V}(K_1, \ldots, K_d) \triangleq \frac{1}{d} \int_{S^{d-1}} \rho(K_1; \xi) \cdots \rho(K_d; \xi) d\omega .$$

The dual mixed volume has the following elementary properties (Lutwak [21] page 532).

1. $\tilde{V}$ is continuous;
2. $\tilde{V}(K_1, \ldots, K_d) > 0$;
3. $\tilde{V}(\lambda_1 K_1, \ldots, \lambda_d K_d) = \lambda_1 \cdots \lambda_d \tilde{V}(K_1, \ldots, K_d)$, $\lambda_i > 0$;
4. If $A_i \subseteq B_i$ for all i then $\tilde{V}(A_1, \ldots, A_d) \leq \tilde{V}(B_1, \ldots, B_d)$ with equality if and only if $A_i = B_i$ for all i ;
5. $\tilde{V}(K, \ldots, K) = V(K)$.

The notation $\tilde{V}_i(K_1, K_2)$ is introduced :

$$\tilde{V}_i(K_1, K_2) \triangleq \tilde{V}(\underbrace{K_1, \ldots, K_1}_{d-i}, \underbrace{K_2, \ldots, K_2}_{i}) .$$

Theorem 6.1 (Lutwak [21] Theorem 1, page 533.)

$$\tilde{V}^m(K_1, \ldots, K_d) \leq \prod_{i=0}^{m-1} \tilde{V}(K_1, \ldots, K_{d-m}, K_{d-i}, \ldots, K_{d-i}) , \quad 1 < m \leq d ,$$

with equality if and only if $K_{d-m+1}, K_{d-m+2}, \ldots, K_d$ are all dilations of each other (with the origin as the center of dilation).

When $m = d$ Theorem 6.1 becomes:

Corollary 6.2 (Lutwak [21] Corollary 1.1, page 534.)

$$\tilde{V}^d(K_1, \ldots, K_d) \leq V(K_1) \cdots V(K_d) ,$$

with equality if and only if $K_1, K_2, \ldots, K_d$ are all dilations of each other (with the origin as the center of dilation).

A special case of this is:

Corollary 6.3 (Lutwak [21] Corollary 2.1, page 535.)

$$\tilde{V}_i(K_1, K_2) \leq V^{(d-i)/d}(K_1) V^{i/d}(K_2) \ , \quad 0 < i < d \ ,$$

with equality if and only if K_1 is a dilation of K_2 (with the origin as the center of dilation).

Combine this with Alexandrov inequality and obtain:

Corollary 6.4 (Lutwak [21] Corollary 1.2, page 534.)

$$\tilde{V}(K_1, \ldots, K_d) \leq V(K_1, \ldots, K_d) \ ,$$

with equality if and only if $K_1, K_2, \ldots, K_d$ are all dilations of each other (with the origin as the center of dilation).

And in particular, we have :

Corollary 6.5 (Lutwak [21] Corollary 1.3, page 534.)

$$\tilde{V}_i(K_1, K_2) \leq V_i(K_1, K_2) \ ,$$

with equality if and only if K_1 is a dilation of K_2 (with the origin as the center of dilation).

Now we turn to solving the problem of attitude determination.

Definition 6.3 The attitude determination problem is defined as finding the rotation R such that $R(K) = K'$, where K and K' are known, K is a prototype, K' is a measured object that is obtained by the unknown rotation R.

There are many ways to represent rotations. The most often used in computational vision research are orthonormal matrices and axis and angle representations. Horn [16] used unit quaternions to represent rotations. One advantage of using quaternion is "that it is much simpler to enforce the constraint that a quaternion have unit magnitude than it is to ensure that a matrix is orthonormal". We choose quaternions to represent rotations. For a set K, the rotated set under the rotation represented by quaternion $\mathring{q}$ is $\mathring{q}\, K\, \mathring{q}^*$.

Define function

$$\chi(\mathring{q}) \triangleq 3\tilde{V}(\mathring{q}\, K_1\, \mathring{q}^*, K_2, K_2) = \int_{S^2} \rho(\mathring{q}\, K_1\, \mathring{q}^*; \xi)\rho^2(K_2; \xi)d\omega ,$$

to measure how close K_1 is to K_2 assuming that K_2 is a rotated image of K_2.

First, we claim that the domain of $\chi(\mathring{q})$ is R^4. Obviously the domain contains the unit sphere in R^4, because rotations do not change the starshapedness of a set and hence the radial function for $\mathring{q}\, K_1\, \mathring{q}^*$ is well defined, and hence is the dual mixed volume. With non-unit quaternions $\mathring{q}$ we do not know whether $\mathring{q}\, K_1\, \mathring{q}^*$ is still starshaped or not. If we look at Definition 4.2 and Definition 6.1, we notice that it is not necessary to require convexity or starshapedness for (1) and (2) to be meaningful. Thus $\chi(\mathring{q})$ is well defined for any $\mathring{q}$ because $\rho(\mathring{q}\, K_1\, \mathring{q}^*; \xi)$ is well defined by (2) whether $\mathring{q}\, K_1\, \mathring{q}^*$ is starshaped or not.

We will lose, however, the uniqueness of distance/radial function if we go beyond starshaped sets (see Section 4). But this does not affect the way we solve the problem if we start with starshaped sets K_1 and K_2.

Corollary 6.2 says that

$$\tilde{V}^3(K_1, K_2, K_2) \le V(K_1)V^2(K_2) ,$$

with equality if and only if K_1, K_2 are dilations of each other (with the origin as the center of dilation). The condition stated in Lutwak's paper is that K_1, K_2 be convex. If we look at the proof of the theorems, we find that an extension of Hölder's Inequality (see Hardy [14] page 22) was used, and thus the requirement should be that $\rho(K_1; \xi), \rho(K_2; \xi)$ be strictly positive continuous functions on S^2. It is not hard to see that for starshaped compact set K with the origin in its kernel, $\rho(K; \xi)$ is strictly positive. Thus we claim that Corollary 6.2 is valid for starshaped sets as well.

This means that among all starshaped sets K_1 of volume 1, those that are dilations of K_2 yield the maximal value of

$$\int_{S^2} \rho(K_1; \xi)\rho^2(K_2; \xi)d\omega .$$

Hence the problem of attitude determination by distance function becomes the following constrained optimization problem:

$$\begin{array}{ll} \text{maximize} & \chi(\mathring{q}) \;,\;\; \mathring{q} \in R^4 \;, \\ \text{subject to} & \| \mathring{q} \| = 1 \;. \end{array} \tag{3}$$

Theorem 6.6 There exist solutions to problem (3), the problem of attitude determination by distance function.

Proof Since the constraint function $\| \mathring{q} \| - 1$ for the optimization problem (3) is continuous, the feasible region of the optimization problem is closed. In addition, the feasible region is obviously bounded. Thus a solution to the optimization problem exists.

□

7 Conclusions

We have surveyed results about necessary and/or sufficient conditions for a set to be starshaped and results concerning kernels of starshaped sets. It is noted that starshaped sets are uniquely determined by their distance functions. The proposal of using distance function to solve attitude determination problem in computational vision has been theoretically justified. A notion of star hull has been introduced and several notions of generalized convexity defined by previous researchers are reviewed. Further experimentations are needed to verify whether the theoretical proposal works in practice.

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