A Tight Lower Bound on the Size of Planar Permutation Networks

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1. INTRODUCTION

We define a **t-permutation network** to be a graph G with t distinguished vertices called terminals, with the property that for any one-to-one pairing $\{(x_i, y_i)\}$ among the terminals there is a set $\{P_i\}$ of vertex-disjoint paths in G with P_i joining x_i to y_i for each i. Permutation networks obviously have many applications in communication networks, but they have also received substantial attention in the context of permutation layouts, a basic tool in the layout of printed circuits and large scale integrated chips (see [CS80], [KKF79], [SH80], [TK82], [AKLLW85], [AKLLW90], [AKS90]).

A **permutation layout** is a permutation network where the graph G is a rectangular (2-dimensional) grid graph. The definition of permutation layout sometimes includes additional assumptions such as that the terminals are partitioned into inputs and outputs with only one-to-one pairings between inputs and outputs considered. Since it is possible to modify our definition and result in a straightforward manner to correspond to these variants we restrict our attention to the case described here.

One of the key questions concerning permutation layouts is how large a rectangle is needed to construct a t-permutation layout, since this influences how densely circuits can be laid out on chips. Examples of rectangular grids with $O(t^3)$ area which contain t-permutation layouts (and simple algorithms for finding the routings of the connecting paths) were given by Cutler and Shiloach in [CS78], who also proved that if all the terminals lie on at most two horizontal lines of the grid, then the rectangle must have area at least $\Omega(t^{2.5})$. Techniques very similar to those given by Cutler and Shiloach are commonly used in circuit layout. In [AKLLW85], Aggarwal et al proved an $\Omega(t^3)$ lower bound on the area of t-permutation layouts, showing that the Cutler-Shiloach techniques are asymptotically optimal. This result raises two obvious questions. Can the area needed be reduced by using multiple layers of grids, or by using some other planar graph instead of rectangular grids? Since area is not an appropriate measure for

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planar graphs, in the second question area is replaced by number of vertices as these two measures essentially agree on grids.

The first question is addressed in [AKLLW90] where the $\Omega(t^3)$ lower bound on area is extended to multi-layer grid permutation networks with the restriction that some (arbitrarily small) fixed fraction of the connecting paths do not change layers. The restriction that a fixed fraction of the paths do not change layers is essential, since the standard crosspoint switch is a two layer t-permutation layout with $O(t^2)$ area, in which every routing path changes layers once. In spite of the reduction of area obtainable with the use of layer changes, the practical advantages of avoiding layer changes continue to make planar permutation networks a useful tool in circuit layout. Thus the second question remains a significant issue. The purpose of this paper, is to answer the second question by proving an $\Omega(t^3)$ lower bound on the number of vertices in a planar t-permutation network, showing that the current grid-based techniques are asymptotically optimal.

Like the lower bound for t-permutation grid graphs in [AKLLW85, AKLLW90], our proof uses the permutation property of the graph to simulate a planar embedding of an expanding graph on $\Omega(t)$ vertices and then applies the quadratic lower bound on the crossing number of expanding graphs to get the desired $\Omega(t^3)$ lower bound. However, we also use an additional tool, namely the existence of weight-balanced separators for planar graphs. Combining these two techniques results in a proof which is more general and simpler than the ones for grid graphs given in [AKLLW85] and [AKLLW90].

2. THE LOWER BOUND

Let G be a t-permutation network with n vertices. We first note that we may assume that G has maximum degree 3 since replacing the edges adjacent to each vertex of higher degree with a binary tree connecting the vertex to its neighbours only increases the number of vertices by at most a constant factor, and does not affect the permutation property. In addition we may assume that G is connected since all the terminals must lie in the same connected component of a permutation network, and the connected component will itself be a permutation network. Finally, we may assume that each terminal has degree 1 since if necessary we can hang a new terminal vertex off each original terminal.

We start the section by describing the weight-balanced separator theorem (2.1) and one of its corollaries, culminating with the formulation we will actually apply, the balanced terminal separator lemma (2.2). We then give the version of the lower bound on crossing number (2.3) which we need, and close with the proof of the lower bound on the number of vertices in a planar permutation network (2.4).

The weight-balanced separator is a generalization of the weighted version of the planar separator theorem given in [LT79]. Specifically, the original theorem in [LT79] proves that if every vertex in an n-vertex planar graph has a weight, then there is a set of $O(\sqrt{n})$ edges and vertices whose removal splits the graph into two subsets so that each subset contains at most half the total weight. In the generalization, vertices have several different weights and we want to find a separator which simultaneously splits all the weights in half. The precise statement is as follows.

Theorem 2.1 (Weight-Balanced Separator). Given an n-vertex planar graph where each vertex has a k-vector of weights, the graph can be split into two subsets by removing $O(k\sqrt{n})$ edges and vertices, such that for each component of the weight vector, the total component weight of each subset is at most half the total component weight of the graph.

A weaker form of this theorem was first proved by Leighton in [L82], using a combinatorial result on splitting necklaces of coloured beads. A stronger and very elegant form of the necklace splitting result was proved by Goldberg and West [GW85], though with a rather lengthy and involved proof. Alon and West [AW86] later gave a very simple proof based on the Borsuk-Ulam "ham-sandwich " theorem from topology. The proof of the weight-balanced separator theorem in its full generality can be found in the last two lectures of [LLS89], though in fact the special cases found in [L82], [BL84], and [GW85, p. 104, thm. 4], would suffice for our purposes.

It is well-known, and easy to prove by iteratively applying the original weighted planar separator theorem, that for any p, and any weighted planar graph G of bounded degree, there exist $O(\sqrt{pn})$ edges whose removal splits G into p pieces each having at most $1/p$ of the total weight. Similarly, by using the weight-balanced separator theorem and assigning each vertex a pair of weights, one the vertex's original weight and the second the number of removed edges which are adjacent to the vertex, is not hard to prove the following stronger result. For any p, and any weighted planar graph G of bounded degree, there exist $O(\sqrt{pn})$ edges whose removal splits G into p pieces each having at most $1/p$ of the total weight, and such that each piece is adjacent to $O(1/p)$ of the removed edges. Such decompositions are called fully balanced decompositions and are discussed in detail in the last two lectures of [LLS89] and in [BL84]. Applying the fully balanced decomposition result in the context of planar permutation networks yields the following lemma.

Lemma 2.2 (Balanced Terminal Separator). Given a bounded degree n-vertex planar graph with t terminals each having degree one, there exist $O(\sqrt{nt})$ edges whose removal results in a graph such that each connected component is incident to $O(\sqrt{n/t})$ removed edges and each terminal is its own component.

Proof. First remove the t edges adjacent to terminals. Assign each vertex a weight equal to the number of removed edges adjacent to it. Now taking $p = t$, a fully-balanced decomposition of this weighted graph has the desired properties.∎

Lemma 2.3 (Crossing Pairs).

There exists a constant $c > 0$ such that for each s there is an s-vertex graph H of degree at most 3, such that for each planar embedding of H there are at least cs^2 distinct pairs of edges which cross each other.

Proof. We first note that in any planar embedding of a graph H with the minimum number of edge-crossings, each pair of edges crosses at most once. To see this, suppose we have an embedding and that e and e' are edges which cross each other more than once. Let x and y be consecutive crossings between e and e'. The crossings at x and y can be eliminated by rerouting e' and e so that each follows the other's path between x and y (see Figure 1), and hence the embedding could not have had the minimum number of crossings. Given this observation the lemma follows immediately from the well-known fact that there are expanding graphs of degree 3 and Leighton's quadratic lower bound on the crossing number of expanding graphs [L84]. ∎

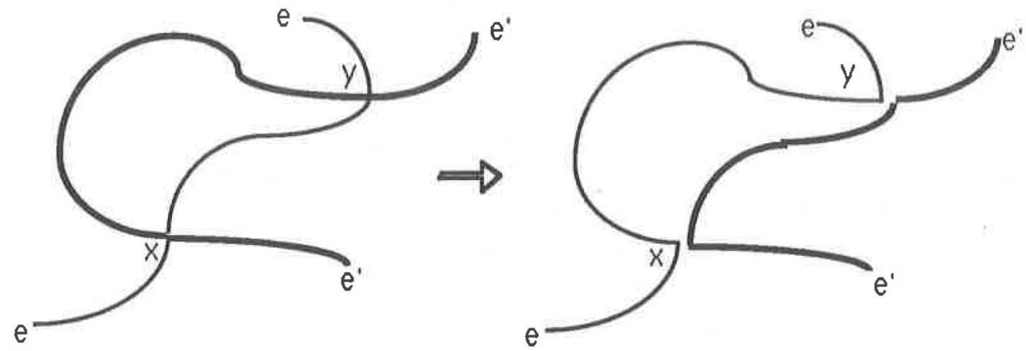

Figure 1.

We are now ready to prove the desired lower bound.

Theorem 2.4. If G is a connected n-vertex planar t-permutation network of degree at most 3, then $n = \Omega(t^3)$.

Proof. By the terminal separator lemma, there is a set R of $O(\sqrt{nt})$ edges of G whose removal results in a graph such that each connected component is incident to $O(\sqrt{n/t})$ removed edges and each terminal is its own component. Let $G \backslash R$ be the graph obtained by removing the edges in R from G, and let G^c be the graph of obtained from G by contracting every edge of G which is not in R, and then removing multiple edges. An example is shown in figure 2. It is easy to see that each vertex of G^c corresponds to a connected component of $G \backslash R$, and that G^c is a connected planar graph with $O(\sqrt{n/t})$ maximal degree. We will refer to a vertex of G^c as a **terminal node** if the corresponding connected component of $G \backslash R$ is a terminal. We will assume that we have a fixed embedding of G in the plane.

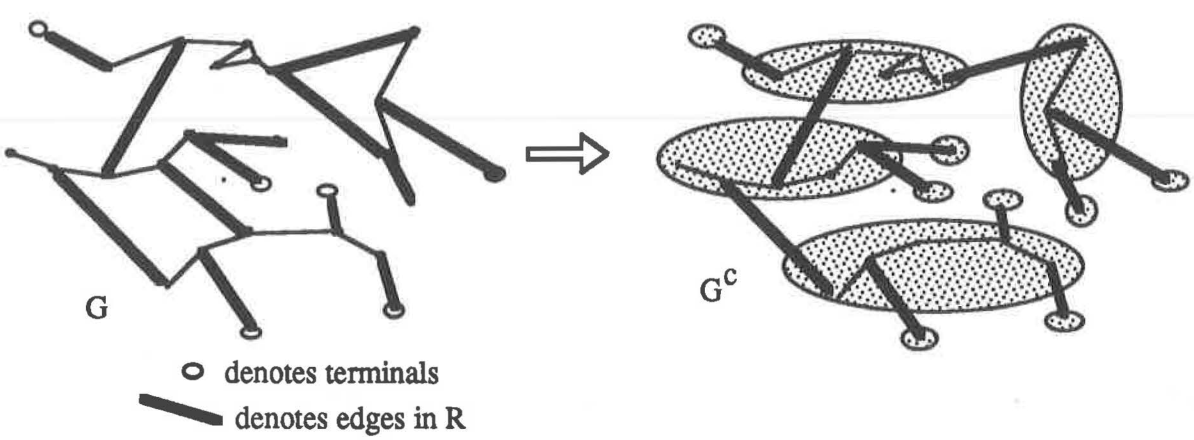

Figure 2

Let T^c be a subtree of G^c whose leaves are the terminal nodes. Such a tree can be obtained, for example, by taking a spanning tree of G^c and chopping off all branches which contain no terminal nodes. Since each terminal node has degree 1 in G and hence in G^c, it must be a leaf of any spanning tree of G^c and hence the leaves of this tree will be exactly the terminal nodes. Let T be a subtree of G which maps onto T^c, i.e. T is obtained by replacing each edge of T^c with a representative edge in R and replacing each vertex of T^c with a subtree of the component corresponding to that vertex in G^c. An example is shown in figure 3. Note that the leaves of T are the terminals.

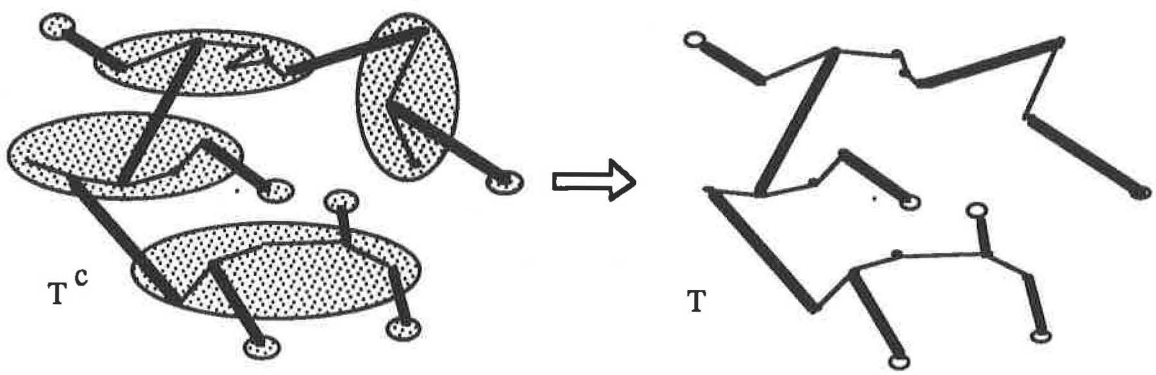

Figure 3

Let Q be a simple curve in the plane connecting the terminals, with Q running alongside the induced embedding of the edges of T in the plane, picking up the terminals as illustrated in Figure 4. Q is assumed to be routed sufficiently closely to T so that it only intersects edges of G when it runs past a vertex of T where it may have to cross an edge in $G\backslash T$ which is adjacent to the vertex. We label the terminals $z_1, \ldots, z_t$ in the order in which they are first visited by Q.

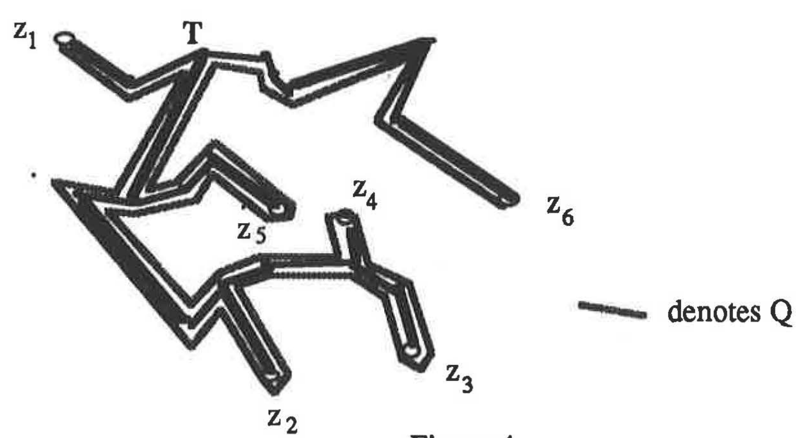

Figure 4

We will call each portion of Q joining a pair of consecutive terminals a **link**, and say that a link **runs through a component** of $G\backslash R$ if it runs past some vertex in the component. It will be important to keep in mind that links are not part of any of the graphs but merely simple curves lying in the plane in which the graphs are embedded. Since each link starts and ends at a terminal, and at most two links can run alongside any edge in T, it is easy to see that if y is a vertex of degree d in T^c, then at most $2d$ links can run through the component C_y represented by y. Let $C_1, \ldots, C_m$ be the components of $G \backslash R$, and for each i let n_i be the number of links which run through C_i. We now show that $\sum_{n_i > 4} n_i \leq 6t$. Let d_i be the degree of the vertex representing C_i in T^c. We already noted that $n_i \leq 2d_i$ and hence it suffices to prove that $\sum_{d_i > 2} n_i \leq 3t$. However this is obvious since it is easy to prove that the number of vertices of degree at least 3 in any tree is at most 3 times the number of leaves, and T^c has exactly t leaves since its leaves are the terminal nodes.

Now taking $s = t/3$ in Lemma 2.3, suppose H is a degree 3 graph on $t/3$ vertices $v_1, \ldots, v_{t/3}$ such that for each planar embedding of H there are at least $c(t/3)^2$ distinct pairs of edges which cross each other. We want to use the planar embeddings of G and Q to produce an embedding of H in the plane. We use z_{3h-1} to represent v_h for each h. Let $Z_h = \{z_{3h-2}, z_{3h-1}, z_{3h}\}$. Now let $\{(x_i, y_i)\}$ be a one-to-one pairing of the terminals of G such that for each edge (v_j, v_k) in H there is some i such that $x_i \in Z_j$ and $y_i \in Z_k$. This is easy to do since H is of degree at most 3 and $|Z_h| = 3$ for each h. Now each edge (v_j, v_k) of H is embedded as the corresponding permutation path P_i plus possibly a link at one or both ends to complete the connection to its endpoints. Note that each link is used by at most one edge of H. Since the P_i are mutually disjoint and the links are also mutually disjoint except at possibly the vertices of H, two embedded edges of H can only cross if one of the edges' permutation paths crosses a link used by the other edge. Thus there are $\Omega(t^2)$ distinct pairs of permutation paths and links which cross each other. By the choice of Q, each such crossing can only occur when the link runs past a vertex which is an endpoint of an edge in the permutation path. We will say a permutation path and link cross inside a component of $G \backslash R$ if the vertex is in that component.

For each i let r_i be the number of edges in R incident to the connected component C_i. Since there is at most one permutation path using each edge in R, the number of permutation paths which pass through C_i is at most r_i, and hence the number of distinct pairs of permutation paths and links which cross inside C_i is at most $n_i r_i$. Let α be the total number of distinct pairs which cross. We have

$$\alpha \leq \sum_{i=1}^{m} n_i r_i \leq 4 \sum_{r_i \leq 4} n_i + \sum_{r_i > 4} n_i r_i.$$

Clearly we have $\sum n_i \leq 2|R|$, and we proved earlier that $\sum_{r_i > 4} r_i \leq 6t$. In addition we have $|R| = O(\sqrt{nt})$ and $\max\{n_i\} = O(\sqrt{n/t})$. Thus we have $4 \sum_{r_i \leq 4} n_i = O(|R|) = O(\sqrt{nt})$ and $\sum_{r_i > 4} n_i r_i \leq 6t \max\{n_i\} = O(\sqrt{nt})$ also. Hence $\alpha = O(\sqrt{nt})$. Finally, combining this with the lower bound $\alpha = \Omega(t^2)$ implies $n = \Omega(t^3)$ as desired.∎

3. REFERENCES

[A87] N. Alon, Splitting necklaces, Advances in Math. 63(1987) pp.247-253.

[AM85] N. Alon and V. Milman, λ_1, isoperimetric inequalities for graphs, and superconcentrators, J. Comb. Theory B 38(1985), pp. 73 - 88.

[AKS90] A. Aggarwal, M. Klawe and P.Shor, Multi-layer grid embeddings for VLSI, to appear in Algorithmica.

[AKLLW85] A. Aggarwal, M. Klawe, D. Lichtenstein, N. Linial, and A. Wigderson, Multi-layer grid embeddings, Proc. IEEE 26th Ann. Symp. Found. Comp. Sci., 1985, pp. 186 - 196.

[AKLLW90] A. Aggarwal, M. Klawe, D. Lichtenstein, N. Linial, and A. Wigderson, A lower bound on the area of permutation layouts,to appear in Algorithmica.

[AW86] N. Alon and D. West, The Borsuk-Ulam theorem and bisection of necklaces, Proc. Amer. Math. Soc., 98(1986) pp. 623-628.

[BL84] S. Bhatt and F. T. Leighton, A Framework for Solving VLSI Graph Layout Problems, JCSS Vol 28 No.2 April 1984, pp. 300 - 343.

[BP77] N. K. Bose and K. A. Prabhu, Thickness of Graphs with Degree Constrained Vertices, IEEE Trans. on Circuits and Systems, Vol. CAS-24, No. 4, April 1977, pp. 184-190.

[CS78] M. Cutler and Y. Shiloach, Permutation Layout, Networks, Vol. 8, 1978, pp. 253-278.

[GW85] C.H.Goldberg and D.B. West, Bisection of circle colorings, SIAM J. Alg. Disc. Math., 6(1985) pp. 93-106.

[KKF79] E. S. Kuh, T. Kashiwabara, and T. Fujisawa, On Optimum Single Row-Routing, IEEE Trans. on Circuits and Systems, Vol. CAS-26, No. 6, June 1979, pp. 361-368.

[Ka80] Y. Kajitani, On Via Hole Minimization of Routings on a Two-Layer Board, Tech. Report, Dept. of Elec. Engineering, Tokyo Institute of Technology, Japan, 1980.

[L81] F. T. Leighton, Layouts for the Shuffle-Exchange Graph and Lower Bound Techniques for VLSI, Ph. D. thesis, Dept. of Math., MIT, 1981.

[L82] F. T. Leighton, A layout strategy for VLSI which is provably good, Proc. ACM Ann. Symp. Theory of Comp., 1982, pp. 85-98.

[L84] F. T. Leighton, New Lower Bound Techniques for VLSI, Math Systems Theory 17(1984), pp. 47-70.

[LLS89] F.T.Leighton, C.E.Leiserson and E.Schwabe, Theory of parallel and VLSI computation, MIT/LCS/RRS 6 Research Seminar Series, Lecture Notes for 18.435 /6.848, March 89.

[LT79] R. J. Lipton and R. E. Tarjan, A Separator Theorem for Planar Graphs, SIAM J. Appl. Math., 36 (1979) 177-189.

[Ri82] R. Rivest, The Placement and Interconnect System, Proc. 19th Design Automation Conference, 1982, pp. 475-481.

[Sh80] I. Shirakawa, Letter to the Editor : Some Comments on Permutation Layout, Networks, Vol. 10, 1980, pp. 179 - 182.

[So74] H.C. So, Some Theoretical Results on the Routing of Multilayer Printed Wiring Boards, 1974 IEEE Int. Symp. on Circuits and Systems, pp. 296-303.

[TKS76] B. S. Ting, E. S. Kuh, and I. Shirakawa, The Multilayer Routing Problem: Algorithms and Necessary and Sufficient Conditions for the SIngle-Row Single Layer Case, IEEE Trans. of Circuits and Systems, Vol. CAS-23, No. 12, 1979, pp. 768-778.

[TK82] S. Tsukiyama and E. S. Kuh, Double-Row Planar routing and Permutation Layout, Networks, Vol. 12, 1982, pp. 287-316.