e-mail addresses: <gilmore@cs.ubc.ca>, <tsiknis@cs.ubc.ca>

A FORMALIZATION OF CATEGORY THEORY
in
NaDSet

by

Paul C. Gilmore* and George K. Tsiknis

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ABSTRACT

This paper was presented to the Sixth Workshop on Mathematical Foundations of Programming Semantics held at Queen's University, May 15-19, 1990.

Because of the increasing use of category theory in programming semantics, the formalization of the theory, that is the provision of an effective definition of what constitutes a derivation for category theory, takes on an increasing importance. Nevertheless, no suitable logic within which the theory can be formalized has been provided. The classical set theories of Zermelo-Fraenkel and Gödel-Bernays, for example, are not suitable because of the use category theory makes of self-referencing abstractions, such as in the theorem that the set of categories forms a category. In this paper, a formalization of category theory and a proof of the cited theorem is provided within the logic and set theory NaDSet. NaDSet definitions for natural transformations and functor categories are given and an equivalence relation on categories is defined. Additional definitions and discussions on products, comma categories, universals limits and adjoints are presented. They provide evidence that any construct, not only in categories, but also in toposes, sheaves, triples and similar theories can be formalized within NaDSet.

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1. INTRODUCTION

Section A1 of [Feferman84] reinforces the argument presented in [Feferman77] that category theory cannot by itself provide a foundation for mathematics since it makes use of prior notions of logic and set abstraction. At the same time the first paper provides motivation for constructing set theories other than the traditional Zermelo-Fraenkel and Gödel-Bernays set theories. An example of a common argument in modern algebra is presented using structures $<A, \otimes, =_A>$ consisting of a set A, a commutative and associative binary operation $\otimes$ and an identity relation $=_A$ over A. If B is the set of all such structures, PR is the Cartesian product on B and ISO isomorphism between the elements of B, then the structure $<B, PR, ISO>$ is itself a member of B. However, a proof of this fact cannot be formalized within the traditional set theories because of the prohibition against self-membership or self-reference.

In [Gilmore89] a natural deduction based set theory NaDSet was described and a proof that $<B, PR, ISO>$ is a member of B was provided within NaDSet. This encouraged the conjecture that NaDSet could provide a logic within which category theory could be formalized. This paper substantiates this conjecture by providing a proof within NaDSet that the set of all categories is itself a category. The significant role that category theory currently plays in the study of programming semantics may be enhanced by such a formalization of the theory, since the provision of a proof theory makes the semantics accessible to mechanization.

Category theory, of course, involves many more primitive concepts than the theory of B-structures. Section 3 presents a definition of a category within NaDSet that is more general in two respects than the definition given in [Barr&Wells85] or in [Mac Lane71]. First, a category is defined in terms of its arrows only with no reference to objects, as suggested in [Lawvere66]. Secondly, the identity relation of a category is an explicit part of its structure. While the first simplification is not fundamental, the second generalization has important repercussions. It allows each category to assume its own identity relation that generally may be different than the extensional identity implied by the traditional definitions.

The definition of category theory in section 3 is typical for definitions of an axiomatic theory within NaDSet. The axioms of the theory are used only to define the set of structures satisfying the axioms, and in no way imply the existence of a structure satisfying the axioms. Therefore, the formalization of the theory within NaDSet has no existential implications for NaDSet. This fact may help to provide an answer to the question posed in [Blass84]: Does category theory necessarily involve existential principles that go beyond those of other mathematical disciplines? When a traditional set theory is used as a foundation for category theory, it is necessary to distinguish between small and large categories [Mac Lane71]. That is not necessary when category theory is formalized within NaDSet. Of course this does not provide an answer to the question: Does the proof of the existence of some categories involve existential principles that go beyond those of other mathematical disciplines?

In section 4 the notion of a functor on categories is formalized. In section 5, which constitutes the main part of the paper, the necessary definitions for the category of categories and a proof of the theorem that this structure is a category itself, is provided in NaDSet. The proof of the theorem as well as of the lemmas of section 4 are long and tedious and they are only outlined in this paper; the complete proofs are provided in [Gilmore & Tsiknis 90a]. However, by examining the outlines of the derivations, readers may gain confidence in the principal result and in the capability of NadSet to provide logical foundations for category theory.

The ubiquitous notions of natural transformations and functor categories are formalized in section 6, while in section 7, definitions and theorems for a variety of basic constructions including comma categories, universals, limits and adjoints are provided. These two sections further demonstrate that NaDSet may be used as the logic for category theory and suggest that any construct in category theory, as well as in the theories of toposes, sheaves, triples etc. can be formalized within NaDSet in a similar way. Finally, possible directions for future work are discribed in section 8.

2. NaDSet

For space reasons, no description of NaDSet is given in this paper; readers are referred to [Gilmore89] or [Gilmore&Tsiknis90b]. Only an outline of the main differences between NaDSet and a conventional set theory, a short discussion on the meaning of definitions, and the rules of deduction for some defined bounded quantifiers will be given.

The logic differs from a conventional presentation of set theory in four respects:

(1) To provide a transparent formalization of the traditional reductionist semantics of [Tarski36], NaDSet is formalized as a natural deduction based set theory. Since in a reductionist semantics the meaning of a complex formula is reduced to that of simpler formulas, the meaning given to the irreducible atomic formulas is critical.

(2) A nominalist interpretation of atomic formulas is used: Only the name of a set, not the set itself, can be a member of another set. To avoid confusions of use and mention, it is necessary that NaDSet be a second order logic, but no higher order form of NaDSet is necessary or consistent.

(3) Although NaDSet is second order, both first and second order quantification is expressed by the same quantifier. It is only necessary that NaDSet have two distinct kinds of parameters (free variables) one first order and the other second order.

(4) A generalized set abstraction term {**ta**|**F**} is admitted in which **ta** may be a term, not just a single variable, and **F** may be any formula.

These features of NaDSet, elaborated upon in [Gilmore89] and [Gilmore&Tsiknis90a], are essential for the formalization of category theory. In [Gilmore86], an earlier version of the logic is described and motivated and the consistency of the logic proved. There is not (yet) a consistency proof for the current NaDSet.

Essential to an understanding of this paper is the proper interpretation of definitions such as

Cat **for** {<Ar, $=_a$, Sr, Tg, Cp> | Category[Ar, $=_a$, Sr, Tg, Cp] }

Category[**Ar**, $\mathbf{=_a}$, **Sr**, **Tg**, **Cp**] **for** <u>axioms</u>

In the first of these definitions, 'Cat' is provided as an abbreviation for the abstraction term

{ <Ar, $=_a$, Sr, Tg, Cp> | Category[Ar, $=_a$, Sr, Tg, Cp] }

This means that any term or formula in which 'Cat' is used as a term, should be understood as the term or formula in which 'Cat' is replaced by the abstraction term.

The second of these definitions is a definition scheme of individual definitions of the first kind. In the second definition, **Ar**, $\mathbf{=_a}$, **Sr**, **Tg** and **Cp** are used as metavariables ranging over the terms of NaDSet. When they are replaced with particular terms, as they are in the formula

Category[Ar, $=_a$, Sr, Tg, Cp]

by variables 'Ar', '$=_a$', 'Sr', 'Tg' and 'Cp', the resulting formula

Category[Ar, $=_a$, Sr, Tg, Cp]

is an abbreviation for the conjunction of all the axioms for categories, described in the next section, in which the terms **Ar**, $\mathbf{=_a}$, **Sr**, **Tg** and **Cp** are replaced by the variables 'Ar', '$=_a$', 'Sr', 'Tg' and 'Cp'.

Bounded quantifiers are used extensively in this paper. It is important that they be properly understood. A formula

[∀x:{**ta**|**F**}]**G**

is to be understood as an abbreviation for

[∀$\underline{v}$]([$\underline{v}$/$\underline{u}$]**ta**:{**ta**|**F**} ⊃ [[$\underline{v}$/$\underline{u}$]**ta**/x]**G**)

where $\underline{u}$ is a sequence of the distinct variables with free occurrences in **ta**, $\underline{v}$ is a sequence of the same length of distinct variables free to replace x in **G** and without free occurrences in {**ta**|**F**}, and [∀$\underline{v}$] is a sequence of quantifiers one for each variable of $\underline{v}$.

Consider, for example, axiom (c2) from section 3:

[∀f,g:**Ar**](f$=_{\mathbf{a}}$g ⊃ g$=_{\mathbf{a}}$f)

in which terms **Ar** and $=_{\mathbf{a}}$ are assumed given, with the usual infix notation for identity, f$=_{\mathbf{a}}$g being written instead of <f,g>:$=_{\mathbf{a}}$. The interpretation of the bounded quantifer depends upon the term **Ar**. For example, in section 5 the category of categories is defined as the tuple < $\mathbb{Ar}$, $=_{\mathbb{a}}$, $\mathbb{Sr}$, $\mathbb{Tg}$, $\mathbb{Cp}$ > of defined terms $\mathbb{Ar}$, $=_{\mathbb{a}}$, $\mathbb{Sr}$, $\mathbb{Tg}$ and $\mathbb{Cp}$, where in particular $\mathbb{Ar}$ is defined to be Func, the set of functors defined in section 4. To prove that this tuple is a category, it is necessary to show that it satisfies all of the axioms of a category, and in particular that the following formula is derivable:

[∀f,g:Func]A,

where A is the formula (f $=_{\mathbb{a}}$ g ⊃ g $=_{\mathbb{a}}$ f). In keeping with common practice, this formula is an abbreviation for

(fl) [∀f:Func][∀g:Func]A.

Now to understand the quantifiers, it is necessary to know the definition of Func, which is

Func **for** { < F, <Ar_C, $=_{aC}$, Sr_C, Tg_C, Cp_C>, <Ar_D, $=_{aD}$, Sr_D, Tg_D, Cp_D> > | G}

where G is a given formula.

The term

(tm) < F, <Ar_C, $=_{aC}$, Sr_C, Tg_C, Cp_C>, <Ar_D, $=_{aD}$, Sr_D, Tg_D, Cp_D> >

is formed from 11 variables 10 of which occur in two quintuples. The variable 'f' in the first quantifier of (fl) is assumed to have the form of the term (tm). That is, (fl) must be understood to be an abbreviation for

$$[\forall F][\forall Ar_C][\forall =_{aC}][\forall Sr_C][\forall Tg_C][\forall Cp_C][\forall Ar_D][\forall =_{aD}][\forall Sr_D][\forall Tg_D][\forall Cp_D]($$

$$< F, <Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C>, <Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D> >:Func$$

$$\supset [< F,<Ar_C,=_{aC},Sr_C,Tg_C,Cp_C>,<Ar_D,=_{aD},Sr_D,Tg_D,Cp_D>>/f][\forall g:Func]A$$

where each of the variables F, Ar_C, $=_{aC}$, Sr_C, Tg_C, Cp_C, Ar_D, $=_{aD}$, Sr_D, Tg_D and Cp_D is assumed to be free to replace the variable f in [∀g:Func]A. Thus, when the quantifier [∀g:Func] is also interpreted, the formula (fl) must be understood as an abbreviation for a formula in which 22 variables are quantified.

3. CATEGORIES

In this section, a NaDSet definition of the set of categories will be given analogous to the definition of the structure B in section 8 of [Gilmore 89]. The terminology provided in the introduction of [Barr&Wells85] will be used with one exception: Instead of using *objects* and *arrows* in defining a category, by following Lawvere's definition [Lawvere66], objects can be dispensed with altogether, and only arrows used. Nevertheless, for the readers who are accustomed to the more traditonal definition of categories, a definition of the objects for a category in terms of its arrows is provided.

The formalization of category theory within NaDSet is typical of the formalization of any axiomatic theory within the logic: The set of structures satisfying the axioms of category theory is defined. The theorems of category theory are then the formulas that can be proven to be true in any member of the set of categories.

Throughout the paper, conventional algebraic notations are used as abstraction variables and as parameters. These notations will be explained as they are introduced. Additionally, metavariables ranging over terms of NaDSet that are intended to represent algebraic concepts, are used. They will always be printed in bold type. For example, the variables of this kind used in this section, together with their intended interpretation are:

Ar the set of arrows or morphisms

$\mathbf{=_a}$ identity of arrows

Sr a binary term with first argument an arrow and second argument its source object

Tg a binary term with first argument an arrow and second argument its target object

Cp a ternary term the third argument of which is the composite of the arrows that are its first two terms.

The first use of these metavariables is in the following definition:

Category[Ar, $=_\mathbf{a}$, Sr, Tg, Cp] for axioms

"axioms" is the conjunction of the formulas listed below. In these axioms, the usual infix notation for $=_\mathbf{a}$ is used instead of the postfix notation of NaDSet:

Identity Axioms

$$[\forall f{:}\mathbf{Ar}]\ f=_\mathbf{a}f \tag{c1}$$

$$[\forall f,g{:}\mathbf{Ar}](\ f=_\mathbf{a}g \supset g=_\mathbf{a}f) \tag{c2}$$

$$[\forall f,g,h{:}\mathbf{Ar}](\ f=_\mathbf{a}g \wedge g=_\mathbf{a}h \supset f=_\mathbf{a}h\ ) \tag{c3}$$

$$[\forall f,g,a{:}\mathbf{Ar}](\ f=_\mathbf{a}g \wedge \langle f,a\rangle{:}\mathbf{Sr} \supset \langle g,a\rangle{:}\mathbf{Sr}\ ) \tag{c4}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ a=_\mathbf{a}b \wedge \langle f,a\rangle{:}\mathbf{Sr} \supset \langle f,b\rangle{:}\mathbf{Sr}\ ) \tag{c5}$$

$$[\forall f,g,a{:}\mathbf{Ar}](\ f=_\mathbf{a}g \wedge \langle f,a\rangle{:}\mathbf{Tg} \supset \langle g,a\rangle{:}\mathbf{Tg}\ ) \tag{c6}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ a=_\mathbf{a}b \wedge \langle f,a\rangle{:}\mathbf{Tg} \supset \langle f,b\rangle{:}\mathbf{Tg}\ ) \tag{c7}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}](\ f=_\mathbf{a}k \wedge \langle f,g,h\rangle{:}\mathbf{Cp} \supset \langle k,g,h\rangle{:}\mathbf{Cp}\ ) \tag{c8}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}](\ g=_\mathbf{a}k \wedge \langle f,g,h\rangle{:}\mathbf{Cp} \supset \langle f,k,h\rangle{:}\mathbf{Cp}\ ) \tag{c9}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}](\ h=_\mathbf{a}k \wedge \langle f,g,h\rangle{:}\mathbf{Cp} \supset \langle f,g,k\rangle{:}\mathbf{Cp}\ ) \tag{c10}$$

Sr, Tg and Cp are functions

$$[\forall f{:}\mathbf{Ar}][\ \exists a{:}\mathbf{Ar}]\ \langle f,a\rangle{:}\mathbf{Sr} \tag{c11}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Sr} \wedge \langle f,b\rangle{:}\mathbf{Sr} \supset a=_\mathbf{a}b) \tag{c12}$$

$$[\forall f{:}\mathbf{Ar}][\ \exists a{:}\mathbf{Ar}]\ \langle f,a\rangle{:}\mathbf{Tg} \tag{c13}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Tg} \wedge \langle f,b\rangle{:}\mathbf{Tg} \supset a=_\mathbf{a}b) \tag{c14}$$

$$[\forall f,g,b{:}\mathbf{Ar}](\ \langle f,b\rangle{:}\mathbf{Tg} \wedge \langle g,b\rangle{:}\mathbf{Sr}\ \supset [\ \exists h{:}\mathbf{Ar}]\langle f,g,h\rangle{:}\mathbf{Cp}\ ) \tag{c15}$$

$$\begin{aligned}&[\forall f,g,h,a,b,c{:}\mathbf{Ar}](\ \langle f,g,h\rangle{:}\mathbf{Cp} \supset ((\ \langle f,a\rangle{:}\mathbf{Sr}\ \supset \langle h,a\rangle{:}\mathbf{Sr}) \wedge \\ &\qquad (\langle g,b\rangle{:}\mathbf{Tg} \supset \langle h,b\rangle{:}\mathbf{Tg}) \wedge (\langle f,c\rangle{:}\mathbf{Tg}\ \equiv \langle g,c\rangle{:}\mathbf{Sr}\ )))\end{aligned} \tag{c16}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}](\ \langle f,g,h\rangle{:}\mathbf{Cp} \wedge \langle f,g,k\rangle{:}\mathbf{Cp} \supset h=_\mathbf{a}k\ ) \tag{c17}$$

Note that compositions are written in the order of the arrows from left to right. Therefore, $\langle f,g,h\rangle{:}\mathbf{Cp}$ if and only if h is the morphism composition of g with h.

Identity Arrows Exist

$$[\forall f,a{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Sr}\ \supset \langle a,a\rangle{:}\mathbf{Sr} \wedge \langle a,a\rangle{:}\mathbf{Tg} \wedge \langle a,f,f\rangle{:}\mathbf{Cp}\ ) \qquad \text{(c18)}$$

$$[\forall f,a{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Tg}\ \supset \langle a,a\rangle{:}\mathbf{Sr} \wedge \langle a,a\rangle{:}\mathbf{Tg} \wedge \langle f,a,f\rangle{:}\mathbf{Cp}\ ) \qquad \text{(c19)}$$

Composition is Associative

$$[\forall f,g,h,fg,gh,fg1h,f1gh{:}\mathbf{Ar}](\ \langle f,g,fg\rangle{:}\mathbf{Cp} \wedge \langle g,h,gh\rangle{:}\mathbf{Cp} \wedge \langle fg,h,fg1h\rangle{:}\mathbf{Cp} \wedge \langle f,gh,f1gh\rangle{:}\mathbf{Cp} \supset fg1h =_{\mathbf{a}} f1gh\ ) \qquad \text{(c20)}$$

The set of categories is now defined:

$$\text{Cat}\ \ \mathbf{for}\ \ \{\ \langle \text{Ar}, =_a, \text{Sr}, \text{Tg}, \text{Cp}\rangle \mid \text{Category}[\text{Ar}, =_a, \text{Sr}, \text{Tg}, \text{Cp}\ ]\}$$

where Ar, $=_a$, Cp, Sr and Tg are all used as variables that are bound in the abstraction term.

Finally, the projections on a tuple that represents a category can be given by the following definitions.

$$\text{Ar}[\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle]\ \ \mathbf{for}\ \ \{\ u \mid u{:}\mathbf{Ar}\ \}$$

$$=_a[\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle]\ \ \mathbf{for}\ \ \{\ \langle u,v\rangle \mid \langle u,v\rangle{:}\ =_{\mathbf{a}}\ \}$$

$$\text{Sr}[\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle]\ \ \mathbf{for}\ \ \{\ \langle u,v\rangle \mid \langle u,v\rangle{:}\ \mathbf{Sr}\ \}$$

$$\text{Tg}[\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle]\ \ \mathbf{for}\ \ \{\ \langle u,v\rangle \mid \langle u,v\rangle{:}\ \mathbf{Tg}\ \}$$

$$\text{Cp}[\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle]\ \ \mathbf{for}\ \ \{\ \langle u,v,w\rangle \mid \langle u,v,w\rangle{:}\ \mathbf{Cp}\ \}$$

3.1 Objects, Hom-Sets and Commutative Diagrams

The axiomatization of category theory presented here does not require the specification of a set of objects, since the objects of a category correspond exactly to its identity arrows. Therefore the set of objects Ob[$\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle$] of a category $\langle \mathbf{Ar}, =_{\mathbf{a}}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}\rangle$ may be defined to be any one of the following extensionally identical terms.

(i) $\{\ x \mid x{:}\mathbf{Ar} \wedge \langle x, x\rangle{:}\mathbf{Sr} \wedge \langle x, x\rangle{:}\mathbf{Tg}\ \}$

(ii) $\{\ x \mid x{:}\mathbf{Ar} \wedge ([\exists f{:}\mathbf{Ar}]\langle f, x\rangle{:}\mathbf{Sr} \vee [\exists f{:}\mathbf{Ar}]\langle f, x\rangle{:}\mathbf{Tg})\ \}$

(iii) $\{\ x \mid x{:}\mathbf{Ar} \wedge [\forall f,g{:}\mathbf{Ar}](\langle f, x, g\rangle{:}\mathbf{Cp} \supset f =_{\mathbf{a}} g)$

$\wedge\ [\forall f,g{:}\mathbf{Ar}](\langle x, f, g\rangle{:}\mathbf{Cp} \supset f =_{\mathbf{a}} g)\ \}$

The hom-set for objects **ob1** and **ob2** can be defined:

$$\mathbf{Hom[ob1,ob2]}\quad \textbf{for}\quad \{\, x \mid x{:}\mathbf{Ar} \wedge \langle x,\mathbf{ob1}\rangle{:}\mathbf{Sr} \wedge \langle x,\mathbf{ob2}\rangle{:}\mathbf{Tg} \,\}$$

Finally, that the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

commutes means

$$\langle f,a\rangle{:}\mathbf{Sr} \wedge \langle f,b\rangle{:}\mathbf{Tg} \wedge \langle g,b\rangle{:}\mathbf{Sr} \wedge \langle g,c\rangle{:}\mathbf{Tg} \wedge \langle f,g,h\rangle{:}\mathbf{Cp},$$

while that the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "k"'] & b \arrow[d, "g"] \\ d \arrow[r, "m"'] & c \end{tikzcd}$$

commutes means

$$\langle f,a\rangle{:}\mathbf{Sr} \wedge \langle f,b\rangle{:}\mathbf{Tg} \wedge \langle g,b\rangle{:}\mathbf{Sr} \wedge \langle g,c\rangle{:}\mathbf{Tg} \wedge \langle k,a\rangle{:}\mathbf{Sr} \wedge \langle k,d\rangle{:}\mathbf{Tg} \wedge$$
$$\langle m,d\rangle{:}\mathbf{Sr} \wedge \langle m,c\rangle{:}\mathbf{Tg} \wedge [\exists h{:}\mathbf{Ar}](\langle f,g,h\rangle{:}\mathbf{Cp} \wedge \langle k,m,h\rangle{:}\mathbf{Cp}),$$

that is, that both the following diagrams commute:

$$\begin{tikzcd} a \arrow[d, "k"'] \arrow[dr, "h"] & \\ d \arrow[r, "m"'] & c \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

4. FUNCTORS

To define the category of categories the notion of *functor* from one category to another is needed. Its definition is given in the typical NaDSet style with the symbols $\mathbf{F}$, $\mathbf{Ar_C}$, $\mathbf{=_{aC}}$, $\mathbf{Sr_C}$, $\mathbf{Tg_C}$, $\mathbf{Cp_C}$, $\mathbf{Ar_D}$, $\mathbf{=_{aD}}$, $\mathbf{Sr_D}$, $\mathbf{Tg_D}$ and $\mathbf{Cp_D}$ used as metavariables ranging over second order terms.

$$\mathrm{Functor}[\mathbf{F},\langle \mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}\rangle,\langle \mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}\rangle]$$

for <u>axioms</u>

where "axioms" consists of the conjunction of the following formulas:

F is a map for categories

$$< \mathbf{Ar_C}, \mathbf{=a_C}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C} >\text{:Cat} \qquad \text{(f1)}$$

$$< \mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D} >\text{:Cat} \qquad \text{(f2)}$$

F maps arrows to arrows, preserving arrow identity

$$[\forall fc{:}\mathbf{Ar_C}][\exists fd{:}\mathbf{Ar_D}] <fc,fd>{:}\mathbf{F} \qquad \text{(f3)}$$

$$[\forall fc,gc{:}\mathbf{Ar_C}][\forall fd,gd{:}\mathbf{Ar_D}](\, fc \mathbf{=_{aC}} gc \wedge <fc,fd>{:}\mathbf{F} \wedge <gc,gd>{:}\mathbf{F} \supset fd \mathbf{=_{aD}} gd\,) \qquad \text{(f4)}$$

$$[\forall fc,gc{:}\mathbf{Ar_C}][\forall fd{:}\mathbf{Ar_D}](\, fc \mathbf{=_{aC}} gc \wedge <fc,fd>{:}\mathbf{F} \supset <gc,fd>{:}\mathbf{F}\,) \qquad \text{(f5)}$$

$$[\forall fc{:}\mathbf{Ar_C}][\forall fd,gd{:}\mathbf{Ar_D}](\, fd \mathbf{=_{aD}} gd \wedge <fc,fd>{:}\mathbf{F} \supset <fc,gd>{:}\mathbf{F}\,) \qquad \text{(f6)}$$

F preserves source, target and composition

$$[\forall fc,c{:}\mathbf{Ar_C}][\forall fd,d{:}\mathbf{Ar_D}]\,(\, <fc,c>{:}\mathbf{Sr_C} \wedge <fc,fd>{:}\mathbf{F} \wedge <c,d>{:}\mathbf{F} \supset <fd,d>{:}\mathbf{Sr_D}\,) \qquad \text{(f7)}$$

$$[\forall fc,c{:}\mathbf{Ar_C}][\forall fd,d{:}\mathbf{Ar_D}]\,(\, <fc,c>{:}\, \mathbf{Tg_C} \wedge <fc,fd>{:}\mathbf{F} \wedge <c,d>{:}\mathbf{F} \supset <fd,d>{:}\, \mathbf{Tg_D}\,) \qquad \text{(f8)}$$

$$[\forall fc1,fc2,fc3{:}\mathbf{Ar_C}]\, [\forall fd1,fd2,fd3{:}\mathbf{Ar_D}](< fc1,fc2,fc3>{:}\mathbf{Cp_C} \wedge$$
$$<fc1,fd1>{:}\mathbf{F} \wedge <fc2,fd2>{:}\mathbf{F} \wedge <fc3,fd3>{:}\mathbf{F} \supset < fd1,fd2,fd3>{:}\mathbf{Cp_D}\,) \qquad \text{(f9)}$$

Functors, following a suggestion of [Lawvere66], are defined as triples that include the source and target categories. The set of functors is defined:

$$\text{Func } \mathbf{for} \ \{ < F, <Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C>, <Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D> > \mid$$
$$\text{Functor}[F,<Ar_C, =_{aC}, Sr_C,Tg_C, Cp_C>,<Ar_D, =_{aD}, Sr_D,Tg_D, Cp_D>]\}$$

The set of functors from a category $<\mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}>$ to a category $<\mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}>$ is defined as

$$\text{Func}[<\mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}>, <\mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}>]$$

for

$\{x \mid \langle x, \langle Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C\rangle, \langle Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D\rangle\rangle{:}Func\}$.

In [Mac Lane71] and [Barr&Wells85] an additional axiom is included in the definition of functors; the axiom states that a functor must map identity arrows to identity arrows. But that axiom is not independent of the seven axioms given here. Since the identity arrows of a category are its objects, they can be defined by one of the three equivalent definitions given in section 3.1. The following lemma, whose proof can be found in [Gilmore & Tsiknis 90a] expresses that the additional axiom is entailed by the preceding functor definition.

4.1 Lemma The sequent

$\rightarrow [\forall x,y{:}Cat][\forall f{:}Func[x,y]][\forall c{:}Ar[x]][\forall d{:}Ar[y]](\, c{:}Id[x] \wedge \langle c,d\rangle{:}f \supset d{:}Id[y]\,)$

is derivable.

5. THE CATEGORY OF CATEGORIES

5.1 Definitions and Preliminaries

The category of categories is defined as the tuple $\langle \mathbb{Ar}, =_{\mathbb{a}}, \mathbb{Sr}, \mathbb{Tg}, \mathbb{Cp}\rangle$ of the second order terms $\mathbb{Ar}, =_{\mathbb{a}}, \mathbb{Sr}, \mathbb{Tg}, \mathbb{Cp}$ whose definitions are given in this section. Because of the great number of variables used in this section, some abbreviations similar to those used in the derivation of lemma 1, are again used here, and later: The capital letters A, B, C, D, E with or without subscripts, are used to abbreviate the tuples $\langle Ar_A, =_{aA}, Sr_A, Tg_A, Cp_A\rangle, \ldots,$ $\langle Ar_E, =_{aE}, Sr_E, Tg_E, Cp_E\rangle$ of the terms $Ar_A, =_{aA}, Sr_A, Tg_A$ and $Cp_A, \ldots, Ar_E, =_{aE}, Sr_E,$ Tg_E and Cp_E respectively. At different occasions these terms can be second order parameters, abstraction variables or metavariables that range over the second order terms. However, what the terms are to be in a particular context will be described prior to their use.

In the following definitions, the letters C and D, with or without subscripts, are abbreviations for the previously mentioned tuples of abstraction variables, while the letters F, G, H possibly subscripted, are regular abstraction variables.

A definition of the set $\mathbb{Ar}$ of arrows for the category of categories will be provided first; it is just the set of functors, as defined in section 4:

$\mathbb{Ar}$ **for** Func

The identity $=_\mathbf{a}$ for members of **Ar** is defined in terms of extensional identity.

$$=_\mathbf{a} \textbf{ for } \{<<F1, C1, D1>, <F2, C2, D2>> \mid C1 =_\mathbf{e} C2 \wedge D1 =_\mathbf{e} D2 \wedge F1 =_\mathbf{e} F2\ \}$$

In this definition $=_\mathbf{e}$ is the coordinate-wise extensional identity among tuples of terms defined by

$$=_\mathbf{e} \textbf{ for } \{<<Ar1, =_{a1}, Sr1, Tg1, Cp1>, <Ar2, =_{a2}, Sr2, Tg2, Cp2>> \mid$$
$$Ar1 =_e Ar2 \wedge =_{a1} =_e =_{a2} \wedge Sr1 =_e Sr2$$
$$\wedge\, Tg1 =_e Tg2 \wedge Cp1 =_e Cp2\ \}$$

where Ar1, ..., Cp2 are all being used as abstraction variables. The definition of extensional identity $=_e$ depends upon the context:

$$Ar1 =_e Ar2 \textbf{ for } [\forall f{:}Ar1]\, f{:}Ar2 \wedge [\forall f{:}Ar2]\, f{:}Ar1$$

$$=_{a1} =_e =_{a2} \textbf{ for } [\forall f,g{:}Ar1](f =_{a1} g \supset f =_{a2} g) \wedge [\forall f,g{:}Ar2](f =_{a2} g \supset f =_{a1} g)$$

$$Sr1 =_e Sr2 \textbf{ for } [\forall f,g{:}Ar1](<f, g>{:}Sr1 \supset <f, g>{:}Sr2) \wedge$$
$$[\forall f,g{:}Ar2](<f, g>{:}Sr2 \supset <f, g>{:}Sr1)$$

$$Tg1 =_e Tg2 \textbf{ for } [\forall f,g{:}Ar1](<f, g>{:}Tg1 \supset <f, g>{:}Tg2) \wedge$$
$$[\forall f,g{:}Ar2](<f, g>{:}Tg2 \supset <f, g>{:}Tg1)$$

$$Cp1 =_e Cp2 \textbf{ for } [\forall f,g,h{:}Ar1](<f, g,h>{:}Cp1 \supset <f, g,h>{:}Cp2) \wedge$$
$$[\forall f,g,h{:}Ar2](<f, g,h>{:}Cp2 \supset <f, g,h>{:}Cp1)$$

$$F1 =_e F2 \textbf{ for } [\forall f{:}Ar_{C1}][\forall g{:}Ar_{D1}](\ <f,g>{:}F1 \supset <f,g>{:}F2\ ) \wedge$$
$$[\forall f{:}Ar_{C2}][\forall g{:}Ar_{D2}](\ <f,g>{:}F2 \supset <f,g>{:}F1\ )$$

Clearly, the source and target of an arrow has to coincide with the identity functor of the source and target category, respectively. Their definitions follow, in a style similar to that of **Ar**.

$$\mathbf{Sr} \textbf{ for } \{<<F1, C1, D1>, <F2, C2, D2>> \mid$$
$$C2 =_\mathbf{e} C1 \wedge D2 =_\mathbf{e} C1 \wedge [\forall f,g{:}Ar_{C1}](\ <f,g>{:}F2 \equiv f =_{aC1} g\ )\ \}$$

Similarly,

$$\mathbf{Tg} \textbf{ for } \{<<F1, C1, D1>, <F2, C2, D2>> \mid$$
$$C2 =_\mathbf{e} D1 \wedge D2 =_\mathbf{e} D1 \wedge [\forall f,g{:}Ar_{D1}](\ <f,g>{:}F2 \equiv f =_{aD1} g\ )\ \}$$

The final definition needed is of $\mathbb{C}\mathrm{p}$, composition of the arrows for the category of categories.

$$\mathbb{C}\mathbf{p}\ \mathbf{for}\ \{ \langle\langle F1, C1, D1\rangle, \langle F2, C2, D2\rangle, \langle F3, C3, D3\rangle\rangle \mid$$
$$C1 =_e C3 \wedge D1 =_e C2 \wedge D2 =_e D3 \wedge$$
$$[\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)) \} .$$

The main goal of this section is to show that the set Cat with the defined constructs is itself a category. The existence of identity and composition functors must be shown first. The following notation will be used:

$$\mathbb{I}\mathrm{d}[C]\ \mathbf{for}\ =_{aC}$$

The next lemma insures that for any category (i.e., an element of Cat) there exists an identity functor from the category to itself.

5.2 Lemma

The sequent

$$\rightarrow [\forall x{:}Cat]\ \langle \mathbb{I}\mathrm{d}[x], x, x\rangle{:}\mathbb{A}\mathrm{r}$$

is derivable.

Proof Outline: If C is any tuple $\langle Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C\rangle$ of second order parameters, the lemma is obtained by an application of $\rightarrow\forall$ to the sequent

$$C{:}Cat \rightarrow \langle \mathbb{I}\mathrm{d}[C], C, C\rangle{:}\mathbb{A}\mathrm{r}$$

whose derivation is obtained as following.

Let Ax[G,A,B] be the result of replacing **F** by G, $\langle \mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}\rangle$ by A, and $\langle \mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}\rangle$ by B in an axiom of (f1) to (f9). From the definition of $\mathbb{A}\mathrm{r}$, it is obvious that a proof of the last sequent follows from a derivation of the sequent

$$C{:}Cat \rightarrow \mathrm{Functor}[\ \mathbb{I}\mathrm{d}[C], C, C\ ]$$

by a single application of $\rightarrow\{\}$. The latter derivation can in turn be obtained if a derivation for the sequent

$$C{:}Cat \rightarrow Ax[\ \mathbb{I}\mathrm{d}[C], C, C\ ] \tag{L1}$$

is provided, when Ax[-,-,-] is in turn each of the axioms (f1) to (f9). Derivations of the sequents (L1) are lengthy and for space reason they are omitted. The interested reader can find the more

difficult ones in the proof of lemma 5.2.1 in [Gilmore & Tsiknis 90a].
End of proof

For the next lemma, the following definition of the composition of two functors is required:

FC[F1,C1,D1,F2,C2,D2] for

$$\{<f,g> \mid [\exists h:\mathbf{Ar_{D1}}](<f,h>:\mathbf{F1} \wedge <h, g>:\mathbf{F2})\}$$

The lemma states that if two functors are composable, their composite is also a functor.

5.3 Lemma

The sequent

$$\rightarrow [\forall f,g:\mathrm{Func}][\forall b,c,d,e:\mathrm{Cat}](<f,b,c>:\mathrm{Ar} \wedge <g,d,e>:\mathrm{Ar} \wedge c=_{\mathbf{e}}d$$
$$\supset <\mathrm{FC}[f,b,c,g,d,e], b, e>:\mathrm{Ar})$$

is derivable.

Proof Outline: If F1, F2 are second order parameters and C1, D1, C1, D1 are the usual tuples of second order parameters, the lemma can be obtained from the sequent

$$<F1,C1,D1>:\mathrm{Ar},\ <F2,C2,D2>:\mathrm{Ar},\ D1=_{\mathbf{e}}C2$$
$$\rightarrow <\mathrm{FC}[F1,C1,D1,F2,C2,D2], C1, D2>:\mathrm{Ar}$$

by successive applications of the $\rightarrow\forall$ rule. The last sequent can be derived from the sequent

$$\mathrm{Functor}[F1,C1,D1],\ \mathrm{Functor}[F2,C2,D2]\ D1=_{\mathbf{e}}C2$$
$$\rightarrow \mathrm{Functor}[\mathrm{FC}[F1,C1,D1,F2,C2,D2], C1, D2]$$

by two applications of $\{\}\rightarrow$ and one of $\rightarrow\{\}$.

From the definition of functor, a derivation of the latter sequent can be obtained if a derivation for the sequent

$$\mathrm{Ax}[F1,C1,D1],\ \mathrm{Ax}[F2,C2,D2]\ D1=_{\mathbf{e}}C2$$
$$\rightarrow \mathrm{Ax}[\mathrm{FC}[F1,C1,D1,F2,C2,D2], C1, D2] \qquad \text{(L2)}$$

is provided for each of the axioms (f1) to (f9), where Ax[-,-,-] has the same meaning as in the proof of lemma 5.2. The more difficult derivations are provided in the proof of lemma 5.3.1 of [Gilmore&Tsiknis 90a].
End of proof

The main theorem of the paper together with an outline of its proof is presented next.

5.4 Theorem

The sequent

$\rightarrow$ < Ar, $=_a$, Sr, Tg, Cp>:Cat

is derivable within NaDSet.

Proof Outline: A derivation of $\rightarrow$ <Ar,$=_a$,Sr,Tg,Cp>:Cat can be obtained from a derivation of $\rightarrow$ Category[Ar,$=_a$,Sr,Tg,Cp] by one application of $\rightarrow${ } rule and the definition of Cat. To derive the latter sequent it is necessary to provide a derivation of each sequent of the form

$\rightarrow$ Ax[Ar, $=_a$, Sr, Tg, Cp] (T1)

where Ax[-,-,-,-,-] is one of the axioms c1 to c20.

Lemma 5.2 provides a term for the existential quantifier in (c11) and (c13) while (c15) uses lemma 5.3. The derivations of the remaining sequents (T1) are lengthy, tedious applications of the definitions and the properties of the extensional identity and are omitted for space reasons. The interested reader is refered to the proof of theorem 5.4.1 of [Gilmore & Tsiknis 90] for a detailed proof of the theorem.
End of proof

6. NATURAL TRANSFORMATIONS and FUNCTOR CATEGORIES

As Eilenberg and Mac Lane observed [MacLane71], "category" has been defined in order to define "functor", and "functor" has been defined in order to define "natural transformation". This notion induces an equivalence relation between categories that allows the comparison of categories that are "alike" but of different "sizes". Moreover, natural transformation is the basic ingredient in the ubiquitous construction of functor categories.

6.1 Natural Transformations

We now proceed with a NaDSet definition of a natural transformation from one functor to another. In this, **T, F, G**, $\mathbf{Ar_B}$, $\mathbf{=_{aB}}$, $\mathbf{Sr_B}$, $\mathbf{Tg_B}$, $\mathbf{Cp_B}$, $\mathbf{Ar_C}$, $\mathbf{=_{aC}}$, $\mathbf{Sr_C}$, $\mathbf{Tg_C}$, $\mathbf{Cp_C}$, $\mathbf{Ar_D}$, $\mathbf{=_{aD}}$, $\mathbf{Sr_D}$, $\mathbf{Tg_D}$, $\mathbf{Cp_D}$ are used as metavariables ranging over second order terms, while **B, C, D** are used as abbreviations of the tuples < $\mathbf{Ar_B}$, $\mathbf{=_{aB}}$, $\mathbf{Sr_B}$, $\mathbf{Tg_B}$, $\mathbf{Cp_B}$ >, <$\mathbf{Ar_C}$, $\mathbf{=_{aC}}$, $\mathbf{Sr_C}$, $\mathbf{Tg_C}$,

$\mathbf{Cp_C}$> and <$\mathbf{Ar_D}$, $\mathbf{=_{aD}}$, $\mathbf{Sr_D}$, $\mathbf{Tg_D}$, $\mathbf{Cp_D}$> respectively.

As with categories and functors, the set of natural transformations is defined in two steps:

NatTransform[**T, F, G, C, D**] **for** axioms

where "axioms" consist of the conjunction of the following formulas:

T is a map for functors

$$\langle \mathbf{F, C, D} \rangle\text{:Func} \tag{t1}$$

$$\langle \mathbf{G, C, D} \rangle\text{:Func} \tag{t2}$$

T is a function from objects in C to arrows in D

$$[\forall c\text{:Ob}[\mathbf{C}]]\ [\exists tc\text{:}\mathbf{Ar_D}]\ \langle c,tc\rangle\text{:}\mathbf{T} \tag{t3}$$

$$[\forall c\text{:Ob}[\mathbf{C}]]\ [\forall tc\text{:}\mathbf{Ar_D}](\langle c,tc\rangle\text{:}\mathbf{T} \supset [\exists fc,gc\text{:}\mathbf{Ar_D}](\langle c,fc\rangle\text{:}\mathbf{F} \wedge \langle c,gc\rangle\text{:}\mathbf{G} \wedge \langle tc,fc\rangle\text{:}\mathbf{Sr_D} \wedge \langle tc,gc\rangle\text{:}\mathbf{Tg_D})) \tag{t4}$$

$$[\forall c1,c2\ \text{Ob}[\mathbf{C}]]\ [\forall tc1,tc2\text{:}\mathbf{Ar_D}](c1 \mathbf{=_{aC}} c2 \wedge \langle c1,tc1\rangle\text{:}\mathbf{T} \wedge \langle c2,tc2\rangle\text{:}\mathbf{T} \supset tc1 \mathbf{=_{aD}} tc2) \tag{t5}$$

$$[\forall c1,c2\text{:Ob}[\mathbf{C}]]\ [\forall tc\text{:}\mathbf{Ar_D}](c1 \mathbf{=_{aC}} c2 \wedge \langle c1,tc\rangle\text{:}\mathbf{T} \supset \langle c2,tc\rangle\text{:}\mathbf{T}) \tag{t6}$$

$$[\forall c\text{:Ob}[\mathbf{C}]]\ [\forall tc1,tc2\text{:}\mathbf{Ar_D}](tc1 \mathbf{=_{aD}} tc2 \wedge \langle c,tc1\rangle\text{:}\mathbf{T} \supset \langle c,tc2\rangle\text{:}\mathbf{T}) \tag{t7}$$

For every arrow $\begin{array}{c} c1 \\ {\scriptstyle h}\big\downarrow \\ c2 \end{array}$ of C, the diagram $\begin{tikzcd} Fc1 \arrow[r, "Tc1"] \arrow[d, "Fh"'] & Gc1 \arrow[d, "Gh"] \\ Fc2 \arrow[r, "Tc2"'] & Gc2 \end{tikzcd}$ commutes

$$[\forall c1,c2\ \text{Ob}[\mathbf{C}]]\ [\forall h\text{:}\mathbf{Ar_C}]\ [\forall tc1,tc2,fh,gh\text{:}\mathbf{Ar_D}](\langle h,c1\rangle\text{:}\mathbf{Sr_C} \wedge \langle h,c2\rangle\text{:}\mathbf{Tg_C} \wedge \langle c1,tc1\rangle\text{:}\mathbf{T} \wedge \langle c2,tc2\rangle\text{:}\mathbf{T} \wedge \langle h,fh\rangle\text{:}\mathbf{F} \wedge \langle h,gh\rangle\text{:}\mathbf{G} \supset [\exists k\text{:}\mathbf{Ar_D}](\langle tc1,gh,k\rangle\text{:}\mathbf{Cp_D} \wedge \langle fh,tc2,k\rangle\text{:}\mathbf{Cp_D})) \tag{t8}$$

The set of natural transformations is defined:

NatTrans **for** { <t,f,g,c,d>| NatTransform[t,f,g,c,d] }

Given the functors **F**, **G**: **C**→ **D**, the sets of natural transformations from **F** to **G** can now be defined:

NatTrans[**F**, **G**, **C**, **D**] **for** {t | <t, **F**, **G**, **C**, **D** >:NatTrans }

6.2 Natural Equivalence

A natural transformation is a *natural isomorphism* (or a *natural equivalence*) if each component of it is an isomorphism in the target category:

NatIsomorphism [**F**, **G**, **C**, **D**] **for**
{ t | t:NatTrans[**F**, **G**, **C**, **D**]
∧ [∀c:Ob[**C**]] [∀tc,d1,d2:$\mathbf{Ar_D}$] (<c,tc>:t ∧ <tc,d1>:$\mathbf{Sr_D}$ ∧ <tc,d2>:$\mathbf{Tg_D}$
⊃ [∃h:$\mathbf{Ar_D}$](<tc,h,d1>:$\mathbf{Cp_D}$ ∧ <h,tc,d2>:$\mathbf{Cp_D}$)) }

Given two categories **C** and **D**, the equivalence relation among functors from **C** to **D** is given by:

NatEq[**C**, **D**] **for** {<f,g> | [∃t:NatTrans[f, g, **C**, **D**] t:NatIsomorphism[f, g, **C**, **D**] }

An equivalence relation ≡ between categories that meets the requirements mentioned at the beginning of the section, can be given by the following definition in which C, D are used as tuples of abstraction variables.

≅ **for** { <C,D> | [∃F:Func[C,D]] [∃G:Func[D,C]] (
< $\mathbb{FC}$[F,C,D,G,D,C], $\mathbb{Id}$[C] >: NatEq[C, C]
∧ < $\mathbb{FC}$[G,D,C,F,C,D], $\mathbb{Id}$[D] >: NatEq[D, D]) }

where $\mathbb{Id}$[_] and $\mathbb{FC}$[_,_,_,_,_,_] are the terms defined prior to lemmas 5.2 and 5.3 respectively.

6.3 Functor Categories

If **C** and **D** are caregories, the category of functors --*functor category* -- from **C** to **D**, denoted by $\mathbf{D^C}$ or FunCat[**C**,**D**], is defined as the tuple

$\mathbf{D^C}$ **for** <*Ar*[C,D], $=_a$[C,D], *Sr*[C,D], *Tg*[C,D], *Cp*[C,D]>

of the parameterized terms *Ar*[C,D], $=_a$[C,D], *Sr*[C,D], *Tg*[C,D], *Cp*[C,D] whose definitions follow.

Obviously, the arrows of this category are the natural transformations among functors from **C** to **D**.

The reader should note that the objects of this category are the functors themselves. Thus we define

$\boldsymbol{Ar}$ **[C,D] for** $\{\langle T,F,G\rangle \mid \text{NatTransform}[\,\mathbf{T}, \mathbf{F}, \mathbf{G}, \mathbf{C}, \mathbf{D}\,]\,\}$

The identity among the members of $\boldsymbol{Ar}$ [C,D] is defined in terms of the extensional identity.

$$=_a [\mathbf{C},\mathbf{D}]\ \textbf{for}\ \{\langle\langle T1,F1,G1\rangle, \langle T2,F2,G2\rangle\rangle \mid F1 =_e F2 \wedge G1 =_e G2 \wedge T1 =_e T2\ \}.$$

The identity $=_e$ for the terms that represent functors (F's and G's) was defined in section 5; it only remains to give its definition for the terms representing natural transformations:

$$\mathbf{T1} =_e \mathbf{T2}\ \textbf{for}\ [\forall c{:}Ob[\underline{C}]]\,[\forall d{:}\mathbf{Ar_D}](\ \langle c,d\rangle{:}T1 \equiv \langle c,d\rangle{:}T2\ ).$$

The source and the target of an arrow coincides with the source and the target functors of the transformation which are viewed as identity natural transformations. Consequently we define

$$\boldsymbol{Sr}\,[\mathbf{C},\mathbf{D}]\ \textbf{for}\ \{\langle\langle T1,F1,G1\rangle, \langle T2,F2,G2\rangle\rangle \mid T2 =_e F1 \wedge F2 =_e F1 \wedge G2 =_e F1\ \}$$

and

$$\boldsymbol{Tg}\,[\mathbf{C},\mathbf{D}]\ \textbf{for}\ \{\langle\langle T1,F1,G1\rangle, \langle T2,F2,G2\rangle\rangle \mid T2 =_e G1 \wedge F2 =_e G1 \wedge G2 =_e G1\ \}.$$

Finally, the composition of

$$\begin{aligned}\boldsymbol{Cp}\,[\mathbf{C},\mathbf{D}]\ \textbf{for}\ \{&\langle\langle T1,F1,G1\rangle, \langle T2,F2,G2\rangle, \langle T3,F3,G3\rangle\rangle \mid \\ &F1 =_e F3 \wedge G =_e F2 \wedge G2 =_e G3 \\ &\quad \wedge [\forall c{:}Ob[C]]\,[\forall d{:}\mathbf{Ar_D}](\ \langle c,d\rangle{:}T3 \\ &\qquad \equiv [\exists d1,d2{:}\mathbf{Ar_D}](\langle c,d1\rangle{:}T1 \wedge \langle c,d2\rangle{:}T2 \wedge \langle d1,d2,d\rangle\mathbf{Cp_D}))\ \}.\end{aligned}$$

The sequent of the following theorem states that for any categories C, D, the set of functors from C to D is itself a category.

6.3.1 Theorem

The sequence

$$\rightarrow\ [\forall x,y{:}Cat]\ \langle \boldsymbol{Ar}\,[x,y],\ =_a [x,y],\ \boldsymbol{Sr}\,[x,y],\ \boldsymbol{Tg}\,[x,y],\ \boldsymbol{Cp}\,[x,y]\rangle{:}Cat$$

is derivable within NaDSet.

A derivation of the theorem can be obtained if a derivation is provided for each sequence of the form

$$\mathrm{Ax}[\mathrm{Ar}_C, =_{aC}, \mathrm{Sr}_C, \mathrm{Tg}_C, \mathrm{Cp}_C],\ \ \mathrm{Ax}[\mathrm{Ar}_D, =_{aD}, \mathrm{Sr}_D, \mathrm{Tg}_D, \mathrm{Cp}_D]$$

$$\rightarrow \mathrm{Ax}[\,\boldsymbol{Ar}\,[C,D], =_{\boldsymbol{a}}[C,D], \boldsymbol{Sr}\,[C,D], \boldsymbol{Tg}\,[C,D], \boldsymbol{Cp}\,[C,D]\,]$$

where $\mathrm{Ar}_C, =_{aC}, \mathrm{Sr}_C, \mathrm{Tg}_C, \mathrm{Cp}_C, \mathrm{Ar}_D, =_{aD}, \mathrm{Sr}_D, \mathrm{Tg}_D, \mathrm{Cp}_D$ are second order parameters, C and D are the tuples $<\mathrm{Ar}_C, =_{aC}, \mathrm{Sr}_C, \mathrm{Tg}_C, \mathrm{Cp}_C>$, $<\mathrm{Ar}_D, =_{aD}, \mathrm{Sr}_D, \mathrm{Tg}_D, \mathrm{Cp}_D>$ and Ax[-,-,-,-,-] is one of the axioms (c1) to (c20). The latter derivations are similar (in structure as well as in length) to those in the proof of theorem 5.4 and are omitted for space reasons.

7. OTHER CONSTRUCTIONS

7.1 Opposites

To each category **C**, we assosiate the *opposite* category, $\mathbf{C}^{\mathbf{op}}$, defined to be the term $<\mathbf{Ar}_{\mathbf{C}}, =_{\mathbf{C}}, \mathrm{Sr}^{op}[\mathbf{C}], \mathrm{Tg}^{op}[\mathbf{C}], \mathrm{Cp}^{op}[\mathbf{C}]>$ with components:

$\mathrm{Sr}^{op}[\mathbf{C}]$ **for** $\{<u,v> \mid <u,v>:\mathbf{Tg}_{\mathbf{C}}\}$

$\mathrm{Tg}^{op}[\mathbf{C}]$ **for** $\{<u,v> \mid <u,v>:\mathbf{Sr}_{\mathbf{C}}\}$

$\mathrm{Cp}^{op}[\mathbf{C}]$ **for** $\{<u,v,g> \mid <u,v,g>:\mathbf{Cp}_{\mathbf{C}}\}$

7.1.1 Lemma

The sequents

$\rightarrow [\forall x:\mathrm{Cat}]\ x^{op}:\mathrm{Cat}$

$\rightarrow [\forall x:\mathrm{Cat}]\ (x^{op})^{op} =_{\mathbf{e}} x$

are derivable.

7.2. Product Categories

Given two categories **B** and **C**, the product of them, **BxC**, is defined to be the term $<\mathrm{Ar}^x[\mathbf{B},\mathbf{C}], =^x[\mathbf{B},\mathbf{C}], \mathrm{Sr}^x[\mathbf{B},\mathbf{C}], \mathrm{Tg}^x[\mathbf{B},\mathbf{C}], \mathrm{Cp}^x[\mathbf{B},\mathbf{C}]>$ with components:

$\mathrm{Ar}^x[\mathbf{B},\mathbf{C}]$ **for** $\{<u,v> \mid u:\mathbf{Ar}_{\mathbf{B}} \wedge v:\mathbf{Ar}_{\mathbf{C}}\}$

$=^x$[B,C] **for** {<<u,v>,<f,g>> | <u,f>: $=_{aB}$ ∧ <v,g>: $=_{aC}$ }

Sr^x[B,C] **for** { <<u,v>,<f,g>> | <u,f>:Sr_B ∧ <v,g>:Sr_C }

Tg^x[B,C] **for** { <<u,v>,<f,g>> | <u,f>:Tg_B ∧ <v,g>:Tg_C }

Cp^x[B,C] **for** { <<u1,v1>,<u2,v2>,<f,g>> | <u1,u2,f>:Cp_B ∧ <v1,v2,g>:Cp_C }.

Given two functors **F** and **G** their product, **FxG** is given by:

FxG for {<<u,v>,<f,g>> | <u,f>:**F** ∧ <v,g>:**G** } .

7.2.1. Lemma

The sequents

→ [∀w,z:Cat] wxz:Cat

→ [∀w1,w2,z1,z2:Cat][∀f:Func[w1,z1]][∀g:Func[w2,z2]]
fxg:Func[w1xw2, z1xz2]

are derivable.

7.3. Comma Categories

If **B**,**C** and **D** are categories and **F**:C→B, **G**:**D**→B functors, the *comma* category (**F**,**G**) is defined to be the term

<Ar'[F,G,B,C,D], =' [F,G,B,C,D], Sr'[F,G,B,C,D], Tg'[F,G,B,C,D], Cp'[F,G,B,C,D] >

with components:

Ar'[F,G,B,C,D] **for** {<u,v,w,x> | u:Ar_C ∧ v:Ar_D ∧ w:Ar_B ∧ x:Ar_B
∧ [∃f,g,h:Ar_B](<u,f>:**F** ∧ <v,g>:**G** ∧ <f,w,h>:Cp_B ∧ <x,g,h>:Cp_B)}

=' [F,G,B,C,D] **for** {<<u1,v1,w1,x1>,<u2,v2,w2,x2>> |
<u1,u2>: $=_{aC}$ ∧ <v1,v2>: $=_{aD}$ ∧ <w1,w2>: $=_{aB}$ ∧ <x1,x2>: $=_{aB}$ }

Sr'[F,G,B,C,D] **for** { <<u1,v1,w1,x1>,<u2,v2,w2,x2>> |
<u1,u2>:Sr_C ∧ <v1,v2>:Sr_D ∧ <w1,w2>: $=_{aB}$ ∧ <w1,x2>: $=_{aB}$ }

Tg'[F,G,B,C,D] **for** { <<u1,v1,w1,x1>,<u2,v2,w2,x2>> |
<u1,u2>:Tg_C ∧ <v1,v2>:Tg_D ∧ <x1,w2>: $=_{aB}$ ∧ <x1,x2>: $=_{aB}$ }

Cp'[F,G,B,C,D] **for** { <<u1,v1,w1,x1>,<u2,v2,w2,x2>, <u3,v3,w3,x3>> |
<u1,u3>: $=_{aB}$ ∧ <v1,u2>: $=_{aB}$ ∧ <v2,v3>: $=_{aB}$
∧ <u1,u2,u3>:Cp_C ∧ <v1,v2,v3>:Cp_D } .

The meticulous reader will have already noticed in the last definition a slight deviation from the traditional one. The arrows of a comma category, according to the above definition, are quadruples instead of pairs. Although such a deviation is immaterial (it only affects the representation of the construct not its properties), it has been found necessary to avoid the explicit use of objects and Hom-sets. Nevertheless, it can be shown that a triple <e,d,f> is an object of (F,G) as defined in [MacLane 71] iff <e,d,f,f> is an object of (F,G) according to our definition. Moreover, an arrow <k,h> : <e,d,f> → <e',d',f'> in [MacLane 71] is exactly the arrow <k,h,f,f'> in our definition. The difference is that in the first case an arrow cannot be determined by the pair <k,h> alone without explicitly giving its source and target, while in our presentation the tuple <k,h,f,f'> uniqely determines an arrow in (F,G).

7.3.1. Lemma

The sequent

$$\rightarrow\ [\forall x,y,z{:}\mathrm{Cat}][\forall f{:}\mathrm{Func}[x,y]][\forall g{:}\mathrm{Func}[z,y]]\ \ (f,g){:}\mathrm{Cat}$$

is derivable.

7.4. Universals and Limits

To improve readability, in the next two sections additional abbreviations will be used that resemble the functional notation used in mathematics. Specifically, if **F** is a functor (or transformation) from **B** to **C** then

$$\mathbf{F[x]_C}\ \ \textbf{for}\ \{y \mid y{:}\mathbf{Ar_C} \wedge \langle \mathbf{x},y\rangle{:}\mathbf{F}\},$$

$$[\mathbf{y}\rightarrow\mathbf{z}]_\mathbf{C}\ \ \textbf{for}\ \{w \mid w{:}\mathbf{Ar_C} \wedge \langle w,\mathbf{y}\rangle{:}\mathbf{Sr_C} \wedge \langle w,\mathbf{z}\rangle{:}\mathbf{Tg_C}\},$$

and combining them

$$[\mathbf{y}\rightarrow\mathbf{F[x]}]_\mathbf{C}\ \ \textbf{for}\ \{w \mid w{:}\mathbf{Ar_C} \wedge \langle w,\mathbf{y}\rangle{:}\mathbf{Sr_C} \wedge [\exists z{:}\mathbf{F[x]_C}]\langle w,z\rangle{:}\mathbf{Tg_C}\}.$$

Similar definition can be given for $[\mathrm{F}[y]\rightarrow x]_\mathrm{C}$ and $[\mathrm{F}[y]\rightarrow \mathrm{F}[x]]_\mathrm{C}$. We can proceed now with the definition of universal arrows.

Given a functor **F**:**D**→C and an object **c** of **C**, the following term defines the set of *universal arrows* from **c** to **F**.

$$\mathrm{UniArrFrom}[\mathbf{F},\mathbf{D},\mathbf{C},\mathbf{c}]\ \ \textbf{for}$$

$$\{\langle r,u\rangle \mid r{:}\mathrm{Ob}[\mathbf{D}] \wedge u{:}[\mathbf{c}\rightarrow\mathbf{F}[r]]_\mathbf{C}$$

$\wedge$ [$\forall$d:Ob[**D**]] [$\forall$g:[c$\rightarrow$**F**[d]]$_{\mathbf{C}}$] [$\exists$g1:[r$\rightarrow$d]$_{\mathbf{D}}$] [$\exists$fg1:**F**[g1]$_{\mathbf{C}}$](<u,fg1,g>:$\mathbf{Cp_C}$

$\wedge$ [$\forall$g2:[r$\rightarrow$d]$_{\mathbf{D}}$] [$\forall$fg2:**F**[g2]$_{\mathbf{C}}$](<u,fg2,g>:$\mathbf{Cp_C}$ $\supset$ g1=$_{\mathbf{aD}}$g2)) }

By duality, the set of universal arrows from the functor **F** to an object **c** is given by:

UniArrTo[F,D,C,c] for

{<r,u> | r:Ob[**D**] $\wedge$ u:[**F**[r]$\rightarrow$**c**]$_{\mathbf{C}}$

$\wedge$ [$\forall$d:Ob[**D**]] [$\forall$g:[**F**[d]$\rightarrow$**c**]$_{\mathbf{C}}$] [$\exists$g1:[d$\rightarrow$r]$_{\mathbf{D}}$] [$\exists$fg1:**F**[g1]$_{\mathbf{C}}$](<fg1,u,g>:$\mathbf{Cp_C}$

$\wedge$ [$\forall$g2:[d$\rightarrow$r]$_{\mathbf{D}}$] [$\forall$fg2:**F**[g2]$_{\mathbf{C}}$](<fg2,u,g>:$\mathbf{Cp_C}$ $\supset$ g1=$_{\mathbf{aD}}$g2)) }

A definition of the diagonal functor must preceed a discusion of limits and colimits. In the following definitions **B** and **C** are categories, **c** an object of **C** and **f** an arrow of **C**:

DF[**B**,**C**,**c**] **for** {<u,v> | u:$\mathbf{Ar_B}$ $\wedge$ v=$_{\mathbf{aC}}$**c** }

DT[**B**,**C**,**f**] **for** { <u,v> | u:Ob[**B**] $\wedge$ v=$_{\mathbf{aC}}$**f** } .

The diagonal functor from **C** to $\mathbf{C^B}$ is defined as

Δ[**B**,**C**] **for** {< u,y> | u:$\mathbf{Ar_C}$ $\wedge$ y =$_e$ DT[**B**,**C**, u] }

The following lemma justifies these definitions:

7.4.1. Lemma

The sequences

$\rightarrow$ [$\forall$j,x:Cat][$\forall$c:Ob[x]] DF[j,x,c]:Func[j,x]

$\rightarrow$ [$\forall$j,x:Cat] [$\forall$c,c':Ar$_x$] [$\forall$f:[c$\rightarrow$c']$_x$]

DT[j,x,f]:NatTrans[DF[j,x,c], DF[j,x,c'],j,x]

$\rightarrow$ [$\forall$j,x:Cat] Δ[j,x]:Func[x,x^j]

are derivable.

Definitions of limits and colimits can now be given. Given a functor **F**:**B**$\rightarrow$**C**, the *limits* for **F** are given by

Limit[F,B,C] for { <u,v> | <u,v>:UniArrowTo[Δ[**B**,**C**], **C**, $\mathbf{C^B}$, **F**] }

and the *colimits* of F by

Colimit[F,B,C] for { <u,v> | <u,v>:UniArrowFrom[Δ[**B**,**C**], **C**, $\mathbf{C^B}$, **F**] } .

Products, powers, equalizers, pullbacks and their duals can easily be defined as special cases of limits and colimits respectively.

7.5. Adjoints

Given two categories **C**, **D**, an *adjunction* from **C** to **D** consists of a pair of functors **F**:**C**→**D**, **G**:**D**→**C** and a natural transformation η from the identity functor of **C** to the composition of **F** and **G** with some additional properties given by the folowing definition.

Adjunction[**C**,**D**,**F**,**G**,η] **for**
F:Func[**C**,**D**] ∧ **G**:Func[**D**,**C**]
∧ η:NatTrans[𝕀d[**C**], 𝔽ℂ[**F**,**C**,**D**,**G**,**D**,**C**], **C**, **C**]
∧ [∀x:Ob[**C**]] [∀y:Ob[**D**]] [∀f:[x→**G**[y]]$_{\mathbf{C}}$] [∃ηx:η[x]$_{\mathbf{C}}$]
[∃f1:[**F**[x]→y]$_{\mathbf{D}}$] [∃gf1:**G**[f1]$_{\mathbf{C}}$] (<ηx,gf1,f>:**Cp**$_{\mathbf{C}}$
∧ [∀f2:[**F**[x]→y]$_{\mathbf{D}}$] [∀gf2:**G**[f2]$_{\mathbf{C}}$] (<ηx,gf2,f>:**Cp**$_{\mathbf{C}}$ ⊃ f1=$_{\mathbf{aD}}$f2))

or equivalently,

Adjunction[**C**,**D**,**F**,**G**,η] **for**
F:Func[**C**,**D**] ∧ **G**:Func[**D**,**C**]
∧ η:NatTrans[𝕀d[**C**], 𝔽ℂ[**F**,**C**,**D**,**G**,**D**,**C**], **C**, **C**]
∧ [∀x:Ob[**C**]] [∃fx:**F**[x]$_{\mathbf{D}}$] [∃ηx:η[x]$_{\mathbf{C}}$] <fx,ηx>:UniArrFrom[**G**,**D**,**C**,x]

Finally, the set of *adjoint pairs* of functors from **C** to **D** is defined as

Adjoint[**C**,**D**] **for**

{<f,g> | [∃η:NatTrans[𝕀d[**C**], 𝔽ℂ[f,**C**,**D**, g,**D**,**C**], **C**, **C**]] Adjunction[**C**,**D**, f, g,η] }.

8. FUTURE DIRECTIONS

Formalizations for most of the main concepts and constructs in category theory have been presented. NaDSet definitions for natural transformations, functor categories, an equivalence relation on categories, products, comma categories, universals limits, adjoints and some related theorems have been provided . This suggests that the variety of constructs defined for categories, toposes, triples and related theories, [Barr Wells 85] can be defined within NaDSet.

There are two kinds of issues that have not been addressed in this paper. The first concerns the definition of the category of sets per se and the second involves notions like completeness that make either an implicitl or an explicit reference to the traditional foundations of the category theory.

Although these issues are topics of future research, some preliminary ideas and directions are presented in the following paragraphs.

In a traditional presentation of category theory [MacLane 71], the category of sets, **Set**, is defined to be a category whose objects consist of every object that a classical set theory accepts as set and whose arrows are the mappings among these sets. In NaDSet, however, set abstraction is introduced via abstraction rules, rather than through a comprehension axiom scheme, and enjoys an equal treatment with the connectives and quantifiers. As a consequence, NaDSet provides a characterization of sound arguments, and not a characterization of acceptable sets. Nevertheless, the category **Set** should be definable within NaDSet. Section 8 of [Gilmore 89] provides a NaDSet formalization of Gödel-Bernays set theory within which every theorem of Gödel-Bernays theory can be derived. Using the formalization, it should be possible to define a term representing the class of Gödel-Bernays sets; the class of mappings among these sets then can be defined as the set of triples with elements the domain, co-domain, and the extension of the mapping. Among the remaining components of **Set**, target, source, and composition should have the expected definitions while arrow identity is taken to be the extensional identity over the mappings. It is expected that the traditional categorical constructions that are related to **Set**, such as hom-functors, functor representations and the Yoneda construction, can also be developed within NaDSet.

Set is not the only meaningful category of sets that can be defined within NaDSet. There are two identity relations definable in the theory: the intensional identity defined by

$$= \textbf{ for } \{\langle u,v\rangle \mid [\forall z](u{:}z \supset v{:}z)\}$$

and the extensional identity

$$=_e \textbf{ for } \{\langle u,v\rangle \mid [\forall z](z{:}u \equiv z{:}v)\} .$$

Each one of them defines a different 'universal' set :

$$V1 \textbf{ for } \{u \mid u=u\}$$

$$V2 \textbf{ for } \{u \mid u=_e u\} .$$

Any one of V1, V2 and $V1 \cap V2$ can be used as the object component of a category of sets **V1Set**, **V2Set**, and **V12Set**, respectively. It should be possible to show that **Set** is a subcategory of **V1Set**. What properties each of them has and whether the classical constructions on **Set** can be carried over to these categories, remains to be seen.

An analogous treatment may given to the second group of issues. The concept of completeness is taken to illustrate the main idea. A category is said to be small if its objects and arrows are Gödel-Bernays sets. Traditionally, a category C is called small-complete if every functor from a small category J to C has a limit [MacLane 71]. Such a notion can be defined in NaDSet given the

NaDSet definition of Gödel-Bernays sets. Moreover it is believed that classical results such as Freyd's proposition (that a small category which is small-complete is a preorder) and Freyd's Adjoint Functor Theorem [MacLane 71] can be proved in this framework. Nonetheless, a more general notion of completeness can be defined in NaDSet. Let R be a unary relation on Cat definable within NaDSet, and define an R-category to be one that satisfies R. A category C is called R-complete if every functor from an R-category to C has a limit. This leads naturally to the question: Does there exist other interesting special cases of R-completeness besides the small-completeness case?

Another class of issues is addressed in [Gilmore&Tsiknis90c].

REFERENCES

The numbers in parentheses refer to the date of publication. (xx) is the year 19xx.

Barr, M. & Wells, C.

(85) Toposes, Triples and Theories, Springer-Verlag

Blass, Andreas

(84) The Interaction between Category Theory and Set Theory, in Mathematical Applications of Category Theory, J.W. Gray editor, *Contemporary Mathematics*, 30, American Mathematical Society, 5-29.

Feferman, Solomon

(77) Categorical Foundations and Foundations of Category Theory, Logic, Foundations of Mathematics and Computability Theory, Editors Butts and Hintikka, D. Reidel, 149-169.

(84) Towards Useful Type-Free Theories, I, *Journal of Symbolic Logic*, March, 75-111.

Gentzen, Gerhard

(34,35) Untersuchungen über das logische Scliessen, *Mathematische Zeitschrift*, 39, 176-210, 405-431.

Gilmore, P.C. (Paul C.),

(86) Natural Deduction Based Set Theories: A New Resolution of the Old Paradoxes, *Journal of Symbolic Logic*, 51, 393-411.

(89) How Many Real Numbers are There?, Department of Computer Science Technical Report TR 89-7, University of British Columbia. Revised July 1990.

Gilmore, Paul C. & Tsiknis George K.

(90a) A Logic for Category Theory, Department of Computer Science Technical Report TR 90-2, University of British Columbia.

(90b) Logical Foundations for Programming Semantics, a paper presented to the Sixth Workshop on the Mathematical Foundations of Programming Semantics, Kingston, Ontario, Canada, May 15-19. Department of Computer Science Technical Report TR 90-22, University of British Columbia.

(90c) Function Spaces in NaDSet, in preparation.

Mac Lane, Saunders

(71) Categories for the Working Mathematician, Springer-Verlag

Lawvere, F.W.

(66) The Category of Categories as a Foundation for Mathematics. Proc. Conf. on Categorical Algebras, pp1-21, Springer

Tarski, Alfred

(56) The Concept of Truth in Formalized Languages appearing in Logic, Semantics, Metamathematics, Papers from 1923 to 1938, Oxford University Press, 152-278.