Mathematical Foundations for Orientation Based Representations of Shape

by
Ying Li

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Department of Computer Science
The University of British Columbia
Vancouver, B. C. V6T 1W5
Canada

email: li@cs.ubc.ca

Abstract

Mathematical foundations for orientation based shape representation are reviewed. Basic tools include support function, mixed volume, vector addition, Blaschke addition, and the corresponding decompositions, as well as some basic facts about convex bodies, are presented. Results on several types of curvature measures such as spherical images, m-th order area functions are summarized . As a case study, the EGI approach is examined to see how the classical results on Minkowski's problem are utilized in computational vision. Finally, results on Christoffel's problem are surveyed, including constructive proofs.

Contents

1 Introduction

An *orientation based representation* is a representation that encodes some object property as a function of surface orientation. More formally, an orientation based representation is a map, $\mathcal{R}$, from connected point sets in Euclidean d-space, R^d, to functions defined on the unit sphere in d-space, S^{d-1}. The following three orientation based representations are considered:

R1: Gaussian curvature.

R2: Mean radius of curvature.

R3: Distance (from an origin) to the closest tangent plane.

The following questions are posed:

Q1: For what class of objects is the representation unique?

Q2: What methods exist to reconstruct the object given its representation?

Q3: How can the representation be used for shape matching?

The first two representations have been studied intensively by mathematicians, and are known as *Minkowski's problem* and *Christoffel's problem* respectively. The support function, **R3**, appears in proofs related to both Minkowski's problem and Christoffel's problem.

In computational vision, the representation, **R1**, was first utilized by Horn [13, 15]. Other researchers have proposed using the mean curvature, as well, as an orientation based representation [2]. The representation, **R2**, is the mean radius of curvature, not the mean curvature. The distinction may not seem significant but the mathematical properties of **R2** have yet to be exploited in computational vision. Nalwa [19] has proposed using representation **R3**. This representation also appears as an internal representation in certain algorithms in computational geometry. (See, for example, Little [17].) As will be shown, representation **R3** also is related to the Legendre transformation used in applied mathematics. The answer to question **Q1** is known for representations **R1**–**R3** when the object involved is convex. The representations **R1**–**R3** are not unique, in general. Special cases have been explored [13]. As well, extentions to the representations to handle nonconvex objects have been considered [17]. Methods have been described to reconstruct an object from its representation **R1**. The most effective of these is the iterative method of Little [17]. There have been no methods described to reconstruct an object from its representation **R2**. As will be shown, however, constructive proofs exist which should lead to effective reconstruction algorithms. The representation **R3** is an effective representation in that most other representations are readily computed, given **R3**. The represerntation **R1** has been used directly in shape matching Ikeuchi [16], Brou [4]. Little [17] developed a matching method based on the mixed volume that combines representations **R1** and **R3**.

Section 2 presents some basic facts about hypersurfaces, particularly convex hypersurfaces. Those facts, along with the tools used to discover them, establish the framework for the remaining sections. Section 3 describes Minkowski's problem, both in theory and in application. Section 4 describes Christoffel's problem, both in general and for the special case of polytopes. Appendix A provides notation and terminology used in this paper.

2 Facts and Tools

Most of the material in this section is excerpted from Bonnesen and Fenchel [3], Busemann [5], Grünbaum [12], Pogorelov [20], and Schneider [22]. Proofs of theorems are not included except where a constructive proof forms the essential basis for a subsequent algorithm.

Many mathematical proofs assume smoothness conditions, which are not determined by the problem but rather by the methods of proof used. Many convexity results are simply due to convexity without smoothness assumptions. Methods without smoothness assumptions are relevant to discrete algorithms.

This section begins with two theorems on the differentiability of convex hypersurfaces (Section 2.1). Section 2.2, 2.3, and 2.4 introduce, respectively, spherical image, support function, and area function of a convex body. As well, results about the existence and uniqueness of a convex body with respect to spherical image, support function, and area function, respectively, are reviewed. Section 2.5 describes two ways of combining convex bodies, vector addition and Blaschke addition. Section 2.6 describes mixed volume and the Brunn-Minkowski Theorem. Section 2.7 presents results associated with m-th order area function. Finally, Section 2.8 introduces distance function, cross sectional measure, breadth, diameter, and width of a convex body, and the Legendre transformation of a convex function. The Legendre transformation of a convex function can be viewed as the support function of a convex body. Moreover, the support function and the distance function of a convex body are Legendre transformations of each other (claimed by Fenchel [6].)

2.1 Basic Facts and Concepts

Theorem 2.1 For a given point p on a convex hypersurface C there is a neighborhood C_p of p and a system of rectangular coordinates $x_1, x_2, \ldots, x_{d-1}, z$ with p as origin such that C_p is representable in the form

$$z = f(x_1, \ldots, x_{d-1}) = f(x) \geq 0, \quad f(0) = 0, \quad | x | \leq \delta, \quad \delta \geq 0.$$

Moreover $f(x)$ is a convex function of x, and its difference quotients

$$\frac{| f(x) - f(y) |}{| x - y |}, \quad x \neq y$$

are bounded. If (x_0, z_0) is a point in C_p where C is differentiable then $f(x)$ possesses at x_0 a differential.

Theorem 2.2 (Reidemeister, 1921) A convex hypersurface is almost everywhere differentiable.

Although simple and basic, Theorems 2.1 and 2.2 are important. Quite often, we see the following types of statements: "Assume the surface of the object is sufficiently smooth, ...", or "Let us consider the behavior of the surface in a local coordinate system around point p, ...", etc. We should always be aware of the extent to which these assumptions are ensured as a result only of convexity.

Definition 2.1 Let $v \in R^d$. A hyperplane $H = \{x \in R^d | \ \langle x, v \rangle = \alpha\}$ is said to *cut* a subset A of R^d if there exist $x_1, x_2 \in A$ such that $\langle x_1, v \rangle < \alpha$ and $\langle x_2, v \rangle > \alpha$. A hyperplane H is said to *support* A if H does not cut A and the distance between H and A is 0. When a hyperplane H that supports A is represented as $H = \{x \in R^d | \ \langle x, v \rangle = \alpha\}$ such that $\langle x, v \rangle \leq \alpha$ for all $x \in A$, H is will be referred as the support hyperplane of A with *outward normal* v.

Theorem 2.3

(a) If A is a bounded set in R^d and if H is a given hyperplane, there exists a support hyperplane of A parallel to H.

(b) If A is, moreover, convex and its interior int$A \neq \emptyset$, there exist exactly two such hyperplanes.

(c) If A is convex and bounded, and $x \notin$ int$A$, there exists a support hyperplane of A which contains x.

Thus, given a convex hypersurface C which, by definition, is the boundary of a convex set K, and a point $p \in C$, we can talk about the support hyperplanes of C at p as the support hyperplanes of K containing p, we also say that p has a support hyperplane H_p. Concerning the number of support hyperplanes at p, we have the following definition.

Definition 2.2 A boundary point p of a convex body K is said to be *singular* if K has more than one support hyperplanes at p. If K has only one support hyperplane at p, p is called *regular*. A support hyperplane of K is said to be *regular* if it intersects K at only one point.

2.2 Spherical Images

Definition 2.3 A set M on S^{d-1} is (*spherical*) *convex* if for every $u, v \in M, u \neq \pm v, M$ contains the shorter arc of the great circle determined on S^{d-1} by u and v.

Theorem 2.4 A convex set on S^{d-1} is either S^{d-1} or contained in a closed hemisphere of S^{d-1}.

Definition 2.4 Consider a convex hypersurface C. The *spherical image* $v'(p)$ of a point $p \in C$ consists of the points of S^{d-1} which, as vectors, are the outward normals of the support hyperplanes of C at p. The *spherical image* $v'(M)$ of a set $M \subseteq C$ is

$$v'(M) = \cup_{p \in M} v'(p) \ .$$

Note Spherical images are not defined for non-convex hypersurfaces, because the support hyperplanes at a concave point are not defined.

Theorem 2.5 For a convex hypersurface C, $v'(C)$ is convex.

Definition 2.5 For any set $M \subseteq C$ for which $v'(M)$ is measurable, the *integral curvature* of M is defined as

$$v(M) = \text{measure of } v'(M) .$$

Theorem 2.6 (Alexandrov) Given a convex hypersurface C, $v'(M)$ is measurable for any $M \in \mathcal{B}(C)$ and the integral curvature $v(M)$ is completely additive on $\mathcal{B}(C)$.

Theorem 2.7 (Alexandrov) For a given completely additive non-negative set function $\alpha(M')$ defined for all $M' \in \mathcal{B}(S^{d-1})$ there exists a closed convex hypersurface C containing origin O in its interior such that $\alpha(M') = v(M)$ for the projection M of M' from O on C, if and only if

1. $\alpha(S^{d-1}) = \omega_d$ (See Appendix A for the definition of ω_d.)
2. $\alpha(K) \leq \omega_d - \beta$ for every convex set K on S^{d-1}, where β is the measure of the spherical image of the cone projecting K from O .

This theorem looks quite analogous to the existence theorem of Minkowski's problem. Actually, the proofs are quite similar. The corresponding uniqueness theorem is stated as follows.

Theorem 2.8 Let C_1 and C_2 be two closed convex hypersurfaces containing O in the interior. If $v(M_1) = v(M_2)$ for any $M_i \in \mathcal{B}(C_i), i = 1, 2$, which are projections of each other from O, then C_2 is obtained from C_1 by a dilation with center O .

2.3 Support Functions

Definition 2.6 Let $K \subseteq R^d$ be a nonempty set. The *support function* $H(v)$ of K is defined for all $v \in R^d$ by

$$H(v) = \sup\{\langle x, v\rangle \mid x \in K\} .$$

Support functions are written as $H(K; v)$ or $H_K(v)$ for the support functions of different point sets.

Let $H(v)$ be the support function of a convex body K, $v \in R^d \setminus \{O\}$. Then the support hyperplane of K with outer normal v can be represented as $\langle x, v \rangle = H(v)$. Actually, this was how Bonnesen defined support function. Support functions following different definitions are defined for different scopes of sets (e.g., convex only), and have different domains (e.g., on S^{d-1} only). The definition we have here is the most general one in the sense that the support functions defined in other ways can be extended naturally to the support function we defined here. Support functions defined only on S^{d-1} can be extended to be defined for any $v \in R^d$ as $H(v) = \|v\| H(v/\|v\|)$. The following result asserts that the extended support functions are the support functions we defined.

Theorem 2.9 The support function $H(v)$ of a nonempty set K is *positively homogeneous* and *convex*, that is, it satisfies

1. $H(\lambda v) = \lambda H(v)$ for all $\lambda \geq 0,\ v \in R^d$,
2. $H(v + w) \leq H(v) + H(w)$ for all $v, w \in R^d$.

Support functions are defined for any nonempty set. However, the support functions of convex sets attracted more attention from mathematicians. From the above theorem, we know that the support function of a convex set is necessarily positively homogeneous and convex. We now prove that positively homogeneity and convexity are sufficient for a function to be the support function of a convex set.

Theorem 2.10 If $H(v)$ is any function defined on R^d such that

1. $H(\lambda v) = \lambda H(v)$ for all $\lambda \geq 0,\ v \in R^d$,
2. $H(v + w) \leq H(v) + H(w)$ for all $v, w \in R^d$,

then there exists a nonempty closed convex set K such that $H(v)$ is the support function of K.

Proof Let $K_u = \{x | \ \langle x, u \rangle \leq H(u)\}$ for $u \in S^{d-1}$, that is, the half space bounded by plane $\langle x, u \rangle = H(u)$ with unit outer normal u. Let K be the intersection of these half spaces, that is,

$$K = \{x | \ \langle x, u \rangle \leq H(u) \ \text{ for all } u \in S^{d-1}\}.$$

We can prove that K is nonempty, closed, and convex.

□

Theorem 2.11 If K_1, K_2 are nonempty closed convex sets in R^d such that $H(K_1; v) = H(K_2; v)$ for every $v \in R^d$, then $K_1 = K_2$.

The uniqueness result does not hold for non-convex bodies. For example, in Figure 1, polygons A and B have the same support function, $H(A; v) = \sum_{i=1}^{d} |v_i|$, if A is the cube having edges parallel to the axes with edge length 2 and is centered at the origin. Here A is convex, B is not, and hull$(B) = A$.

Figure 1: Two polygons with the same support function.

Theorem 2.12 Let K_1 and K_2 be two convex bodies. Then $H(K_1; v) \leq H(K_2; v)$ holds for all $v \in R^d$ if and only if $K_1 \subseteq K_2$.

Theorem 2.13 If a convex body K has only regular support planes, then its support function $H(K; v)$ has continuous partial derivatives of the first order, and

$$x_i = \frac{\partial H(K; v)}{\partial v_i}, \; i = 1, \ldots, d,$$

where x_i is the coordinate of its boundary points.

2.4 Area Functions

Definition 2.7 Let K be a convex body in R^d, $\omega \in \mathcal{B}(S^{d-1})$. Denote, by $S(K; \omega)$, the $(d-1)$-content (area when $d = 3$) of the set of all those boundary points of K, each of which has a support hyperplane with outward normal in ω. Set function $S(K; \omega)$ is called the *area function* (or *primary area function* by W. Firey) of K.

Theorem 2.14 Let μ be a positive measure on $\mathcal{B}(S^{d-1})$ not concentrated on a great sphere, and suppose that

$$\int_{S^{d-1}} u d\mu(u) = 0 .$$

Then there exists a convex body K, unique up to translation, with $S(K;\omega) = \mu(\omega)$.

Obviously, the area function of a polytope P is a discrete system of vectors $\mathcal{A}(P) = \{a_i \mid 1 \leq i \leq f(P)\}$, where $f(P)$ is the number of facets of P. For each facet F_i of P, the direction of a_i is that of the outward normal of F_i and the length of a_i is equal to the $(d-1)$-content of F_i.

Definition 2.8 A system $\mathcal{V} = \{v_i | 1 \leq i \leq n\}$ of non-zero vectors in R^d is called *equilibrated* if $\sum_{i=1}^{n} v_i = 0$ and no two of the vectors in $\mathcal{V}$ are positively proportional. $\mathcal{V}$ is called *fully equilibrated* in R^d when it is equilibrated and spans R^d.

Theorem 2.15 (Minkowski's Fundamental Theorem)

(1) If P is a polytope in R^d, then $\mathcal{A}(P)$ is equilibrated. If P is a k-polytope, then $\mathcal{A}(P)$ is fully equilibrated in the subspace R^k parallel to the affine space spanned by P.

(2) If $\mathcal{V}$ is a fully equilibrated system in $R^k, k \geq 2$, there exists a polytope P, unique within a translation, such that $\mathcal{V} = \mathcal{A}(P)$.

2.5 Combinations of Convex bodies

There are many ways to combine convex bodies. Two ways described here are vector addition (or mixture) and Blaschke addition. The two definitions are closely related to the support function and the area function respectively. Methods of combination lead naturally to questions of decomposition. This subsection also presents associated decomposition results.

2.5.1 Vector Addition

Definition 2.9 For two set Q and R in R^d, the *vector sum* $Q + R$ of Q and R is defined as

$$Q + R = \{x + y | x \in Q, y \in R\}$$

If $P = Q + R$, then Q and R are called summands of P. The process is called *vector addition.* Associated with the vector addition, is a scalar multiplication λQ defined as $\lambda Q = \{\lambda x | x \in Q\}$. The set $\{a\} + \lambda Q, a \in R^d$, is said to be *homothetic* to Q, and *positively homothetic* to Q if $\lambda > 0$.

Theorem 2.16

(1) A set P is the vector sum of Q and R if and only if

$$H(P;v) = H(Q;v) + H(R;v), \text{ for all } v \in R^d.$$

(2) Let Q and R be two polytopes, $q_i, i = 1,\ldots,n$ and $r_j, j = 1,\ldots,m$, be the vertices of Q and R, respectively. A set P is the vector sum of Q and R if and only if P is the convex hull of the set

$$\{q_i + r_j | i = 1,\ldots,n,\ j = 1,\ldots,m\}.$$

Assertions (1) and (2) can be regarded as equivalent ways of defining vector sum. Assertion (1) says that $H(Q + R;v) = H(Q;v) + H(R;v)$. Assertion (2) implies that the vector sum of two polytopes is again a polytope.

In the same manner as we define vector sum of two convex bodies, we can define the *linear combinations* of convex bodies. Let $\lambda_i \geq 0, i = 1,\ldots,r$, the linear combination of convex bodies K_i, $i = 1,\ldots,r$, is defined as

$$K = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \mid x_i \in K_i\}$$

and is represented as

$$K = \lambda_1 K_1 + \cdots + \lambda_r K_r = \sum_{i=1}^{r} \lambda K_i.$$

The position of K gencrally depends on the choice of origin O . If O is replaced by another point O', K will be translated by $(\sum_{i=1}^{r} \lambda_i - 1)\ \overrightarrow{OO'}$, where $\overrightarrow{OO'}$ is the vector from O to O'. Hence, if $\sum_{i=1}^{r} \lambda_i = 1$, linear combinations will be independent of the coordinate system. Of particular interest is the linear combination

$$K_\theta = (1-\theta)K_0 + \theta K_1\ ,\ 0 \leq \theta \leq 1$$

of two convex bodies K_0 and K_1.

Similarly to the argument of Theorem 2.16, the support function $H(v)$ of the linear combination of convex bodies K_i, $i = 1,\ldots,r$ is the linear combination of the support functions $H_i(v)$ of K_i, i.e.,

$$H(v) = \lambda_1 H_1(v) + \cdots + \lambda_r H_r(v).$$

2.5.2 Blaschke Addition

Definition 2.10 Let K_1 and K_2 be two convex bodies with area function $S(K_1;\omega)$ and $S(K_1;\omega)$, respectively. The Blaschke sum of K_1 and K_2 is defined as the convex body K whose area function equals to $S(K_1;\omega) + S(K_1;\omega)$, and is represented as $K = K_1 \# K_2$. The process is called *Blaschke addition.* The notion of scalar multiplication associated with Blaschke addition is defined as the convex body with area function $\lambda S(K;\omega)$, denoted as $\lambda \times K$.

To see that Blaschke sum is well defined, recall Theorem 2.14 in Section 2.4. That the area functions $S(K_1;\omega)$ and $S(K_2;\omega)$ satisfy the condition of the theorem implies that their sum also satisfies the same condition. Thus a convex body is uniquely determined, up to translation, as having the sum of the two area functions as its area function. By definition, $S(K_1 \# K_2;\omega) = S(K_1;\omega) + S(K_2;\omega)$.

Similar to vector sum, we can define the weighted Blaschke sum of convex bodies. Let K_i, $i = 1, \ldots, r$ be convex bodies, $\lambda_i \geq 0, i = 1, \ldots, r$, define

$$K = \lambda_1 \times K_1 \# \cdots \# \lambda_r \times K_r = \#_{i=1}^{r} \lambda_i \times K_i$$

to be the convex body having function $S(K;\omega) = \lambda_1 S(K_1;\omega) + \cdots + \lambda_r S(K_r;\omega)$ as its area function. Similarly, $K_\theta = (1-\theta) \times K_0 \# \theta \times K_1$, $0 \leq \theta \leq 1$, may draw our special attention under certain circumstance.

2.5.3 Vector Addition vs. Blaschke Addition

The area function of $K_\theta = (1-\theta)K_0 + \theta K_1$, where K_0 and K_1 are convex bodies, is given by the ***generalized Steiner formula*** (quoted by Firey [8]), as,

$$S(K_\theta;\omega) = (1-\theta)^2 S(K_0,\omega) + 2\theta(1-\theta)S_{01}(\omega) + \theta^2 S(K_1;\omega) \ .$$

By Theorem 2.14, there is a unique convex body which has S_{01} as its area function. Firey called this convex body the ***mixed convex body*** resulting from K_0 and K_1, and denoted it as $C(K_0, K_1)$. Then

$$(1-\theta)K_0 + \theta K_1 = (1-\theta)^2 \times K_0 \# 2\theta(1-\theta) \times C(K_0, K_1) \# \theta^2 \times K_1 \ .$$

2.5.4 Decomposition of Polytopes

In the plane, every polygon is the vector sum of finitely many summands of a simple type (segments and triangles), and every convex set is the limit of finite vector sums of triangles. Both assertions fail to have analogues in higher dimensions. The decomposition of d-polytopes with respect to Blaschke addition is well behaved, as the following theorems show.

Theorem 2.17 (Firey-Grünbaum) Every polytope P is expressible in the form

$$P = \#_{i=1}^{m} P_i$$

where each P_i is a simplex. Further, if P is d-dimensional and the number of facets of P is $f(P) = n \geq d+1$, then there is a representation with $m \leq n-d$.

Theorem 2.18 (Firey-Grünbaum) Every d-polytope P is representable in the form $P = \#_{i=1}^{m} P_i$ where each P_i is a d-polytope with the number of facets of P is $f(P_i) \leq 2d$.

2.6 Mixed Volumes

Definition 2.11 The d-dimensional measure (or volume) of K is denoted by $V(K)$. For variable $\lambda_i \geq 0$ the volume of $K = \sum_{i=1}^{r} \lambda_i K_i$ is a form

$$V(K) = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_d=1}^{r} V_{i_1 \ldots i_d} \lambda_{i_1} \ldots \lambda_{i_d}$$

of degree d in the λ_i, where the coefficients $V_{i_1 \ldots i_d}$ are uniquely determined by requiring that they are symmetric in their subscripts. Then $V_{i_1 \ldots i_d}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_d}$ and not on the remaining bodies K_j, hence we may write $V(K_{i_1}, \ldots, K_{i_d})$ for $V_{i_1 \ldots i_d}$, and call it the *mixed volume* of $K_{i_1}, \ldots, K_{i_d}$.

The most important case in studying mixed volumes is the case which involves only two distinct convex bodies. Thus the notation $V_m(K_1, K_2)$ was introduced as :

$$V_m(K_1, K_2) = V(\underbrace{K_1, \ldots, K_1}_{d-m}, \underbrace{K_2, \ldots, K_2}_{m}) = V_{d-m}(K_2, K_1)$$

Then

$$V(\lambda_1 K_1 + \lambda_2 K_2) = \sum_{m=0}^{d} \begin{pmatrix} d \\ m \end{pmatrix} \lambda_1^{d-m} \lambda_2^m V_m(K_1, K_2) \qquad .$$

Since $V(\sum \lambda_i K) = (\sum \lambda_i)^d V(K)$, we have $V(K, \ldots, K) = V(K)$.

All in all, the notion of mixed volumes does not seem to be easy to grasp. The following relationships are helpful in understanding this notion.

Consider a non-degenerate polytope P. Denote by $p^i, i = 1, \ldots, N$, the facets of P, by $\mathcal{A}(p^i)$ the area of p^i, by u^i the unit outer normal of the support plane of P containing p^i. It can be proved that

$$V(P) = d^{-1} \sum_{i=1}^{N} H(P; u^i) \mathcal{A}(p^i).$$

This can be generalized to mixed volumes of a polytope P and an arbitrary convex body K^* as

$$V_1(P, K^*) = d^{-1} \sum_{i=1}^{N} H(K^*; u^i) \mathcal{A}(p^i). \tag{1}$$

Furthermore, if K is an arbitrary convex body with interior points, bd$K$ its boundary, dS the area element of bd$K$, then a limit process of the above equation will give us

$$V_1(K, K^*) = d^{-1} \int_{\mathrm{bd}K} H(K^*; u(x)) dS \tag{2}$$

The flexibility of the mixed volumes makes it easy to get various geometric quantities of K_1 by substituting suitable $K_2, \ldots, K_d$ in $V(K_1, \ldots, K_d)$. For example, let U be the unit ball, $dV_1(K, U)$ is the area of bd$K$ by Equation 2, because $H(U; u) \equiv 1$. In fact, Bonnesen and Fenchel [3] defines the surface area $S(K)$ of an arbitrary convex body K as $d \cdot V_1(K, U)$. Furthermore, if $K_1, K_2, \ldots, K_{d-1}$ are convex bodies, then $d \cdot V(U, K_1, \ldots, K_{d-1})$ is designated as their mixed surface area.

Theorem 2.19 (Brunn-Minkowski Theorem) If K_0 and K_1 are convex bodies in R^d, then

$$g(\alpha) = V^{1/d}((1-\alpha)K_0 + \alpha K_1)\ , \ \ 0 \leq \alpha \leq 1$$

is a concave function of α, which is linear if and only if K_0 and K_1 are homothetic or lie in parallel hyperplanes.

Corollary 2.20 (Minkowski's Inequalities)

$$V_1^d(K_0, K_1) \geq V^{d-1}(K_0)V(K_1) ,$$

$$V_{d-1}^d(K_0, K_1) \geq V(K_0)V^{d-1}(K_1) .$$

If K_0 and K_1 do not lie in parallel hypersurfaces the equality sign holds only when K_0 and K_1 are homothetic.

Corollary 2.21 (Quadratic Inequalities of Minkowski)

$$V_1^2(K_0, K_1) \geq V(K_0)V_2(K_0, K_1) ,$$

$$V_{d-1}^2(K_0, K_1) \geq V_{d-2}(K_0, K_1)V(K_1) .$$

These inequalities are used to solve many extremal and uniqueness problems.

2.7 m-th Order Area Functions

Definition 2.12 Let K be a convex body, U be the unit ball, $\omega \in \mathcal{B}(S^{d-1})$. Then for $\lambda \geq 0, S(K + \lambda U; \omega)$ is a polynomial

$$\sum_{i=1}^{d-1} \binom{d-1}{i} S_{d-1-i}(K; \omega)\lambda^i.$$

This defines the measure $S_{d-1-i}(K;\omega)$ over $\mathcal{B}(S^{d-1})$ for $i = 0, 1, \ldots, d-1$. Call $S_{d-1-i}(K;\omega)$ the area function of order $d-1-i$ of K.

If K is a regular convex body and $R_1, R_2, \ldots, R_{d-1}$ are the principal radii of curvature of the surface of K, then

$$S_m(K;\omega) = \int_\omega \{R_1, \ldots, R_m\} d\omega / \binom{d-1}{m} , \omega \in \mathcal{B}(S^{d-1}) , \qquad (3)$$

where $\{R_1, \ldots, R_m\}$ is the m-th elementary symmetric function of $R_1, R_2, \ldots, R_{d-1}$, denoted as D_m. In particular, $D_1 = R_1 + R_2+, \cdots, +R_{d-1}$ and $D_{d-1} = R_1 R_2 \cdots R_{d-1}$.

Theorem 2.22 (Alexandrov-Fenchel-Jessen) If K_1 and K_2 are convex bodies in R^d, $i \in \{1, \ldots, d-1\}, dimK_1, dimK_2 \geq i+1$, then $S_i(K_1; \cdot) = S_i(K_2; \cdot)$ if and only if K_1 and K_2 are translates of each other.

Theorem 2.23 (Alexandrov 1961) Suppose K_1 and K_2 are convex bodies in R^d for which

$$S_i(K_1; \cdot) \leq S_i(K_2; \cdot) \text{ and } S_{i+1}(K_1; S^{d-1}) \geq S_{i+1}(K_2; S^{d-1})$$

for some $i \in \{1, \ldots, d-1\}$ (where for $i = d-1$ the second condition has to be replaced by $V(K_1) \geq V(K_2)$). Then K, K' are translates of each other.

The above theorems are about the uniqueness of convex bodies with respect to m-th order area functions. Results about the existence of a convex body, for which m-th order area function is given, are stated in terms of the necessary and sufficient conditions for a measure defined over $\mathcal{B}(S^{d-1})$ to be m-th order area function of a convex body. The problem is called Minkowski-Christoffel problem, see Firey [11]. For $m = d-1$, we have seen the answer in Section 2.4. Firey [10] solved this problem for the case that $m = 1$, and we will see how in Section 4.2. Thus for $d = 3$, the Minkowski-Christoffel problems are solved, although the problems are still open for $d > 3$. We know that $S_0(K; \omega)$ is the area of ω. For $S_1(K; \omega)$, the problem can be thought of as a generalization of Christoffel's problem, which will be discussed in Section 4. $S_2(K; \omega)$ is the area function of K, and the problem can be thought of as a generalization of Minkowski's problem which is the topic of Section 3.

Let $\mathcal{M}$ be the set of finite Borel measures on S^{d-1} which have barycentre O, $\mathcal{S}_m$ be the set of m-th order area functions of convex bodies in R^d, $m = 1, \ldots, d-1$. Weil [23] proved that $\mathcal{S}_{d-1}$ is dense in M (in the weak topology).

2.8 Other Functions and Combinations

So far, we have seen quite a few functions that are associated with convex bodies, like support function, area function, and so on. They basically measure certain properties of the convex bodies involved. Meanwhile, we have seen two ways of combining convex bodies, vector addition and Blaschke addition. They can be thought of as combining certain property measurements of the convex bodies involved. There are a few other functions that are associated with convex bodies in terms of measuring certain properties of the convex bodies involved. And of course, there are other ways of combining convex bodies.

2.8.1 Distance Function and Polar Means

Definition 2.13 Let K be a convex body with interior points. Suppose the origin O is chosen in the interior of K. For any $x \in R^d \backslash \{O\}$, let ξ_x be the (unique) intersection point of the ray $\overrightarrow{Ox}$ with the boundary of K. The distance function $F(x)$, $x \in R^d$, of K is defined as

1. $F(O) = 0$, and
2. $F(x) = \|x\|/\|\xi_x\|$, $x \in R^d \backslash \{O\}$.

The following observations regarding distance function follow immediately from the definition. 1) The points that satisfy the inequality $F(x) \leq 1$ are precisely the points of K. 2) If two convex sets with O as their common interior point have the same distance function, the two must be the same. 3) If the distance function of K is $F(x)$, that of λK is $F(x)/\lambda$. 4) Suppose $F_0(x)$ and $F_1(x)$ are the distance functions of convex sets K_0 and K_1 respectively. $F_0(x) \geq F_1(x), \forall x \in R^d$ if and only $K_0 \subseteq K_1$.

Theorem 2.24 The distance function $F(x)$ of a convex set K has the following properties :

a) $F(x) \geq 0$	with equality if and only if $x = O$,
b) $F(\mu x) = \mu F(x)$	for all $\mu > 0,\ x \in R^d$,
c) $F(x+y) \leq F(x) + F(y)$	for all $x, y \in R^d$.

Proof Properties a) and b) follow immediately from the definition of F. To prove c), let x and y be different from O (the assertion holds obviously when x or y are O), we have

$$F(\tfrac{x}{F(x)}) = F(\tfrac{y}{F(y)}) = 1 .$$

Therefore,

$$F((1-\theta)\tfrac{x}{F(x)} + \theta\tfrac{y}{F(y)}) \leq 1$$

because of the convexity of K. Choose $\theta = \frac{F(y)}{F(x)+F(Y)}$ in the above inequality, together with the positive homogeneity of F, we get $F(x+y) \leq F(x)+F(y)$.

□

Theorem 2.25 If $F(x)$ is any function defined on R^d such that

a) $F(x) \geq 0$ with equality if and only if $x = O$,
b) $F(\mu x) = \mu F(x)$ for all $\mu > 0,\ x \in R^d$,
c) $F(x+y) \leq F(x) + F(y)$ for all $x, y \in R^d$,

then there exists a nonempty closed convex set K such that $F(x)$ is the distance function of K.

Proof Let $K = \{x|\ F(x) \leq 1\}$ If $x \in K$, $y \in K$, and $0 \leq \theta \leq 1$, then $F(x) \leq 1, F(y) \leq 1$. By c) and b)

$$F((1-\theta)x + \theta y) \leq (1-\theta)F(x) + \theta F(y) \leq 1 \ .$$

Thus K is convex. Furthermore, if x is an arbitrary point different from O, ξ_x is a boundary point of K that lies on the ray $\overrightarrow{Ox}$, then b) and $F(\xi_x) = 1$ imply

$$F(x) = F(\tfrac{\|x\|}{\|\xi_x\|} \cdot \xi_x) = \tfrac{\|x\|}{\|\xi_x\|} \ .$$

□

The concept and properties of the distance function are due to Minkowski. The above materials are extracted from Bonnesen and Fenchel [3]. As with vector addition and Blaschke addition, we may define another combination of convex bodies via a combination of the distance functions of the convex bodies involved. It was studied by Firey [7]. We will cite the definition and a theorem (of Brunn-Minkowski type, claimed by Firey) to finish this subsection.

If $F_0(x)$ and $F_1(x)$ are the distance functions of convex sets K_0 and K_1 containing O as a common interior point, then

$$F_\theta^{(1)}(x) = (1-\theta)F_0(x) + \theta F_1(x)\ , \quad 0 \leq \theta \leq 1\ ,$$

and more generally,

$$F_\theta^{(p)}(x) = \sqrt[p]{(1-\theta)F_0^p(x) + \theta F_1^p(x)}\,, \quad 1 \le p \le \infty\,,$$

satisfy conditions a) through c) stated in Theorem 2.25. By $F_\theta^{(\infty)}(x)$ we mean

$$\lim_{p\to\infty} F_\theta^{(p)}(x) = \max(F_0(x), F_1(x))$$

for $0 < \theta < 1$ with $F_i^{(\infty)}(x) = F_i(x)$ for $i = 1, 2$. Then we can speak of a unique convex body $K_\theta^{(p)}$ having the distance function $F_\theta^{(\infty)}$. We will call this body the pth *dot-mean* of K_0 and K_1. The convex body with distance function $\sqrt[p]{F_0^p(x) + F_1^p(x)}$ is called the pth *dot-sum* of K_0 and K_1.

Theorem 2.26

$$V^{1/d}(K_0 \cap K_1) \le V^{1/d}(K_\theta^{(p)}) \le 1/\sqrt[p]{(1-\theta)V^{-p/d}(K_0) + \theta V^{-p/d}(K_1)}\,,$$

for $1 \le p < \infty$. There is equality on the left if and only if $K_0 = K_1$ and on the right if and only if $K_0 = \lambda K_1$ with center of homothety at O. Further

$$V^{1/d}(K_0 \cap K_1) = V^{1/d}(K_\theta^{(\infty)}) \le \min(V^{1/d}(K_0), V^{1/d}(K_1))$$

with equality on the right if and only if $K_0 = K_1$.

2.8.2 Cross Sectional Measure

Let K be a convex body and v an arbitrary direction. The $(d-1)$-dimensional volume of the orthogonal projection of K onto a hyperplane with normal direction v is called the (*(d-1)-dimensional*) *cross sectional measure* of K in the direction v, denoted as $\sigma(K; v)$. For $d = 3$, $\sigma(K; v)$ is called the *brightness function* of K, i.e., the area of orthogonal projection of K onto a plane with normal direction v.

It can be justified that

$$\sigma(K; v) = d \cdot V_1(K, v)\,.$$

From Firey [8], we can represent the brightness function $\sigma(K; v)$ of K in terms of its area function $S(K; \omega)$:

$$\sigma(K; v) = \frac{1}{2}\int_{S^2} \mid \langle v, w\rangle \mid S(K; d\omega(w))\,.$$

From this formula it can be inferred that the brightness, in any given direction, of a Blaschke sum is the sum of the corresponding brightnesses of the summands.

Theorem 2.27 (Alexandrov 1937) If two centrally symmetric convex bodies have the same brightness function, then they differ at most by a translation.

It follows that spheres are the only central bodies of constant brightness. Let K' be the reflection of K, $K\#K'$ is defined as the *areal domain* of K. Since $K\#K'$ is central, and K, K' have the same brightness function, we say that K has constant brightness function if and only if $K\#K'$ is a sphere.

Given the (d-1)-dimensional cross sectional measure $\sigma(K;v)$ of convex body K, the surface area $S(K;S^{d-1})$ of K can be computed via Cauchy's formula

$$S(K;S^{d-1}) = \frac{1}{\pi_{d-1}} \int_{S^{d-1}} \sigma(K;u) d\omega \ .$$

2.8.3 Breadth, Diameter, Width

Let K be a convex body and v an arbitrary direction. The distance between the two support planes of a convex body K with normal direction v is called the *breadth* of K in the direction v, and is denoted as $B(K;v)$. The maximum of $B(K;v)$ over all v is called the diameter of K, denoted as $D(K)$. The minimum of $B(K;v)$ over v is called the width of K, and is denoted as $\Delta(K)$. A convex body is called a *body of constant breadth* if it has the same breadth in all directions, hence if the diameter is equal to the width.

Since $B(K;v) = H(K;v) + H(K;-v)$ for all direction v, for any convex body K, it follows that the breadth, in any given direction, of a vector sum is the sum of the corresponding breadths of the summands.

Let K' be the reflection of K, $K + K'$ is called the *vector domain* of K. A convex body K is a body of constant breadth if and only if its vector domain is a sphere.

2.8.4 Legendre Transformation

Legendre transformation is a very useful mathematical tool in mathematical analysis. It establishes a duality between objects in dual spaces. It is also used in the theory of partial differential equations to reduce the order

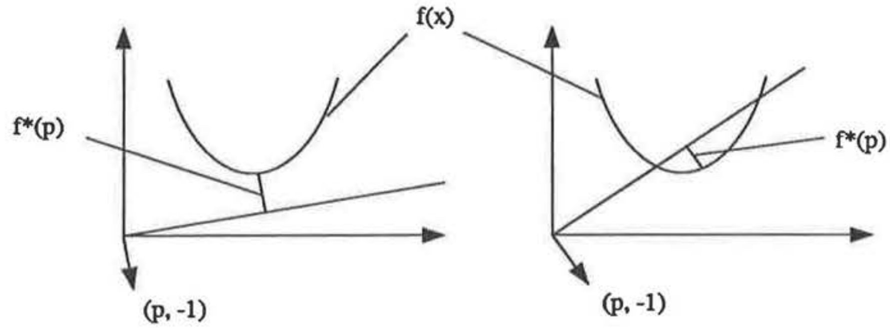

Figure 2: Legendre Transformation.

of partial differential equations (By means of a Legendre transformation, a Lagrangian system of second-order differential equations is converted into a symmetrical system of first-order equations called a Hamiltonian system of equations. See Arnold [1].)

Among many versions of the definition of Legendre Transformation, we will use the one that best fits in the context of our discussion.

Definition 2.14 Let $f(x), x \in A$, be a convex function defined on $A \subseteq R^d$ ($\det(\partial^2 f/\partial x^i \partial x^j)$ is not zero on A). The Legendre transformation f^* of f is defined as

$$f^*(x^*) = \max(\langle x, x^* \rangle - f(x)) ,$$

where the domain A^* of f^* is the set of points x^* at which the extreme value exists.

Given a convex function f defined on $A \subseteq R^d$, let A^+ be the point set in R^{d+1} bounded by $\{(x, f(x)) \in R^{d+1} | x \in A\}$. A^+ is convex because f is convex. Let $v \in R^d, v^+ = (v, -1) \in R^{d+1}$, then

$$H(A^+; v^+) = max\{\langle x^+, v^+ \rangle | x^+ \in A^+\} = f^*(v) .$$

This means that $f^*(v)$ is the value of the support function of A^+ in the orientation v^+. Figure 2 demonstrates the relation between f and f^*, and gives an visual explanation of the above statement.

Interestingly enough, the distance function and the support function of a convex body are Legendre transformations of each other (called conjugate by Fenchel [6].) Fenchel did not provide any evidence for the validity of this claim. A set of conditions for the existence and the uniqueness of a function conjugate to a given function is given instead.

3 Minkowski's Problem and the EGI

We have already seen a glimpse of Minkowski's problem in Section 2.4. This section presents a more detailed historical account of the problem and how the results have been utilized, to date, in computational vision.

3.1 Minkowski's Problem

Minkowski's problem concerns the existence and uniqueness of a closed convex surface C such that the Gauss curvature of C at a point with unit outward normal vector u is a given function $K(u)$. Minkowski solved the problem in a certain generalized sense and also appropriately modified for polyhedra, which we have seen in Section 2.4. Alexandrov and Fenchel-Jessen showed independently that the problem can be solved concerning set functions. Now let us look at the solution to the original Minkowski's problem.

Theorem 3.1 Let $K(u)$ be a given positive continuous function defined for $u \in S^{d-1}$ which satisfies the condition

$$\int_{S^{d-1}} \frac{u d\omega(u)}{K(u)} = 0 ,$$

where $d\omega(u)$ is the area element on S^{d-1}. Then there exists a closed convex surface Φ, unique up to translation, whose Gauss curvature at the point with outward normal u is $K(u)$.

Theorem 2.14 solved a generalized Minkowski's problem, i.e., a problem concerning area function instead of Gauss curvature. The sense of generalization is based on the relation between area function and Gauss curvature. Recall Equation 3, we have

$$S(K;\omega) = \int_{\omega} R_1 R_2 \cdots R_{d-1} d\omega$$

where $R_1, R_2, \ldots, R_{d-1}$ are the principal radii of curvature. To see that Theorem 2.15 also solves a generalized Minkowski's problem, recall that Gauss curvature can be described as

$$K(p) = \lim_{|E| \mapsto 0} \frac{|G(E)|}{|E|}$$

where E is a compact portion of the surface containing p, $G(E)$ is the image of E under Gauss map. Thus the principal radii of curvature and the area of the preimage of Gauss map are related somehow. For a polytope, the preimage of Gauss map is either the area of a facet of the polytope or zero, i.e., a discrete system of non-coplanar vectors.

3.2 The EGI Representation in Computational Vision

EGI stands for Extended Gaussian Image, which is defined, for a surface C, as a map which associates the inverse of Gauss curvature at a boundary point of a surface to the orientation of the surface at the point:

$$G_C : S^{d-1} \longmapsto R^1, \; \forall u \in S^{d-1}, \, G_C(u) = \frac{1}{K(x(u))},$$

where $x(u)$ is the point on C with unit outward normal u, and $K(x(u))$ is the Gauss curvature of C at point $x(u)$. Such a map G_C is called the EGI of C.

In computational vision research, we need to use a discrete version of the EGI, i.e., the EGI of polytopes. Since the Gaussian curvature at the faces of a polytope vanishes, the above definition of EGI can no longer be used. In fact, the definition in use in computational vision research is the same as the definition of the area function of a polytope mentioned in Section 2.4, i.e., a discrete system of vectors $\mathcal{A}(P) = \{a_i \mid 1 \leq i \leq f(P)\}$, where $f(P)$ is the number of facets of P, the direction of a_i is the same as that of the outward normal of face F_i and the length of a_i is equal to the $(d-1)$-content of F_i.

The EGI has the following properties:

1. It is insensitive to the position of an object ;
2. It is unique for a convex body (cf. Theorem 2.14 2.15) ;

3. It can be computed easily from needle diagrams obtained using photometric stereo, or depth maps obtained using the binocular stereo method (cf. [14]) .

The EGI is useful for recognition and attitude determination. For the rest of this section, we will review the work of Little [17]. Other work on the EGI is found in Horn [13, 14].

Little made full use of Brunn-Minkowski's Theorem (Theorem 2.19). Now we are only concerned with R^3, i.e., $d = 3$. Let P and Q be two convex bodies, $R = \lambda P + (1-\lambda)Q$. By the Brunn-Minkowski Theorem,

$$V^{\frac{1}{3}}(\lambda P + (1-\lambda)Q) \geq \lambda V^{\frac{1}{3}}(P) + (1-\lambda)V^{\frac{1}{3}}(Q) .$$

If the left hand side is replaced by its representation in terms of mixed volumes, one obtains

$$V_1^3(Q,P) \geq V(P)V^2(Q) .$$

The Brunn-Minkowski Theorem states that the polytope P, having unit volume, that minimizes $V_1(Q,P)$ is homothetic to Q. Now suppose Q has area function $\mathcal{A}(Q) = \{\mathcal{A}(q_i) \mid 1 \leq i \leq N\}$, where q_i are faces of Q and with outward unit normal ω_i. Recall, from Equation 1, that

$$V_1^3(Q,P) = (\frac{1}{3}\sum_{i=1}^{N} H(P;\omega_i)\mathcal{A}(q_i))^3 .$$

Using this relation, Little developed an iterative method which combines the techniques of constructing a polytope from its support vector (values of support function at the facet orientations of the polytope) and minimization techniques to construct the support function of P such that $\mathcal{A}(P) = \mathcal{A}(Q)$. This is to say that given a sensed EGI $\mathcal{A}(Q)$, its corresponding polytope can be reconstructed.

In attitude determination, we are given a sensed EGI $\mathcal{A}(Q)$ and we want to find the attitude which rotates the sensed EGI into correspondence with the prototype EGI. Think of the P in the above inequalities as the prototype. Then to determine object attitude, Little minimized

$$\sum_{i=1}^{N} H(P;R(\omega_i))\mathcal{A}(q_i)$$

over all rotation operations R.

4 Christoffel's Problem

Christoffel's problem concerns the existence and uniqueness of a closed convex surface C such that the sum of the principal radii of curvature at a point with unit outward normal vector u is a given function $\phi(u)$ defined on S^{d-1}.

The problem has been attacked by quite a few mathematicians, including Christoffel himself. A recent solution to the existence problem is the constructive proof by Firey [9], which the author claimed corrects and complements the incomplete treatment to the problem of earlier results.

Although a constructive proof potentially is a big help, the smoothness requirement of Firey's theorem essentially excludes many objects which we may be interested in, for example, polytopes. Firey said that his treatment is rather unsatisfactory in that the smoothness restrictions are set by the method rather than the problem. In another paper [10], Firey generalized the original Christoffel problem to the following : what are necessary and sufficient conditions on a set function Φ defined over $\mathcal{B}(S^{d-1})$ in order that Φ be the first order area function for some convex body K.

This generalization inspired us to think of constructing set functions that measure the "bendness" of a surface. Schneider [21] determined necessary and sufficient conditions for a Borel measure ϕ on S^{d-1} to be the first order area function of a convex polytope. Schneider claimed that his result was not deducible directly from Firey's general results.

The rest of this section is divided into three subsections in which the three treatments are sketched respectively.

4.1 Original Christoffel's Problem

Let K be a convex body having only regular support hyperplanes, $u \in S^{d-1}$, $x(u)$ be the point on the boundary bd$K$ of K at which the unit outer normal is u. Function x is called the normal representation of bd$K$, and it can be extended to be defined on $R^d \backslash \{O\}$ by $x(v) = x(v/\|v\|)$.

Theorem 4.1 In order that x, assumed to be continuously differentiable, be the normal representation of the boundary of a non-degenerate convex body with regular support planes, it is necessary and sufficient that, when extended to be defined over $R^d \setminus \{O\}$, its Jacobi matrix

$$\left(\frac{\partial x_i(v)}{\partial v_j}\right) \qquad (A)$$

does not vanish identically and is symmetric and non-negative definite on S^{d-1}.

The proof of this theorem is based on the relation between the normal representation x of a convex body and its support function, i.e. (Theorem 2.13),

$$x_i(v) = \frac{\partial H(v)}{\partial v_i}.$$

Suppose x is the normal representation of the boundary of convex body K with support function H. Assume H is at least three times continuously differentiable. Then the $d-1$ principal radii of curvature, $R_i(u), i = 1, \dots, d-1$, at $x(u)$ are, together with zero, the eigenvalues of the Hessian matrix of H. Hence the sum of the principal radii of curvature at $x(u)$ is the trace of the Hessian matrix of H, i.e.,

$$\Delta H(u) = R_1(u) + \cdots + R_{d-1}(u),\ u \in S^{d-1}.$$

Thus the Christoffel's problem is to seek solutions to the partial differential equation

$$\Delta x_i(v) = \frac{\partial \phi(v)}{\partial v_i},\ i = 1, \dots, d \qquad (B)$$

which are regular over S^{d-1} given ϕ.

Define spherical distance s as

$$s(u', u) = \text{Arc cos } (\langle u', u\rangle / \|u\| \|u'\|),$$

and set

$$\gamma(s) = -\frac{1}{\omega_d} \int_{\pi/2}^{s} \text{cosec}^{d-2} t (\int_{\pi}^{t} \sin^{d-2} t' dt') dt.$$

The Green function constructed for the equation is defined for $u' \neq u$ by

$$G(u', u) = \gamma(s(u', u)). \qquad (4)$$

For $d = 3$,

$$G(u', u) = \frac{1}{4\pi} \ln \left[1 - \frac{\langle u', u \rangle}{\|u'\| \, \|u\|} \right]. \tag{5}$$

The solution to the equation is then

$$x(u') = \int_{S^{d-1}} G(u', u) \nabla \phi(u) d\omega(u).$$

Here is the main theorem.

Theorem 4.2 Let ϕ be a continuously differentiable function over S^{d-1}. There exists a non-degenerate convex body K with regular support hyperplanes such that $\phi(u)$ is the sum of the principal radii of curvature at that boundary point of K at which u is the outer normal if and only if ϕ satisfies the following conditions. Let

$$\Theta(u', u) = [1 - \langle u', u \rangle^2]^{\frac{1}{2}(1-d)} \int_{\pi}^{\text{arc cos}\langle u', u \rangle} \sin^{d-2} t \, dt / \omega_d,$$

then

$$\int_{S^{d-1}} u \phi(u) d\omega(u) = 0 \, , \tag{C}$$

$$\int_{S^{d-1}} \langle u, u'' \rangle \Theta(u', u) \langle \nabla \phi(u), u'' \rangle d\omega(u) \geq 0 \, , \tag{C'}$$

for all u' on S^{d-1} and u'' for which $\langle u', u'' \rangle = 0$ with strict inequality for some such choices.

Condition (C) is necessary and sufficient in order for equation (B) to have a regular solution on S^{d-1}. In order for the solution of (B) to be indeed the normal representation of a convex body, condition (C') is necessary and sufficient, which is obtained through forcing the quadratic form of the Jacobi matrix to be larger than or equal to zero. For $d = 3$, condition (C') can be simplified as

$$\int_{S^2} \frac{\langle u, u'' \rangle \langle \nabla \phi(u), u'' \rangle}{1 - \langle u, u' \rangle} d\omega(u) \leq 0 \, .$$

4.2 Christoffel's Problem for General Convex Bodies

To feel the sense of generalization, let ϕ be a continuous function defined on S^{d-1}, $\omega \in \mathcal{B}(S^{d-1})$, and define

$$\Phi(\omega) = \int_{\omega} \phi(u) d\omega(u) .$$

Then Φ is a completely additive set function over $\mathcal{B}(S^{d-1})$. In case ϕ is the sum of the principal radii of curvature function associated with convex body K, then Φ is the first order area function of K (recall Equation 3).

Theorem 4.3 A completely additive set function Φ over $\mathcal{B}(S^{d-1})$ is the first order area function of a convex body if and only if it satisfies

$$\int_{S^{d-1}} u\Phi(d\omega) = 0,$$

$$\left| \int_{S^{d-1}} g_1(u', u)\Phi(d\omega) \right| < +\infty,$$

where g_1 is the fundamental singularity

$$g_1(u', u) = \begin{cases} \frac{1}{2\pi}\ln \text{ Arc } \cos(\langle u', u\rangle / \|u\|\|u'\|) & \text{if } d = 3, \\ -\frac{1}{(d-3)\omega_{d-1}}[\text{Arc } \cos(\langle u', u\rangle / \|u\|\|u'\|)]^{3-d} & \text{if } d > 3, \end{cases}$$

and

$$\int_{S^{d-1}} \Lambda(u', v', u)\Phi(d\omega) \geq 0,$$

where

$$\Lambda(u', v', u) = \Gamma(u', u) + \Gamma(v', u) - \Gamma(u' + v', u),$$
$$\Gamma(u', u) = (d-2)\langle u', u\rangle G(u', u) - \langle u', \nabla G(u', u)\rangle,$$

and G is the Green function constructed for the original Christoffel's problem (see Equation 4 and 5).

4.3 Christoffel's Problem for Polytopes

Before we examine the first order area function of polytope, we need to define spherical polytope and spherical complex. We will use the definition by McMullen and Shephard [18].

Definition 4.1 A *spherical polytope* in S^{d-1} is the intersection of a finite number of closed hemispheres which is not empty and contains no pair of antipodal points of S^{d-1}

Definition 4.2 A *spherical complex* $\mathcal{C}$ is a finite set $\mathcal{C} = \{c_1, \ldots, c_r\}$ of distinct spherical polytopes (cells) c_i on S^{d-1} which satisfies the following two conditions :

1. $\cup_{i=1}^{r} c_i = S^{d-1}$,
2. For each i, j, the intersection $c_i \cap c_j$ is a face (proper or improper) of both c_i and c_j.

Note 1 Spherical images of all non-empty faces of a polytope determine a spherical complex on S^{d-1}.

Note 2 Each d-polytope P corresponds to a spherical complex by the mapping

$$\Phi \ : x \in bdP, x \ \rightarrow \overrightarrow{Ox} \cap S^{d-1} \ ,$$

where $\overrightarrow{Ox}$ is the ray passing x with endpoint O. Φ maps proper faces to spherical polytopes. It has been shown that the reverse is not necessarily correct, i.e., there are spherical complexes that are not radial projections of any polytopes.

The first order area function $S_1(P, \omega)$ of a polytope P can be represented as

$$S_1(P, \omega) = \frac{1}{d-1} \sum_{e} \lambda(e) \mu_{d-2}(\omega \cap v(e))$$

for any $\omega \in \mathcal{B}(S^{d-1})$, where the summation goes over all edges e of P, $\lambda(e)$ is the length of edge e, $v(e)$ is the spherical image of e, and $\mu_{d-2}(\omega \cap v(e))$ is the (d-2)-dimensional content of $\omega \cap v(e)$ on S^{d-1}.

Theorem 4.4 (Schneider 1977) For a Borel measure ϕ on S^{d-1} there exists a d-polytope P such that $\phi(\omega) = S_1(P, \omega)$ if and only if ϕ satisfies the following conditions:

1. The support of ϕ is the union of the (d-2)-dimensional elements of a spherical complex $\mathcal{S}$.

2. For each (d-2)-element ζ of $\mathcal{S}$, there exists a positive number $\lambda(\zeta)$ such that $\phi(\omega) = \lambda(\zeta)\mu_{d-2}(\omega)$ for each $\omega \subseteq \zeta$.

3. For each (d-3)-element $\eta \in \mathcal{S}$,

$$\sum_{\zeta} \lambda(\zeta) u(\eta, \zeta) = 0$$

   where the summation goes over all (d-2)-elements $\zeta \in \mathcal{S}$ for which η is a side, and $u(\eta, \zeta) = v_\zeta / \|v_\zeta\|$ such that v_ζ is the orthogonal projection of ζ on the two dimensional linear subspace orthogonal to η.

Proof The necessity of the conditions is not difficult to prove. We will only provide the proof of the sufficiency of the conditions.

Let $\xi_1, \ldots, \xi_k$ be the (d-1)-elements of $\mathcal{S}$. By an *element path* ρ, we mean an ordered sequence $(\xi_{i_1}, \ldots, \xi_{i_m})$ of (d-1)-elements of $\mathcal{S}$ such that $\xi_{i_r} \cap \xi_{i_{r+1}}$ is a (d-2)-element, $r = 1, \ldots, m-1$. Assign a vector $v(\rho)$ to an element path ρ as

$$v(\rho) = \sum_{r=1}^{m-1} \lambda(\xi_{i_r} \cap \xi_{i_{r+1}}) v(i_r, i_{r+1})$$

where $v(i, j)$ is the unit vector which is orthogonal to the linear hull of $\xi_i \cap \xi_j$ and pointing into the interior of the halfspace containing ξ_j. Define the following combinatorial deformations of element paths:

(α) The element path $(\xi_{i_1}, \ldots, \xi_{i_m})$ is replaced by

$$(\xi_{i_1}, \ldots, \xi_{i_r}, \xi_j, \xi_{i_r}, \xi_{i_{r+1}}, \ldots, \xi_{i_m})$$

where $\xi_{i_r} \cap \xi_j$ is a (d-2)-element or vice versa.

(β) The element path $(\xi_{i_1}, \ldots, \xi_{i_m})$ is replaced by

$$(\xi_{i_1}, \ldots, \xi_{i_r}, \xi_{j_1}, \ldots, \xi_{j_s}, \xi_{i_{r+k}}, \ldots, \xi_{i_m})$$

where

$$\xi_{i_r}, \xi_{i_{r+1}}, \ldots, \xi_{i_{r+k}}, \xi_{j_s}, \xi_{j_{s-1}} \ldots, \xi_{j_1}$$

are exactly those (d-1)-elements that contain a specific (d-3)-element η and are arranged in a natural cyclic order.

Obviously, deformations of type (α) do not change the vector $v(\rho)$. By condition 3, deformations of type (β) also do not change $v(\rho)$. Hence, an element path from ξ_{i_1} to ξ_{i_m} can be deformed to any other element paths with the same endpoints, by a finite number of deformation of type (α) and type (β), without changing $v(\rho)$. That is to say that the vector $v(\rho)$ only depends on the start point and end point of element path ρ. For $j = 1, \ldots, k$, define $x_j = v(\rho_j)$ by choosing an arbitrary element path ρ_j from ξ_1 to ξ_j. We will show that the convex hull of the points $x_1, \ldots, x_k$ is the polytope we are looking for.

Define a function $h(\cdot) : R^d \rightarrow R$ as follows: $h(0) = 0$. For every $v \in R^d \setminus \{0\}$, choose a ($d$-1)-element $\xi_{i(v)} \in \mathcal{S}$ such that $v/\|v\| \in \xi_{i(v)}$, then set $h(v) = \langle x_{i(v)}, v \rangle$.

We need to verify that h is well defined, i.e., the definition of h does not depend on the selection of $\xi_{i(v)}$. Let ξ_j be another (d-1)-element such that $v/\|v\| \in \xi_j$. Then there exists an element path $(\xi_{i_1}, \ldots, \xi_{i_m})$ with $i_1 = i_{i(v)}, i_m = j$, and $v/\|v\| \in \xi_{i_r}, r = 1, \ldots, m$. According to the definition of vectors x_i, we have

$$\langle x_{i_m}, v \rangle - \langle x_{i_1}, v \rangle = \sum_{r=1}^{m-1} \lambda(\xi_{i_r} \cap \xi_{i_{r+1}}) \langle v(i_r, i_{r+1}), v \rangle = 0$$

because $v(i_r, i_{r+1})$ is orthogonal to the linear hull of $\xi_{i_r} \cap \xi_{i_{r+1}}$, and by condition $v/\|v\| \in \xi_{i_r} \cap \xi_{i_{r+1}}$, $v(i_r, i_{r+1})$ is perpendicular to v. Thus h is well defined.

It follows from above that h is continuous: The continuity of h at 0 is obvious. Let $u \in R^d \setminus \{0\}$, and U be a neighborhood of $u/\|u\|$ in S^{d-1} that does not intersect any (d-1)-elements of $\mathcal{S}$ that are disjoint to $u/\|u\|$. For $v \in U$, we have $u/\|u\| \in \xi_{i(v)}$, and therefore

$$h(u) = \langle x_{i(v)}, u \rangle = h(v) + \langle x_{i(v)}, u - v \rangle.$$

Thus h is continuous.

We now prove that h is convex. Let $u_1, u_2, \alpha_1 u_1 + \alpha_2 u_2 \in S^{d-1}, \alpha_r > 0$, $r = 1, 2$ be such that u_r lies in the interior of a (d-1)-element ξ_{i_r}, whereby $\alpha_1 u_1 + \alpha_2 u_2 \in \xi_{i_1} \cap \xi_{i_2}$, and $\xi_{i_1} \cap \xi_{i_2}$ is a (d-2)-element. Then

$$h(u_2) = \langle x_{i_2}, u_2 \rangle = \langle x_{i_1}, u_2 \rangle + \lambda(\xi_{i_1} \cap \xi_{i_2}) \langle v(i_1, i_2), u_2 \rangle \; > \; \langle x_{i_1}, u_2 \rangle \, ,$$

and therefore

$$h(\alpha_1 u_1 + \alpha_2 u_2) = \langle x_{i_1}, \alpha_1 u_1 + \alpha_2 u_2 \rangle < \alpha_1 h(u_1) + \alpha_2 h(u_2).$$

Choose $u_1, u_2 \in S^{d-1}$ such that $u_1 \neq \pm u_2$ and the shorter arc connecting u_1 and u_2 does not intersect any element of $\mathcal{S}$ of dimension less than $d - 2$. The restriction of h on the convex cone defined by u_1 and u_2 is piecewise linear and from the continuity and local convexity of h, the convexity of h follows.

Since h is positively homogeneous, convex, and piecewise linear, there must exist a polytope P such that h is the support function of P. From the definition of h, it follows that $x_1, \ldots, x_k$ are vertices of P, and ξ_i is exactly the spherical image of the vertex $x_i, i = 1, \ldots, k$. Therefore $\mathcal{S}$ is the spherical image of the polytope P. Especially, the (d-2)-elements of $\mathcal{S}$ are the spherical images of the edges of P with the (d-2)-element $\xi_i \cap \xi_j$ being the spherical image of the edge with the endpoints x_i and x_j, and this edge has length $\lambda(\xi_i \cap \xi_j)$. Now by definition, the first order area function of P is indeed the given measure ϕ, which completes our proof.

□

The above proof is a direct translation from the German paper [21]. We would like to make a few points which may help understanding Schneider's idea. Suppose we start with a polytope P. We calculate its first order area function, and try to construct a polytope following the above constructive proof.

1. It is assumed that we have a coordinate system in which a polytope will be constructed. Since the vector assigned to an element path $(\xi_1, \xi_{i_1}, \ldots, \xi_{i_m}, \xi_1)$ from ξ_1 to ξ_1 is always a null vector, x_1 is actually the origin of the coordinate system. The choice of ξ_1 among all (d-1)-elements of $\mathcal{S}$ determines the position of the polytope constructed. The reconstructed polytope is unique up to a translation.

2. From its definition, the vector $v(i_r, i_{r+1})$ is parallel to the edge e of P that has $\xi_{i_r} \cap \xi_{i_{r+1}}$ as its spherical image. Then, $\lambda(\xi_{i_r} \cap \xi_{i_{r+1}})v(i_r, i_{r+1})$ is a vector that is parallel to e and has the same length as of e. Therefore, x_i is indeed the vertex of P that has ξ_i as its spherical image.

3. A minor complement to the proof of the function h being convex is needed. Since

$$\langle x_{i_2}, u_1\rangle = \langle x_{i_1}, u_1\rangle + \lambda(\xi_{i_1} \cap \xi_{i_2})\langle v(i_1, i_2), u_2\rangle \ < \ \langle x_{i_1}, u_1\rangle ,$$

we have

$$h(\alpha_1 u_1 + \alpha_2 u_2) = \langle x_{i_2}, \alpha_1 u_1 + \alpha_2 u_2\rangle < \alpha_1 h(u_1) + \alpha_2 h(u_2).$$

This covers the situation when $\alpha_1 u_1 + \alpha_2 u_2$ lies in ξ_{i_2}.

5 Summary

Most of the results surveyed in this paper is about the existence and uniqueness of convex bodies whose property measurements in consideration are given functions, the variables of which are usually interpreted as surface orientations. Having constructive proofs of existence may result in algorithms for reconstructions of convex bodies from their property measurements. Recall the results about support functions, distance functions, and first order area functions, for example.

The characteristic properties of the following functions are studied: spherical image, area function, m-th order area function, distance function, cross sectional measure, breadth, Gaussian curvature, and sum of principal radii of curvature. Those functions encode certain properties of convex bodies as functions of surface orientation. We are particularly interested in support function, first order area function, and sum of principal radii of curvature. Questions **Q1** and **Q2** raised in Section 1 are theoretically answered. We have seen, however, little hint on how **Q3** could be answered.

Support function gets lots of our attention because of its nice properties and the role it plays in solving many problems. Nalwa [19] proposed support function to be an orientation based representation, but gave no explanation about how he might go about it. We think it is still early to determine the role that support function might play in the research of computational vision.

Finally, all of the results are about convex bodies. We certainly hope that we could, in the future, have some results that do not require convexity.

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A Notation and Terminology

We do not intend to unify the terminologies for the subject of discourse. Also, we do not want to present the concepts from scratch. The intention of this section is to provide the definition of the concepts we used (or had in mind) when we wrote this paper.

Our subjects (objects, shapes, surfaces, etc.) are restricted to be connected point sets in $R^d, d = 2, 3, \ldots$. We write int$A$ for the interior of A for any $A \subseteq R^d$. If needed, we will just stay in R^2 and R^3, i.e., we will make no effort to generalize any result to an arbitrary space.

Denote by S^1 and S^2 the unit circle in R^2 and the unit sphere in R^3 respectively. They are also called the Gaussian circle and Gaussian sphere. (S^{d-1} denotes the unit sphere in R^d.) $\pi_d = \pi^{d/2}/\Gamma(d/2+1)$. Hence the area ω_d of S^{d-1} is equal to $d\pi_d = 2\pi^{d/2}/\Gamma(d/2)$. Denote by $\mathcal{B}(A)$ the σ-algebra of Borel subset of A. Let $\mathcal{F}(S^d)$ denote the set of functions defined on S^d, and $C^r(S^d)$ the set of r-order differentiable functions defined on S^d. Denote, by O, the origin of the coordinate systems.

Definition A.1 A set $K \subseteq R^d$ is said to be *convex* if for each pair of distinct point $x, y \in K$ the closed segment with endpoints x and y is contained in K. A convex set has dimension r if it is contained in a r-flat and does not lie in a (r-1)-flat. Denote the dimension of K by $dimK$.

Definition A.2 A convex hypersurface in R^d is the boundary of a d-dimensional convex set K in R^d provided it is non-empty and connected.

Theorem A.1 A convex hypersurface in R^d is either homeomorphic to S^{d-1} or to R^{d-1}, or to a product $S^{d-1-r} \times R^r, 1 \leq r \leq d-2$ (and hence is, respectively called closed or open or cylindrical).

Definition A.3 The *convex hull* hull(A) of a set $A \subseteq R^d$ is defined as the intersection of all convex sets in R^d which contain A. A compact convex set $K \subseteq R^d$ is called a *polytope* provided it is the convex hull of a finite set. A d-polytope is a polytope of dimension d. A *d-simplex* is defined as the convex hull of some $d+1$ affinely independent points.

Definition A.4 A set function on $\mathcal{B}(A)$ is said to be completely additive if for any two set $E_1, E_2 \in \mathcal{B}(A)$ that $E_1 \cap E_2 = \emptyset$ then $f(E_1 + E_2) = f(E_1) + f(E_2)$.

Definition A.5 The support of a measure μ defined in space Ω is defined as the closed set $\Omega - \bigcup\{G : \mu(G) = 0, G \text{ open }\}$.