Convergence Properties of Curvature and Torsion Scale Space Representations

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Abstract

Multi-scale, curvature-based shape representation techniques for planar curves and multi-scale, torsion-based shape representation techniques for space curves have been proposed to the computer vision community by Mokhtarian & Mackworth [1986], Mackworth & Mokhtarian [1988] and Mokhtarian [1988]. These representations are referred to as the regular, renormalized and resampled curvature and torsion scale space images and are computed by combining information about the curvature or torsion of the input curve at a continuum of detail levels.

Arc length parametric representations of planar or space curves are convolved with Gaussian functions of varying standard deviation to compute *evolved* versions of those curves. The process of generating evolved versions of a curve as the standard deviation of the Gaussian function goes from 0 to ∞ is referred to as the *evolution* of that curve. When evolved versions of the curve are computed through an iterative process in which the curve is reparametrized by arc length in each iteration, the process is referred to as *arc length evolution.*

This paper contains a number of important results on the convergence properties of curvature and torsion scale space representations. It has been shown that every closed planar curve will eventually become simple and convex during evolution and arc length evolution and will remain in that state. This result is very important and shows that curvature scale space images are well-behaved in the sense that we can always expect to find a scale level at which the number of curvature zero-crossing points goes to zero and know that new curvature zero-crossing points will not be created beyond that scale level which can be considered to be the high end of the curvature scale space image.

It has also been shown that every closed space curve will eventually tend to a closed planar curve during evolution and arc length evolution and that every closed space curve will eventually enter a state in which new torsion zero-crossing points will not be created during evolution and arc length evolution and will remain in that state.

Furthermore, the proofs are not difficult to comprehend. They can be understood by readers without an extensive knowledge of mathematics.

I. Introduction

A multi-scale representation for one-dimensional functions was first proposed by Stansfield [1980] and later developed by Witkin [1983]. The function $f(x)$ is convolved with a Gaussian function as its variance σ^2 varies from a small to a large value. The zero-crossings of the second derivative of each convolved function are extracted and marked in the x–σ plane. The result is the *scale space image* of the function.

The *curvature scale space image* was introduced by Mokhtarian & Mackworth [1986] as a new shape representation for planar curves. The representation is computed by convolving a path-based parametric representation of the curve with a Gaussian function, as the standard deviation of the Gaussian varies from a small to a large value, and extracting the curvature zero-crossing points of the resulting curves. The representation is essentially invariant under rotation, uniform scaling and translation of the curve. This and a number of other properties makes it suitable for recognizing a noisy curve of arbitrary shape at any scale or orientation.

Mackworth and Mokhtarian [1988] introduced a modification of the curvature scale space image referred to as the *renormalized curvature scale space image*. This representation is computed in a similar fashion but the curve is reparametrized by arc length after convolution. As was demonstrated in [Mackworth & Mokhtarian 1988], the renormalized curvature scale space image is more suitable for recognizing a curve with non-uniform noise added to it. However, unlike the regular curvature scale space representation, the renormalized curvature scale space applies only to closed curves.

The *resampled curvature scale space image* is a substantial refinement of the curvature scale space which is based on the concept of *arc length evolution* [Mokhtarian & Mackworth 1989]. It was shown that the resampled curvature scale space image is more suitable than the renormalized curvature scale space image for recognition of curves with added non-uniform noise or when local shape differences exist.

Given a planar curve

$$\Gamma = \{(x(w), y(w)\}$$

where w is the arc length parameter, an *evolved* version of that curve is defined by

$$\Gamma_\sigma = \{(X(u,\sigma), Y(u,\sigma)\}$$

where

$$X(u,\sigma) = x(u) \circledast g(u,\sigma)$$

and

$$Y(u,\sigma) = y(u) \circledast g(u,\sigma).$$

Function $g(u,\sigma)$ denotes a Gaussian of width σ [Marr & Hildreth 1980]. The process of generating the ordered sequence of curves $\{\Gamma_\sigma | \sigma \geq 0\}$ is referred to as the *evolution* of Γ.

The generalized evolution which maps Γ to Γ_σ is defined by:

$$\Gamma \rightarrow \Gamma_\sigma = \{(X(W,\sigma),\, Y(W,\sigma))\}$$

where

$$X(W,\sigma) = x(W) \circledast g(W,\sigma)$$

and

$$Y(W,\sigma) = y(W) \circledast g(W,\sigma).$$

Note that

$$W = W(w,\sigma)$$

and $W(w,\sigma_0)$, where σ_0 is any value of σ, is a continuous and monotonic function of w. When W always remains the arc length parameter of the evolved curve, the evolution of Γ is referred to as *arc length evolution*. $W(w,\sigma)$ is given explicitly by [Gage & Hamilton 1986]:

$$W(w,\sigma) = -\int_0^\sigma \int_0^W \kappa^2(W,\sigma)\, dW\, d\sigma. \tag{i.1}$$

The curvature and torsion functions of a space curve determine that curve uniquely modulo a rigid motion [DoCarmo 1976]. Mokhtarian [1988a] generalized the concepts of multi-scale representation to propose a curvature and *torsion scale space* representation for space curves. The torsion scale space image is computed by again convolving an arc length parametric representation of the curve with a Gaussian function, as the standard deviation of the Gaussian varies from a small to a large value, and locating the torsion zero-crossing points of the resulting curves. The curvature scale space image of a space curve is computed in a similar way but curvature level-crossings rather than zero-crossings are used. Renormalized and resampled curvature and torsion scale space images of space curves can also be computed by generalizing the concepts employed to compute the renormalized and resampled curvature scale space images of planar curves [Mokhtarian 1990]. Finally, evolution and arc length evolution of space curves are defined in a similar fashion. An evolved version of a space curve

$$\Gamma = \{(x(\omega), y(\omega), z(\omega))\}$$

where ω is again the arc length parameter, is defined by

$$\Gamma_\sigma = \{(X(u,\sigma), Y(u,\sigma), Z(u,\sigma)\}$$

and an arc length evolved version of Γ is defined by

$$\Gamma_\sigma = \{(X(W,\sigma), Y(W,\sigma), Z(W,\sigma))\}$$

where W is defined by equation (i.1).

A number of evolution and arc length evolution properties of planar and space curves have been studied [Mackworth & Mokhtarian 1988, Mokhtarian 1988, Mokhtarian 1989a, Mokhtarian 1989b, Mokhtarian 1990]. As a result, the local and global properties of planar and space curves during evolution and arc length evolution are well understood. The existing results help create a strong foundation for the multi-scale, curvature- and torsion-based shape representation techniques proposed earlier.

A number of results on evolution and arc length evolution of planar and space curves have been used in this paper to prove new results. The existing results which have been used are as following:

Theorem i.1: Simple planar curves remain simple during arc length evolution.

Theorem i.2: Let Γ be a planar curve in C_2. If all evolved and arc length evolved curves Γ_σ are in C_2, then all extrema of contours in the regular, renormalized and resampled curvature scale space images of Γ are maxima.

Theorem i.3: Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions of w. Let Γ_σ be an arc length evolved version of Γ with a cusp point at w_0. There is a $\delta > 0$ such that $\Gamma_{\sigma+\delta}$ has two new curvature zero-crossings in a neighborhood of w_0.

Theorem i.4: Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions of w. Let Γ_σ be an arc length evolved version of Γ with a cusp point at w_0. There is a $\delta > 0$ such that $\Gamma_{\sigma+\delta}$ intersects itself in a neighborhood of point w_0.

Theorem i.5: New torsion zero-crossings can appear on a smooth space curve during evolution or arc length evolution in a neighborhood of a point of zero curvature.

Theorem i.6: Let $\Gamma = (x(w), y(w), z(w))$ be a space curve in C_1 and let $x(w)$, $y(w)$ and $z(w)$ be polynomial functions of w. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma), Z(W,\sigma))$ be an evolved or arc length evolved version of Γ with a cusp point at W_0, then either $\Gamma_{\sigma+\delta}$ has two new torsion zero-crossings in a neighborhood of W_0 or a torsion zero-crossing point exists at W_0 on $\Gamma_{\sigma-\delta}$ and $\Gamma_{\sigma+\delta}$.

Theorem i.7: Let $\Gamma = (x(w),y(w),z(w))$ be a space curve in C_1 and let $x(w)$, $y(w)$ and $z(w)$ be polynomial functions of w. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma), Z(W,\sigma))$ be an evolved or arc length evolved version of Γ with a cusp point at W_0. There is a $\delta>0$ such that either $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point W_0 or two projections of $\Gamma_{\sigma-\delta}$ intersect themselves in a neighborhood of point W_0.

This paper contains a number of important results on the convergence properties of curvature and torsion scale space representations. It has been shown that every closed planar curve will eventually become simple and convex during evolution and arc length evolution and will remain in that state. This result is very important and shows that curvature scale space images are well-behaved in the sense that we can always expect to find a scale level σ_0 at which the number of curvature zero-crossing points goes to zero and know that new curvature zero-crossing points will not be created beyond that scale level. σ_0 can be considered to be the high end of the curvature scale space image.

It has also been shown that every closed space curve will eventually tend to a closed planar curve during evolution and arc length evolution and that every closed space curve will eventually enter a state in which new torsion zero-crossing points will not be created during evolution and arc length evolution and will remain in that state.

Furthermore, the proofs are not difficult to comprehend. They can be understood by readers without an extensive knowledge of mathematics.

II. Convergence properties of 1-D functions

This section contains a number of results on the properties of various 1-D functions when they are convolved with Gaussian functions with large standard deviations. The results of this section will be used in sections III and IV to prove important results on the convergence properties of planar and space curves respectively.

The first two theorems look at the convergence properties of polynomial functions.

Theorem ii.1: Let $f(u)$ be a polynomial function

$$f(u) = a_n u^n + a_{n-1} u^{n-1} + \cdots + a_1 u + a_0$$

and let n be even. Let

$$F(u,\sigma) = f(u) \circledast g(u,\sigma)$$

be the function obtained by convolving $f(u)$ with a Gaussian function of width σ. When σ is large, $F(u,\sigma)$ does not have any zeros of the second derivative.

Proof: It follows from the assumptions that n–2, the highest power of u in

$$f''(u) = b_{n-2}u^{n-2} + b_{n-3}u^{n-3} + \cdots + b_1u + b_0$$

is also even.

Note that when function $h(u) = u^k$ is convolved with a Gaussian function $g(u,t)$, where t is $\sigma^2/2$, the convolved function $H(u,t)$ can be expressed as [Mokhtarian 1988b]:

$$H(u,t) = \sum_{\substack{p=0 \\ (p\ even)}}^{k} 1.3 \cdots (p-1)\frac{(2t)^{p/2}k(k-1)\cdots(k-p+1)}{p!}u^{k-p}.$$

It follows that:

$$F''(u,t) = b_{n-2}(u^{n-2} + c_1tu^{n-4} + \cdots + c_{\frac{n}{2}-1}t^{\frac{n}{2}-1})$$

$$+ b_{n-3}(u^{n-3} + d_1tu^{n-5} + \cdots + d_{\frac{n}{2}-2}t^{\frac{n}{2}-2}u)$$

$$+ \cdots + b_1u + b_0$$

where all constants c_i and d_i are positive. Assume w.l.o.g. that b_{n-2} is also positive.

As $u \rightarrow \infty$ or $u \rightarrow -\infty$, $b_{n-2}u^{n-2}$ becomes the dominant term. Since n–2 is even, $F''(u,t)$ is positive as u grows in magnitude. When u is not large in magnitude, the term of $F''(u,t)$ with the highest power of t is dominant since t is large by assumption. That term is: $c_{\frac{n}{2}-1}t^{\frac{n}{2}-1}$. Since $c_{\frac{n}{2}-1}$ is positive, $c_{\frac{n}{2}-1}t^{\frac{n}{2}-1}$ is also positive. It follows that $F''(u,t)$ is always positive therefore $F(u,t)$ has no zeros of the second derivative. □

Theorem ii.2: Let $f(u)$ be a polynomial function

$$f(u) = a_nu^n + a_{n-1}u^{n-1} + \cdots + a_1u + a_0$$

and let n be odd. Let

$$F(u,\sigma) = f(u) \circledast g(u,\sigma)$$

be the function obtained by convolving $f(u)$ with a Gaussian function of width σ. When σ is large, $F(u,\sigma)$ has only one zero of the second derivative.

Proof: It follows from the assumptions that n–2, the highest power of u in

$$f''(u) = b_{n-2}u^{n-2} + b_{n-3}u^{n-3} + \cdots + b_1 u + b_0$$

is also odd.

Note that when function $h(u) = u^k$ is convolved with a Gaussian function $g(u,t)$, where t is $\sigma^2/2$, the convolved function $H(u,t)$ can be expressed as [Mokhtarian 1988b]:

$$H(u,t) = \sum_{\substack{p=0 \\ (p\ even)}}^{k} 1.3 \cdots (p-1) \frac{(2t)^{p/2}\, k(k-1) \cdots (k-p+1)}{p!} u^{k-p}.$$

It follows that:

$$F''(u,t) = b_{n-2}(u^{n-2} + c_1 t u^{n-4} + \cdots + c_{\frac{n-3}{2}} t^{\frac{n-3}{2}} u)$$

$$+ b_{n-3}(u^{n-3} + d_1 t u^{n-5} + \cdots + d_{\frac{n-3}{2}} t^{\frac{n-3}{2}})$$

$$+ \cdots + b_1 u + b_0$$

where all constants c_i and d_i are positive. Assume w.l.o.g. that b_{n-2} is also positive.

As $u \rightarrow \infty$ or $u \rightarrow -\infty$, $b_{n-2}u^{n-2}$ becomes the dominant term. Since n–2 is odd, $F''(u,t) \rightarrow \infty$ as $u \rightarrow \infty$ and $F''(u,t) \rightarrow -\infty$ as $u \rightarrow -\infty$. Therefore $F''(u,t)$ becomes zero at least once. We must show that $F''(u,t)$ becomes zero only once.

Assume by contradiction that $F''(u,t)$ becomes zero at least twice. It follows that there is an extremum between the two zeros. At the point u_0 where the extremum occurs, $F'''(u,t)$ goes to zero. But $F'''(u,t)$ is a polynomial in which the term with the highest power has even power. This is a contradiction of theorem ii.1. It follows that $F''(u,t)$ goes to zero only once. □

The next two theorems look at the behaviour of sine and cosine functions when convolved with Gaussians.

Theorem ii.3: When a sine function is convolved with a Gaussian, the result is another sine function with the same frequency and lower amplitude.

Proof: The convolution of a function $f(u)$ with a Gaussian function can be expressed as:

$$F(u,t) = f(u) \circledast g(u,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} f(u+2v\sqrt{t}\,)\, dv.$$

It follows that when

$$f(u) = a\sin(bu)$$

$F(u,t)$ can be expressed as:

$$F(u,t) = \int_{-\infty}^{\infty} \frac{a}{\sqrt{\pi}} e^{-v^2} \sin(b(u+2v\sqrt{t}\,))\, dv$$

or

$$F(u,t) = \frac{a}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \sin(bu+2bv\sqrt{t}\,)\, dv.$$

Using the formula:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

it follows that:

$$F(u,t) = \frac{a}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} (\sin(bu)\cos(2bv\sqrt{t}\,) + \cos(bu)\sin(2bv\sqrt{t}\,))\, dv$$

and

$$F(u,t) = \frac{a\sin(bu)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \cos(2bv\sqrt{t}\,)\, dv + \frac{a\cos(bu)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \sin(2bv\sqrt{t}\,)\, dv.$$

We now make use of the formulae:

$$\int_{-\infty}^{\infty} e^{-v^2} \cos(nv)\, dv = 2\int_{0}^{\infty} e^{-v^2} \cos(nv)\, dv = \sqrt{\pi}\; e^{-n^2/4} \tag{ii.1}$$

and

$$\int_{-\infty}^{\infty} e^{-v^2} \sin(nv)\, dv = 0 \tag{ii.2}$$

to conclude that:

$$F(u,t) = \frac{a}{e^{b^2 t}} \sin(bu).$$

Therefore $F(u,t)$ and $f(u)$ have the same frequency but the amplitude of $F(u,t)$ is smaller than the amplitude of $f(u)$. □

Theorem ii.4: When a cosine function is convolved with a Gaussian, the result is another cosine function with the same frequency and lower amplitude.

Proof: The convolution of a function $f(u)$ with a Gaussian function can be expressed as:

$$F(u,t) = f(u) \circledast g(u,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} f(u+2v\sqrt{t}\,)\, dv.$$

It follows that when

$$f(u) = a\cos(bu)$$

$F(u,t)$ can be expressed as:

$$F(u,t) = \int_{-\infty}^{\infty} \frac{a}{\sqrt{\pi}} e^{-v^2} \cos(b(u+2v\sqrt{t}\,))\, dv$$

or

$$F(u,t) = \frac{a}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \cos(bu+2bv\sqrt{t}\,)\, dv.$$

Using the formula:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

it follows that:

$$F(u,t) = \frac{a}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} (\cos(bu)\cos(2bv\sqrt{t}\,) - \sin(bu)\sin(2bv\sqrt{t}\,))\, dv$$

and

$$F(u,t) = \frac{a\cos(bu)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \cos(2bv\sqrt{t}\,)\, dv - \frac{a\sin(bu)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \sin(2bv\sqrt{t}\,)\, dv.$$

We now make use of formulae (ii.1) and (ii.2) to conclude that:

$$F(u,t) = \frac{a}{e^{b^2 t}} \cos(bu).$$

Therefore $F(u,t)$ and $f(u)$ have the same frequency but the amplitude of $F(u,t)$ is smaller than the amplitude of $f(u)$. □

The next two theorems express facts about the convergence properties of an infinite sum of sine or cosine functions when convolved with Gaussians with large widths.

Theorem ii.5: Let

$$f_s(u) = \sum_{k=1}^{\infty} a_k \sin(2k\pi u)$$

and let

$$F_s(u,t) = f_s(u) \circledast g(u,t)$$

be the function obtained by convolving $f_s(u)$ with a Gaussian function of width t. $F_s(u)$ has only two inflection points and two zero-crossing points for large values of t.

Proof: It follows from theorem ii.3 that

$$F_s(u,t) = \sum_{k=1}^{\infty} \frac{a_k}{e^{4k^2\pi^2 t}} \sin(2k\pi u).$$

The first sine function in F_s (corresponding to k=1) is the dominant one when t is large and determines the qualitative shape of F_s. The first sign function goes to zero only at u=0.5 and at u=1. All other sine functions in F_s also go to zero at u=0.5 and at u=1. Therefore F_s has at least two zero-crossing points at u=0.5 and u=1. If F_s has any additional zeros, they must be in small neighborhoods of u=0.5 and u=1 since at all other points, the first sine function which determines the shape of F_s is away from zero. Assume w.l.o.g. that the sign of the first sine function is positive. F_s can be approximated to any degree of accuracy by considering only the first N sine functions in the infinite sum. Therefore there is a small neighborhood of point u=0.5 in which all the first N sine functions in F_s are monotonically increasing or monotonically decreasing. Let F_s^+ be the sum of those sine functions which are monotonically decreasing and let F_s^- be the sum of those sine functions which are monotonically increasing. Note that in the neighborhood being considered, F_s^+ is positive when u<0.5, is zero at u=0.5 and is negative when u>0.5. Note also that in the same neighborhood, F_s^- is negative when u<0.5, is zero at u=0.5 and is positive when u>0.5. For any value of u in the neighborhood under consideration, the absolute value of F_s^+ is much larger than the absolute value of F_s^- since the first sine function belongs to F_s^+. It follows that $F_s^+ + F_s^-$ is also positive for u<0.5 and negative for u>0.5. Hence no additional zeros exist in a small neighborhood of u=0.5. A similar argument shows that no additional zeros exist in a neighborhood of u=1. Therefore F_s has only two zero-crossing points. Since the second derivative of F_s has the same form as F_s, the same proof also holds about the second derivative of F_s. It follows that the second derivative of F_s also has only two zero-crossing points. Therefore F_s has only two inflection points. □

Theorem ii.6: Let

$$f_c(u) = \sum_{k=1}^{\infty} b_k \cos(2k\pi u)$$

and let

$$F_c(u,t) = f_c(u) \circledast g(u,t)$$

be the function obtained by convolving $f_c(u)$ with a Gaussian function of width t. $F_c(u)$ has only two inflection points and two zero-crossing points for large values of t.

Proof: It follows from theorem ii.4 that

$$F_c(u,t) = \sum_{k=1}^{\infty} \frac{b_k}{e^{4k^2\pi^2 t}} \cos(2k\pi u).$$

The first cosine function in F_c (corresponding to k=1) is dominant when t is large and determines the qualitative shape of F_c. The first cosine function goes to zero at u=0.25 and u=0.75. Therefore F_c has at least two zero-crossing points in small neighborhoods of those two points. If F_c has any additional zeros, they must also be in small neighborhoods of u=0.25 and u=0.75. Assume w.l.o.g. that the sign of the first cosine function is positive. F_c can be approximated to any degree of accuracy by considering only the first N cosine functions in the infinite sum. Therefore there is a small neighborhood of point u=0.25 in which all cosine functions are either monotonic or have only one extremum. All cosine functions in this neighborhood can be divided into the following four groups:

(1) Cosine functions which go to zero at u=0.25 and are monotonically decreasing. Denote their sum by F_1.
(2) Cosine functions which go to zero at u=0.25 and are monotonically increasing. Denote their sum by F_2.
(3) Cosine functions which have a maximum at u=0.25. Denote their sum by F_3.
(4) Cosine functions which have a minimum at u=0.25. Denote their sum by F_4.

Let $F = F_1 + F_2$. Since the first cosine function belongs to group (1), for any value of u in the neighborhood under consideration, the magnitude of F_1 is much larger than the magnitude of F_2. Furthermore, the rate of change of F_1 is much larger than the rate of change of F_2. It follows that F is also monotonically decreasing and has a zero at u=0.25. Note also that the rate of change of F is much larger than the rates of change of F_3 and F_4 in the considered neighborhood. It follows that when F_3 and F_4 are added to F, the result (which is F_c) is a monotonically decreasing function in a neighborhood of u=0.25. Therefore F_c goes to zero only once in that neighborhood. A similar argument shows that F_c goes to zero only once in a neighborhood of u=0.75. Hence F_c goes to zero only twice. Since the second derivative of F_c has the same form as F_c, the same proof also holds about the second derivative of F_c. It follows that the second derivative of F_c also goes to zero only twice. Therefore F_c has only two inflection points. □

The following theorem expresses the main convergence theorem of section II.

Theorem ii.7: Let $f(u)$ be a periodic function and let

$$F(u,t) = f(u) \circledast g(u,t)$$

be the function obtained by convolving $f(u)$ with a Gaussian function of width t. $F(u,t)$ has only two inflection points for large values of t.

Proof: Since $f(u)$ is a periodic function, it can be expressed as [Duff & Naylor 1966]:

$$f(u) = \sum_{k=1}^{\infty} a_k \sin(2k\pi u) + \sum_{k=1}^{\infty} b_k \cos(2k\pi u).$$

Therefore

$$f'(u) = \sum_{k=1}^{\infty} 2\, a_k\, k\, \pi \cos(2k\pi u) + \sum_{k=1}^{\infty} -2\, b_k\, k\, \pi \sin(2k\pi u)$$

and

$$f''(u) = \sum_{k=1}^{\infty} -4\, a_k\, k^2 \pi^2 \sin(2k\pi u) + \sum_{k=1}^{\infty} -4\, b_k\, k^2 \pi^2 \cos(2k\pi u).$$

Therefore all derivatives of $f(u)$ have essentially the same form: an infinite sum of sine functions plus an infinite sum of cosine functions. It follows from theorems ii.3 and ii.4 that:

$$F''(u,t) = \sum_{k=1}^{\infty} \frac{-4\, a_k\, k^2 \pi^2}{e^{4 k^2 \pi^2 t}} \sin(2k\pi u) + \sum_{k=1}^{\infty} \frac{-4\, b_k\, k^2 \pi^2}{e^{4 k^2 \pi^2 t}} \cos(2k\pi u).$$

Theorem ii.5 showed that, when t is large, the sum of the sine functions in $F''(u,t)$ is a function F_s with only two inflection points and two zero-crossing points. Theorem ii.6 showed that, when t is large, the sum of the cosine functions in $F''(u,t)$ is a function F_c with only two inflection points and two zero-crossing points. We will now show that the sum of F_s and F_c is also a function with two zero-crossing points and two inflection points. Assume that the signs of the first sine function in F_s and the first cosine function in F_c are positive. The arguments used in other cases are similar. F_s has two zero-crossing points at u=0.5 and u=1 and F_c has two zero-crossing points at approximately u=0.25 and u=0.75. Now consider the range of values [0,0.25] for u. F_s+F_c is positive in that range therefore there are no zero-crossings. Now consider the range [0.25,0.5]. F_s+F_c is positive at u=0.25 and negative at u=0.5 so there is at least one zero-crossing point between those two values of u. Since both F_s and F_c are monotonically decreasing in that range, there can be only one zero-crossing point between u=0.25 and u=0.5. There are no zero-crossing points between u=0.5 and u=0.75 since both F_s and F_c are negative in that range. Finally, consider the range [0.75,1]. F_s+F_c is negative at u=0.75 and positive at u=1 so there is at least one zero-crossing point between those two values of u. Since both F_s and F_c are monotonically increasing in that range, there can be only one zero-crossing point between u=0.75 and u=1. Therefore $F'' = F_s+F_c$ has only two zero-crossing points. Therefore $F(u,t)$ has only two inflection points. □

III. Convergence Properties of Planar Curves

This section contains important results on the convergence properties of evolution and arc length evolution of planar curves as defined in section I. These results show that evolution and arc length evolution of planar curves are well-behaved processes.

The first theorem expresses the convergence properties of polynomially-represented, open planar curves.

Theorem iii.1. Let $\Gamma = (x(u), y(u))$ be an open, planar curve represented polynomially and let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma))$ be an evolved version of Γ. When σ is large, **(a)** Γ_σ has one or no curvature zero-crossing points and **(b)** Γ_σ is simple.

Proof: Assume that σ is large. It follows from theorems ii.1 and ii.2 that each of functions $X(u,\sigma)$ and $Y(u,\sigma)$ has either one or no zeros of the second derivative. Note that when Γ_σ is reparametrized by arc length s, functions $X(s,\sigma)$ and $Y(s,\sigma)$ have the same number of zeros of the second derivative as functions $X(u,\sigma)$ and $Y(u,\sigma)$.

(a) Since the magnitude of curvature on Γ_σ is given by:

$$|\kappa(s,\sigma)| = \sqrt{\ddot{X}(s,\sigma)^2 + \ddot{Y}(s,\sigma)^2}$$

at every point where $\kappa(s,\sigma)$ goes to zero, $\ddot{X}(s,\sigma)$ and $\ddot{Y}(s,\sigma)$ also go to zero. It follows that Γ_σ can have at most one curvature zero-crossing point.

(b) Assume by contradiction that Γ_σ is self-intersecting. Let P be the point of self-intersection. Since for two different values of u, u_1 and u_2, Γ_σ goes through point P, it follows that each of functions $X(u,\sigma)$ and $Y(u,\sigma)$ must have at least one extremum inside the self-intersection loop. Since each of functions $X(u,\sigma)$ and $Y(u,\sigma)$ can have at most one extremum, there can be no other extrema of either $X(u,\sigma)$ or $Y(u,\sigma)$ outside the self-intersection loop. We will consider only function $X(u,\sigma)$. There are two cases:

i. The extremum of $X(u,\sigma)$ inside the loop is a minimum.
Let Q be the point of Γ_σ where the minimum occurs. Q divides Γ_σ into two curves. One starts at Q and goes to Q' (at which $x=+\infty$), the other starts at Q and goes to Q'' (at which $x=-\infty$). Consider the latter curve. Since Q is a minimum, x-coordinates of points of this curve will increase in a neighborhood of Q. However, eventually the curve goes to Q''. It follows that there must be a maximum on this curve after point Q. Therefore Γ_σ has at least two extrema. A contradiction has been reached.

ii. The extremum of $X(u,\sigma)$ inside the loop is a maximum.
Let R be the point of Γ_σ where the maximum occurs. R divides Γ_σ into two curves. One starts at R and goes to R' (at which $x=+\infty$), the other starts at R and goes to R'' (at which $x=-\infty$). Consider the former curve. Since R is a maximum, x-coordinates of points of this curve will decrease in a neighborhood of R. However, eventually the curve goes to R'. It follows that there must be a minimum on this curve after point R. Therefore Γ_σ has at least two extrema. Again a contradiction has been reached.

It follows that Γ_σ must be simple. □

Theorems iii.2 through iii.6 examine the convergence properties of closed, planar curves during evolution.

Theorem iii.2. Let Γ be a closed planar curve. Γ becomes convex during evolution and remains convex.

Proof: Since Γ is closed, its coordinate functions, $x(u)$ and $y(u)$, are periodic functions. It follows that $x(u)$ can be expressed as [Duff & Naylor 1966]:

$$x(u) = \sum_{k=1}^{\infty} a_k \sin(2k\pi u) + \sum_{k=1}^{\infty} b_k \cos(2k\pi u).$$

Therefore

$$x'(u) = \sum_{k=1}^{\infty} 2\, a_k\, k\, \pi \cos(2k\pi u) + \sum_{k=1}^{\infty} -2\, b_k\, k\, \pi \sin(2k\pi u)$$

and

$$x''(u) = \sum_{k=1}^{\infty} -4\, a_k\, k^2\, \pi^2 \sin(2k\pi u) + \sum_{k=1}^{\infty} -4\, b_k\, k^2\, \pi^2 \cos(2k\pi u).$$

Let

$$X'(u,t) = x'(u) \circledast g(u,t)$$

and

$$X''(u,t) = x''(u) \circledast g(u,t).$$

It follows from theorems ii.3 and ii.4 that

$$X'(u,t) = \sum_{k=1}^{\infty} \frac{2\, a_k\, k\, \pi}{e^{4\, k^2 \pi^2 t}} \cos(2k\pi u) + \sum_{k=1}^{\infty} \frac{-2\, b_k\, k\, \pi}{e^{4 k^2 \pi^2 t}} \sin(2k\pi u)$$

and

$$X''(u,t) = \sum_{k=1}^{\infty} \frac{-4\, a_k\, k^2\, \pi^2}{e^{4\, k^2 \pi^2 t}} \sin(2k\pi u) + \sum_{k=1}^{\infty} \frac{-4\, b_k\, k^2\, \pi^2}{e^{4\, k^2 \pi^2 t}} \cos(2k\pi u).$$

Due to the exponential term in the denominators, as t grows large the first component in each sum (corresponding to $k=1$) becomes dominant. Therefore, for large t, $X'(u,t)$ and $X''(u,t)$ can be expressed as:

$$X'(u,t) = \frac{2\, \pi\, a_1}{e^{4\, \pi^2 t}} \cos(2\pi u) - \frac{2\, \pi\, b_1}{e^{4\, \pi^2 t}} \sin(2\pi u)$$

and

$$X''(u,t) = - \frac{4\, \pi^2\, a_1}{e^{4\, \pi^2 t}} \sin(2\pi u) - \frac{4\, \pi^2\, b_1}{e^{4\, \pi^2 t}} \cos(2\pi u).$$

Similarly

$$Y'(u,t) = \frac{2\, \pi\, c_1}{e^{4\, \pi^2 t}} \cos(2\pi u) - \frac{2\, \pi\, d_1}{e^{4\, \pi^2 t}} \sin(2\pi u)$$

and

$$Y''(u,t) = - \frac{4\, \pi^2\, c_1}{e^{4\, \pi^2 t}} \sin(2\pi u) - \frac{4\, \pi^2\, d_1}{e^{4\, \pi^2 t}} \cos(2\pi u).$$

Curvature on Γ_t is given by [Goetz 1970]:

$$\kappa(u,t) = \frac{X'(u,t)\,Y''(u,t) - Y'(u,t)\,X''(u,t)}{(X'(u,t)^2 + Y'(u,t)^2)^{3/2}}.$$

Since the denominator of the expression above is always positive, we will investigate the numerator only. Let

$$\alpha(u,t) = X'(u,t)\,Y''(u,t) - Y'(u,t)\,X''(u,t).$$

It follows that

$$\alpha(u,t) = \frac{8\,\pi^3}{e^{8\,\pi^2\,t}}(-a_1\,c_1 \sin(2\pi u)\cos(2\pi u) - a_1\,d_1 \cos^2(2\pi u) + b_1\,c_1 \sin^2(2\pi u)$$

$$+ b_1\,d_1 \sin(2\pi u)\cos(2\pi u) + a_1\,c_1 \sin(2\pi u)\cos(2\pi u) - a_1\,d_1 \sin^2(2\pi u)$$

$$+ b_1\,c_1 \cos^2(2\pi u) - b_1\,d_1 \sin(2\pi u)\cos(2\pi u)).$$

Using the formula:

$$\sin^2(A) + \cos^2(A) = 1$$

it follows that:

$$\alpha(u,t) = \frac{8\,\pi^3}{e^{8\,\pi^2\,t}}(b_1\,c_1 - a_1\,d_1).$$

Hence there are no curvature zero-crossings on Γ_t. Therefore Γ_t is convex. Since this proof holds for all large values of t, Γ remains convex once it becomes convex. □

Theorem iii.3: Let $\Gamma = (X(u,\sigma), Y(u,\sigma))$ be a simple and convex closed planar curve. Coordinate functions $X(u,\sigma)$ and $Y(u,\sigma)$ (which are periodic) each have two inflection points.

Proof: Since Γ is a simple and convex closed planar curve, the total change in the direction of its tangent vector as Γ is traversed one full cycle is equal to 2π. Let θ be the angle that the tangent vector makes with the positive x-axis. θ takes on each value in the range [0,360] only once. It follows that the tangent vector becomes horizontal only twice and becomes vertical only twice. Each time the tangent vector is horizontal, function $Y(u,\sigma)$ has an extremum and each time the tangent vector is vertical, function $X(u,\sigma)$ has an extremum. Hence each of the functions $X(u,\sigma)$ and $Y(u,\sigma)$ has two extrema. Since these functions are periodic, each also has two inflection points. □

Theorem iii.4: Let $\Gamma = (x(u), y(u))$ be a closed planar curve such that functions $x(u)$ and $y(u)$ are periodic functions each of whom has two inflection points. Γ is simple and convex.

Proof: Reparametrize Γ by arc length s. Since $\ddot{x}(u)$ and $\ddot{y}(u)$ each intersect zero only twice and u is a monotonic function of s, $\ddot{x}(s)$ and $\ddot{y}(s)$ also intersect zero only twice. Therefore each of $x(s)$ and $y(s)$ also has two inflection points. Note that on a curve with arc length parametrization, the magnitude of curvature κ is given by:

$$|\kappa| = \sqrt{(\ddot{x})^2 + (\ddot{y})^2}.$$

Therefore at each inflection point of the curve, $\ddot{x}(s) = \ddot{y}(s) = 0$.

Assume by contradiction that Γ is not simple and convex. Suppose that Γ is not convex. There can be only two inflection points on Γ. Let P and P' be those inflection points (figure iii.1). Let T be the tangent line at point P and let T' be the tangent line at P'. Let L be the line going through P and P'. The curvature of Γ changes sign at P therefore Γ will turn to the right of line T. Since curvature of Γ does not change sign again, Γ will continue to turn until it intersects line L at point Q different from P. Similarly, Γ changes sign of curvature at P' and will intersect L at point Q' different from P'. Suppose line L is not vertical. Note that the x-coordinate of Q is larger than the x-coordinate of P and that the x-coordinate of Q' is smaller than the x-coordinate of P'. It follows that the maximum and minimum of $x(s)$ lie outside segment PP'. So there is at least one inflection point of $x(s)$ which is not at P or P'. Hence $x(s)$ has more than two inflection points which is a contradiction. If line L is vertical, a similar argument can be applied to $y(s)$ and a contradiction will be reached again. It follows that Γ must be convex.

Now suppose that Γ is not simple. Therefore Γ intersects itself in at least one point. Let P be the point of self-intersection (figure iii.2) and let T and T' be the tangent vectors at P. Note that function $x(s)$ has at least one extremum inside the loop since the value of that function is equal to the x-coordinate of point P for two different values of s, s_1 and s_2, and takes on different values when s is between s_1 and s_2. Follow the curve in the direction of T. Since Γ is convex, it will continue to turn in the same direction until the tangent to the curve becomes vertical. Let that point be Q. Also follow the curve in the direction of T'. Again Γ will continue to turn in the same direction until its tangent becomes vertical at point Q'. There is an extremum of $x(s)$ at each of Q and Q'. Hence there are at least three extrema on $x(s)$. This is a contradiction since $x(s)$ is a periodic function which has only two inflection points and therefore two extrema. It follows that Γ must also be simple. It was shown earlier that Γ is convex. It follows that Γ is simple and convex. $\square$

Theorem iii.5: Simple and convex planar curves remain simple and convex during evolution.

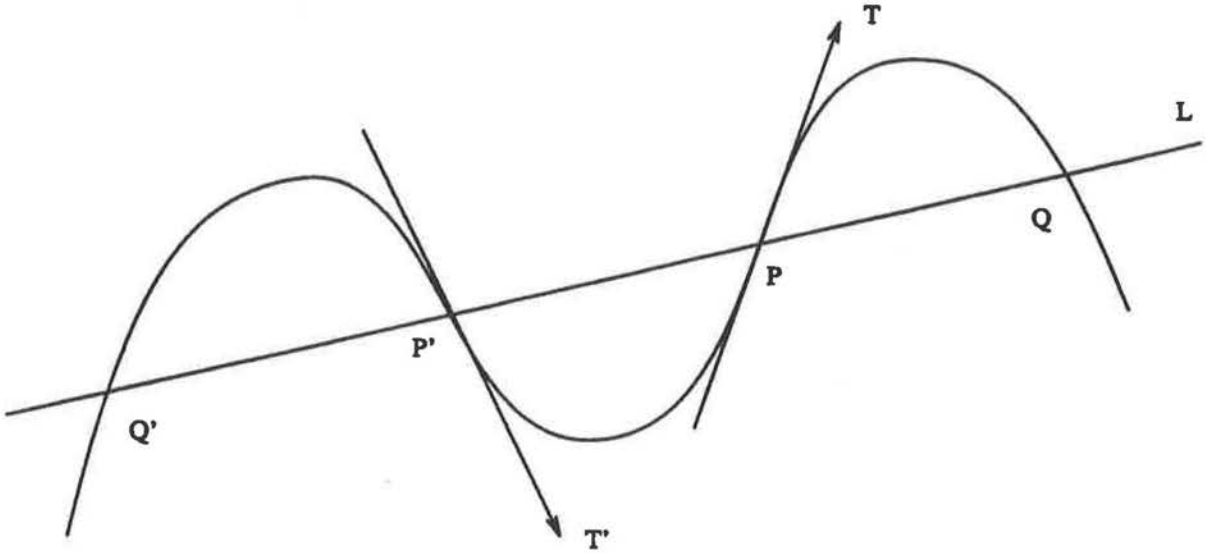

Figure iii.1

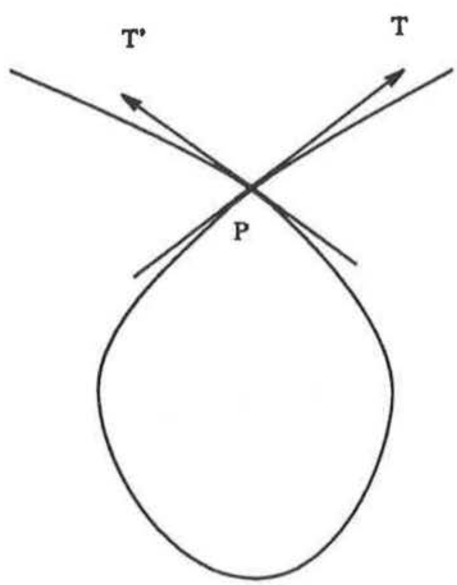

Figure iii.2

Proof: Let $\Gamma = (x(u), y(u))$ be a simple and convex closed planar curve. It follows from theorem iii.3 that coordinate functions $x(u)$ and $y(u)$ each has only two inflection points. It is known that the number of inflection points on $x(u)$ and $y(u)$ will not increase when they are convolved with Gaussian functions [Yuille & Poggio 1986]. Therefore each of the coordinate functions of evolved versions of Γ also has only two inflection points. It follows from theorem iii.4 that evolved versions of Γ are also simple and convex. Hence Γ remains simple and convex during evolution. □

The following theorem expresses the main result of this section on evolution. Theorem iii.2 showed that closed planar curves become convex during evolution and remain convex. That theorem is sufficient to guarantee that there exists a high end of the curvature scale space image, however, it does not impose further constraints on the limiting shape of planar curves during evolution. The following theorem is more difficult to prove but it sheds more light on the evolution process. It follows from this theorem that the correct stopping criterion when computing the curvature scale space image of a planar curve is when the curve becomes simple and convex.

Theorem iii.6: Let Γ be a closed planar curve. Γ becomes simple and convex during evolution and remains simple and convex.

Proof: Let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma))$ be an evolved version of Γ and let σ be large. Theorem ii.7 showed that $X(u,\sigma)$ and $Y(u,\sigma)$ each have only two inflection points. It follows from theorem iii.4 that Γ_σ must be simple and convex. It follows from theorem iii.5 that Γ remains simple and convex. □

The remaining theorems in this section explore the convergence properties of planar curves during arc length evolution.

Theorem iii.7: Let $\Gamma = (x(w), y(w))$ be an open, planar curve represented polynomially and let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of Γ. When σ is large, **(a)** Γ_σ has one or no curvature zero-crossing points and **(b)** Γ_σ is simple.

Proof: The proof of theorem iii.1 also applies to functions $X(W,\sigma)$ and $Y(W,\sigma)$. The theorem follows. □

Note that the following theorem has also been proven in [Gage & Hamilton 1986] using a much longer geometric proof.

Theorem iii.8: Simple and convex planar curves remain simple and convex during arc length evolution.

Proof: Suppose Γ is a simple and convex planar curve. Theorem i.1 states that simple curves remain simple during arc length evolution. Therefore Γ remains simple during arc length evolution. To become non-convex, Γ must form new curvature zero-crossings. It follows from theorems i.2 and i.3 that every planar curve must form a cusp point just before formation of new curvature zero-crossings during arc length evolution and theorem i.4 states that every planar curve must intersect itself just before formation of a cusp point during arc length evolution. It follows that every planar curve must intersect itself just before formation of new curvature zero-crossings during arc length evolution. Since Γ remains simple during arc length evolution, formation of new curvature zero-crossings on Γ is not possible. Therefore Γ also remains convex. □

The following theorem expresses the main result of this section on arc length evolution.

Theorem iii.9: Let Γ be a closed planar curve. Γ becomes simple and convex during arc length evolution and remains simple and convex.

Proof: Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of Γ and let σ be large. It follows from theorem ii.7 that each of $X(W,\sigma)$ and $Y(W,\sigma)$ has only two inflection points. From theorem iii.4, it follows that Γ_σ must be simple and convex, and from theorem iii.8, it follows that Γ remains simple and convex. □

IV. Convergence Properties of Space Curves

This section contains important results on the convergence properties of evolution and arc length evolution of space curves as defined in section I. These results show that evolution and arc length evolution of space curves are well-behaved processes.

The first theorem shows that a space curve gradually flattens into a planar curve as it evolves.

Theorem iv.1: Let Γ be a closed space curve. Γ_t tends to a planar curve during evolution as t grows large.

Proof: Since Γ is closed, its coordinate functions, $x(u)$, $y(u)$ and $z(u)$ are periodic functions. Let

$$X(u,t) = x(u) \circledast g(u,t)$$

$$Y(u,t) = y(u) \circledast g(u,t)$$

and

$$Z(u,t) = z(u) \circledast g(u,t).$$

As shown in the proof of theorem iii.2, when t is large, $X'(u,t)$ and $Y'(u,t)$ can be expressed as:

$$X'(u,t) = \frac{2\pi a_1}{e^{4\pi^2 t}} \cos(2\pi u) - \frac{2\pi b_1}{e^{4\pi^2 t}} \sin(2\pi u)$$

and

$$Y'(u,t) = \frac{2\pi c_1}{e^{4\pi^2 t}} \cos(2\pi u) - \frac{2\pi d_1}{e^{4\pi^2 t}} \sin(2\pi u).$$

Similarly

$$Z'(u,t) = \frac{2\pi e_1}{e^{4\pi^2 t}} \cos(2\pi u) - \frac{2\pi f_1}{e^{4\pi^2 t}} \sin(2\pi u).$$

Furthermore

$$X''(u,t) = -\frac{4\pi^2 a_1}{e^{4\pi^2 t}} \sin(2\pi u) - \frac{4\pi^2 b_1}{e^{4\pi^2 t}} \cos(2\pi u)$$

$$Y''(u,t) = -\frac{4\pi^2 c_1}{e^{4\pi^2 t}} \sin(2\pi u) - \frac{4\pi^2 d_1}{e^{4\pi^2 t}} \cos(2\pi u)$$

$$Z''(u,t) = -\frac{4\pi^2 e_1}{e^{4\pi^2 t}} \sin(2\pi u) - \frac{4\pi^2 f_1}{e^{4\pi^2 t}} \cos(2\pi u)$$

$$X'''(u,t) = -\frac{8\pi^3 a_1}{e^{4\pi^2 t}} \cos(2\pi u) + \frac{8\pi^3 b_1}{e^{4\pi^2 t}} \sin(2\pi u)$$

$$Y'''(u,t) = -\frac{8\pi^3 c_1}{e^{4\pi^2 t}} \cos(2\pi u) + \frac{8\pi^3 d_1}{e^{4\pi^2 t}} \sin(2\pi u)$$

and

$$Z'''(u,t) = -\frac{8\pi^3 e_1}{e^{4\pi^2 t}} \cos(2\pi u) + \frac{8\pi^3 f_1}{e^{4\pi^2 t}} \sin(2\pi u).$$

Torsion on Γ_t is given by [Goetz 1970]:

$$\tau(u,t) = \frac{Z'''X'Y'' - Z'''Y'X'' + Y'''Z'X'' - Y'''X'Z'' + X'''Y'Z'' - X'''Z'Y''}{(Y'Z'' - Z'Y'')^2 + (Z'X'' - X'Z'')^2 + (X'Y'' - Y'X'')^2}.$$

Since the denominator of the expression above is always positive, we will investigate the numerator only. Let $\beta(u,t)$ equal the numerator of the expression for $\tau(u,t)$. It follows that

$$\beta(u,t) = X''(Z'Y''' - Y'Z''') + Y''(X'Z''' - Z'X''') + Z''(Y'X''' - X'Y''').$$

Note that

$$Z'Y''' = Y'Z''' = \frac{16\pi^4}{e^{8\pi^2 t}}(-c_1e_1\cos^2(2\pi u) + c_1f_1\sin(2\pi u)\cos(2\pi u)$$

$$+ d_1e_1\sin(2\pi u)\cos(2\pi u) - d_1f_1\sin^2(2\pi u))$$

and

$$X'Z''' = Z'X''' = \frac{16\pi^4}{e^{8\pi^2 t}}(-a_1e_1\cos^2(2\pi u) + b_1e_1\sin(2\pi u)\cos(2\pi u)$$

$$+ a_1f_1\sin(2\pi u)\cos(2\pi u) - b_1f_1\sin^2(2\pi u))$$

and

$$Y'X''' = X'Y''' = \frac{16\pi^4}{e^{8\pi^2 t}}(-a_1c_1\cos^2(2\pi u) + a_1d_1\sin(2\pi u)\cos(2\pi u)$$

$$+ b_1c_1\sin(2\pi u)\cos(2\pi u) - b_1d_1\sin^2(2\pi u)).$$

Therefore torsion goes to zero at every point of Γ_t at t grows large. It follows that as t becomes large, Γ_t tends to a planar curve. □

The following theorem expresses the main result of this section on evolution of space curves.

Theorem iv.2: Every closed space curve reaches a state during evolution in which new torsion zero-crossings will not be created and will remain in that state.

Proof: Let $\Gamma = (x(u), y(u), z(u))$ be a closed space curve and let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma))$ be an evolved version of Γ. By theorem ii.7, when σ is large, functions $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$ (which are periodic) will each have only two inflection points. Therefore by theorem iii.4, the planar curves defined by the coordinate function pairs $(X(u,\sigma), Y(u,\sigma))$, $(X(u,\sigma), Z(u,\sigma))$ and $(Y(u,\sigma), Z(u,\sigma))$ are simple and convex. By theorem i.5, new torsion zero-crossing points can appear on a space curve during evolution in a neighborhood of a point of zero curvature. On Γ_σ, the magnitude of curvature is given by:

$$|\kappa| = \frac{\sqrt{A^2 + B^2 + C^2}}{((\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2)^{3/2}}$$

where

$$A = \dot{y}\ddot{z} - \dot{z}\ddot{y}$$

$$B = \dot{z}\ddot{x} - \dot{x}\ddot{z}$$

and

$$C = \dot{x}\ddot{y} - \dot{y}\ddot{x}.$$

Note that A, B and C determine the signs of the curvatures of the planar curves defined by Γ_σ. It follows that if a point of zero curvature exists on Γ_σ, there are also curvature zero-crossing points on the planar curves defined by Γ_σ. Since those curves are simple and convex, Γ_σ can not have any zeros of curvature either. Theorem i.6 states that the only other time torsion zero-crossings can appear on a space curve during evolution is right after the formation of a cusp point. Theorem i.7 states that a space curve or two of the planar curves defined by that space curve must intersect themselves just before the formation of a cusp point during evolution. If the space curve intersects itself, then all three planar curves defined by that curve intersect themselves. However, all the planar curves defined by Γ_σ are simple and convex. It follows that cusp points can not exist on Γ_σ either. Hence none of the conditions necessary for the creation of new torsion zero-crossing points are realized on Γ_σ. Furthermore, when the value of σ grows even larger, each of the functions $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$ will continue to have only two inflection points. Hence all the arguments above will continue to apply and Γ_σ will remain in a state in which new torsion zero-crossing points will not be created. □

The remaining theorems of this section explore the convergence properties of space curves during arc length evolution.

Theorem iv.3: Let Γ be a closed space curve. Γ_t tends to a planar curve during arc length evolution as t grows large.

Proof: Let

$$\Gamma_t = (X(W,t), Y(W,t), Z(W,t)).$$

The proof of theorem iv.1 also applies to functions $X(W,t)$, $Y(W,t)$ and $Z(W,t)$. The theorem follows. □

The following theorem expresses the main result of this section on arc length evolution of space curves.

Theorem iv.4: Every closed space curve reaches a state during arc length evolution in which new torsion zero-crossings will not be created and will remain in that state.

Proof: Let Γ be a closed space curve and let

$$\Gamma_t = (X(W,t), Y(W,t), Z(W,t))$$

be an arc length evolved version of Γ. Assume t is large. The proof of theorem iv.2 also applies to functions $X(W,t)$, $Y(W,t)$ and $Z(W,t)$. The theorem follows. $\square$

V. Conclusions

A number of important results on the convergence properties of curvature and torsion scale space representations were presented in this paper.

It was shown that every closed planar curve eventually becomes simple and convex during evolution and arc length evolution and will remain in that state. This result shows that evolution and arc length evolution of planar curves are well-behaved processes in terms of convergence to a stable state.

It was also shown that every closed space curve eventually tends to a closed planar curve during evolution and arc length evolution and that every closed space curve eventually enters a state in which new torsion zero-crossing points will not be created during evolution and arc length evolution and will remain in that state. This result shows that evolution and arc length evolution of space curves are also well-behaved processes.

The convergence properties of curvature and torsion scale space representations presented in this paper strengthen the theoretical foundation of these multi-scale representations.

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References

DoCarmo, M., *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.

Duff, G. F. D. and D. Naylor, *Differential equations of applied mathematics*, John Wiley & sons, 1966.

Gage, M. and R. S. Hamilton, "The heat equation shrinking convex plane curves," *J. Differential Geometry*, vol. 23, pp. 69-96, 1986.

Goetz, A., *Introduction to differential geometry*, Addison Wesley, Reading, MA, 1970.

Mackworth, A. K. and F. Mokhtarian, "The renormalized curvature scale space and the evolution properties of planar curves," *Proc. IEEE CVPR*, pp. 318-326, Ann Arbor, Michigan, 1988.

Marr, D. and E. C. Hildreth, "Theory of edge detection," *Proc. R. Soc. Lond. B*, vol. 207, pp. 187-217, 1980.

Mokhtarian, F. and A. K. Mackworth, "Scale-based description and recognition of planar curves and two-dimensional shapes," *IEEE PAMI*, vol. 8, pp. 34-43, 1986.

Mokhtarian, F., "Multi-scale description of space curves and three-dimensional objects," *Proc. IEEE CVPR*, pp. 298-303, Ann Arbor, Michigan, 1988a.

Mokhtarian, F., "Evolution properties of space curves," *Proc. ICCV*, pp. 100-105, Tarpon Springs, Florida, 1988b.

Mokhtarian, F., "Fingerprint theorems for curvature and torsion zero-crossings," *Proc. IEEE CVPR*, pp. 269-275, San Diego, California, 1989a.

Mokhtarian, F., "A theory of multi-scale, curvature-based shape representation for planar curves," Technical Report 89-14, Dept. Computer Science, Univ. British Columbia, Vancouver, B.C., Canada, 1989b.

Mokhtarian, F., "A theory of multi-scale, curvature- and torsion-based shape representation for space curves," Technical Report 90-1, Dept. Computer Science, Univ. British Columbia, Vancouver, B.C., Canada, 1990.

Stansfield, J. L., "Conclusions from the commodity expert project," AI memo 601, MIT AI Lab, Cambridge, MA, 1980.

Witkin, A. P., "Scale space filtering," *Proc. IJCAI*, pp. 1019-1023, Karlsruhe, W. Germany, 1983.

Yuille, A. L. and T. A. Poggio, "Scaling theorems for zero-crossings," *IEEE PAMI*, vol. 8, pp. 15-25, 1986.